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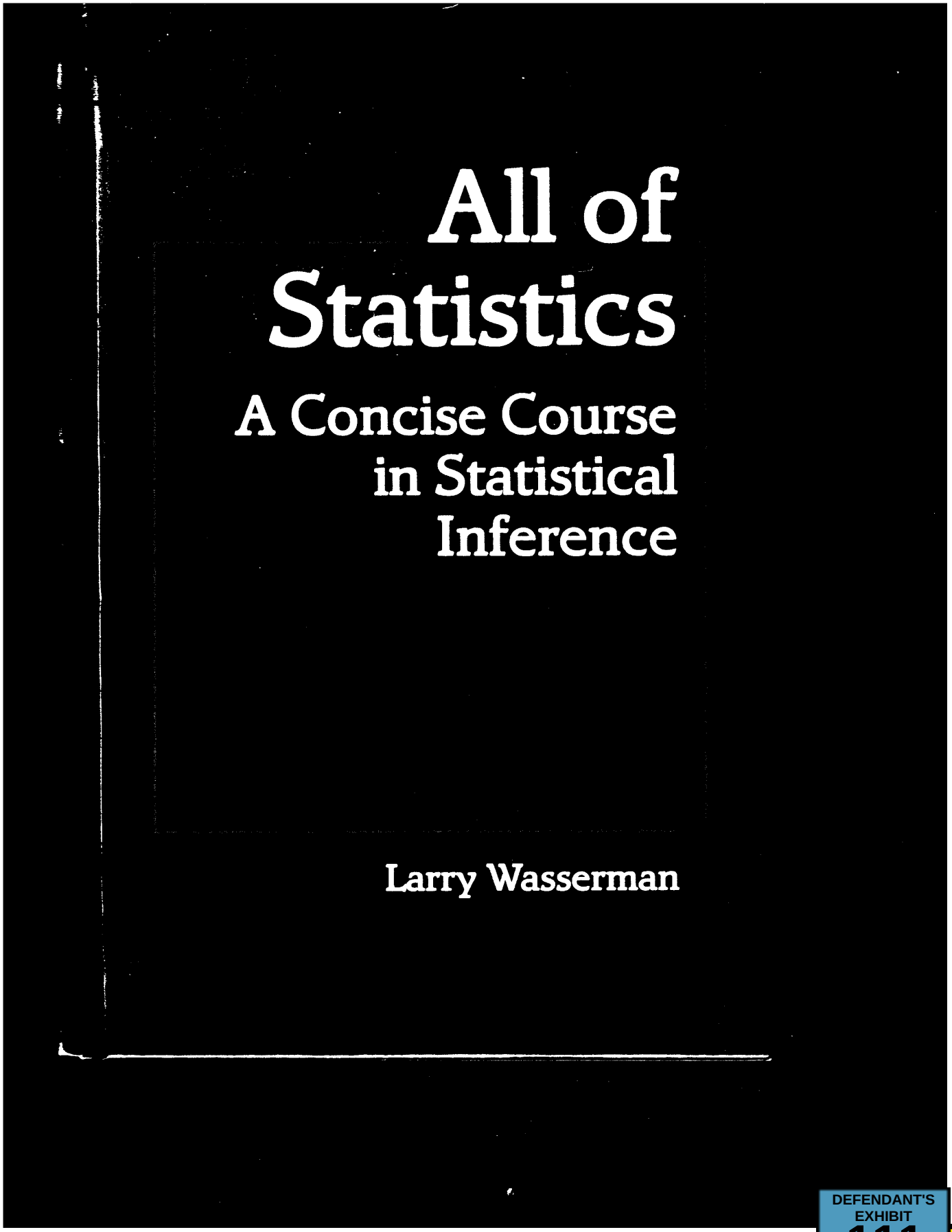

DEFENDANT'S EXHIBIT 111
BIT 2

Springer Texts in Statistics

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Formulae
Control of Stochastic Systems

(continued after index)

Larry Wasserman

All of Statistics

A Concise Course in Statistical Inference

With 95 Figures

Springer

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6.12 Definition. *An estimator is* **asymptotically Normal** *if*

$$\frac{\widehat{\theta}_n - \theta}{\mathsf{se}} \rightsquigarrow N(0,1). \tag{6.8}$$

6.3.2 Confidence Sets

A $1-\alpha$ **confidence interval** for a parameter θ is an interval $C_n = (a,b)$ where $a = a(X_1,\ldots,X_n)$ and $b = b(X_1,\ldots,X_n)$ are functions of the data such that

$$\mathbb{P}_\theta(\theta \in C_n) \geq 1-\alpha, \quad \text{for all } \theta \in \Theta. \tag{6.9}$$

In words, (a,b) traps θ with probability $1-\alpha$. We call $1-\alpha$ the **coverage** of the confidence interval.

Warning! C_n is random and θ is fixed.

Commonly, people use 95 percent confidence intervals, which corresponds to choosing $\alpha = 0.05$. If θ is a vector then we use a **confidence set** (such as a sphere or an ellipse) instead of an interval.

Warning! There is much confusion about how to interpret a confidence interval. A confidence interval is not a probability statement about θ since θ is a fixed quantity, not a random variable. Some texts interpret confidence intervals as follows: if I repeat the experiment over and over, the interval will contain the parameter 95 percent of the time. This is correct but useless since we rarely repeat the same experiment over and over. A better interpretation is this:

> On day 1, you collect data and construct a 95 percent confidence interval for a parameter θ_1. On day 2, you collect new data and construct a 95 percent confidence interval for an unrelated parameter θ_2. On day 3, you collect new data and construct a 95 percent confidence interval for an unrelated parameter θ_3. You continue this way constructing confidence intervals for a sequence of unrelated parameters $\theta_1, \theta_2, \ldots$. Then 95 percent of your intervals will trap the true parameter value. There is no need to introduce the idea of repeating the same experiment over and over.

6.13 Example. Every day, newspapers report opinion polls. For example, they might say that "83 percent of the population favor arming pilots with guns." Usually, you will see a statement like "this poll is accurate to within 4 points

...of the time." They ...
...for the true but unk...
...with guns. If you form ...
...your life, 95 percent ...
...is true even though you ...
...question) every day. ■

...14 Example. The fact that ... about θ is confusing. Consi... Let θ be a fixed, known ... variables such that $\mathbb{P}(X_i = ...) = ... - X_i$ and suppose that you ... "confidence interval" which ...

$$C = \begin{cases} \{Y_1 - 1 \\ \{Y_1 - 1 \end{cases}$$

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...struct a 95 percent confidence ... you collect new data and con... for an unrelated parameter θ_2, ... construct a 95 percent confi...meter ... You continue this way ... a sequence of unrelated param... intervals will trap the true ... introduce the idea of repeating

... opinion polls. For example, they ... favor arming pilots with guns." ... is accurate to within 4 points ... percent of the time." They are saying that 83 ± 4 is a 95 percent confidence interval for the true but unknown proportion p of people who favor arming pilots with guns. If you form a confidence interval this way every day for the rest of your life, 95 percent of your intervals will contain the true parameter. This is true even though you are estimating a different quantity (a different poll question) every day. ■

6.14 Example. The fact that a confidence interval is not a probability statement about θ is confusing. Consider this example from Berger and Wolpert (1984). Let θ be a fixed, known real number and let X_1, X_2 be independent random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. Now define $Y_i = \theta + X_i$ and suppose that you only observe Y_1 and Y_2. Define the following "confidence interval" which actually only contains one point:

$$C = \begin{cases} \{Y_1 - 1\} & \text{if } Y_1 = Y_2 \\ \{(Y_1 + Y_2)/2\} & \text{if } Y_1 \neq Y_2. \end{cases}$$

You can check that, no matter what θ is, we have $\mathbb{P}_\theta(\theta \in C) = 3/4$ so this is a 75 percent confidence interval. Suppose we now do the experiment and get $Y_1 = 15$ and $Y_2 = 17$. Then our 75 percent confidence interval is $\{16\}$. However, we are certain that $\theta = 16$. If you wanted to make a probability statement about θ you would probably say that $\mathbb{P}(\theta \in C | Y_1, Y_2) = 1$. There is nothing wrong with saying that $\{16\}$ is a 75 percent confidence interval. But it is not a probability statement about θ. ■

In Chapter 11 we will discuss Bayesian methods in which we treat θ as if it were a random variable and we do make probability statements about θ. In particular, we will make statements like "the probability that θ is in C_n, given the data, is 95 percent." However, these Bayesian intervals refer to degree-of-belief probabilities. These Bayesian intervals will not, in general, trap the parameter 95 percent of the time.

6.15 Example. In the coin flipping setting, let $C_n = (\widehat{p}_n - \epsilon_n, \widehat{p}_n + \epsilon_n)$ where $\epsilon_n = \sqrt{\log(2/\alpha)/(2n)}$. From Hoeffding's inequality (4.4) it follows that

$$\mathbb{P}(p \in C_n) \geq 1 - \alpha$$

for every p. Hence, C_n is a $1-\alpha$ confidence interval. ■

As mentioned earlier, point estimators often have a limiting Normal distribution, meaning that equation (6.8) holds, that is, $\widehat{\theta}_n \approx N(\theta, \widehat{\mathsf{se}}^2)$. In this case we can construct (approximate) confidence intervals as follows.

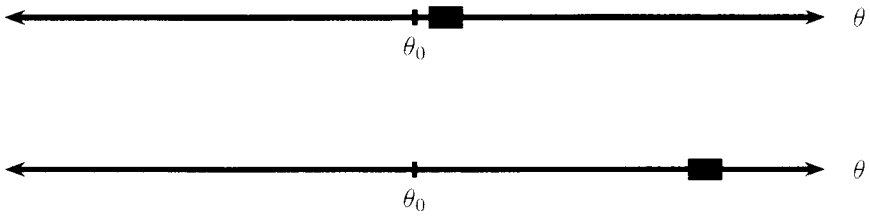


FIGURE 10.2. Scientific significance versus statistical significance. A level α test rejects $H_0 : \theta = \theta_0$ if and only if the $1 - \alpha$ confidence interval does not include θ_0. Here are two different confidence intervals. Both exclude θ_0 so in both cases the test would reject H_0. But in the first case, the estimated value of θ is close to θ_0 so the finding is probably of little scientific or practical value. In the second case, the estimated value of θ is far from θ_0 so the finding is of scientific value. This shows two things. First, statistical significance does not imply that a finding is of scientific importance. Second, confidence intervals are often more informative than tests.

10.2 p-values

Reporting "reject H_0" or "retain H_0" is not very informative. Instead, we could ask, for every α, whether the test rejects at that level. Generally, if the test rejects at level α it will also reject at level $\alpha' > \alpha$. Hence, there is a smallest α at which the test rejects and we call this number the p-value. See Figure 10.3.

10.11 Definition. *Suppose that for every $\alpha \in (0, 1)$ we have a size α test with rejection region R_α. Then.*

$$\text{p-value} = \inf\left\{\alpha : T(X^n) \in R_\alpha\right\}.$$

That is, the p-value is the smallest level at which we can reject H_0.

Informally, the p-value is a measure of the evidence against H_0: the smaller the p-value, the stronger the evidence against H_0. Typically, researchers use the following evidence scale:

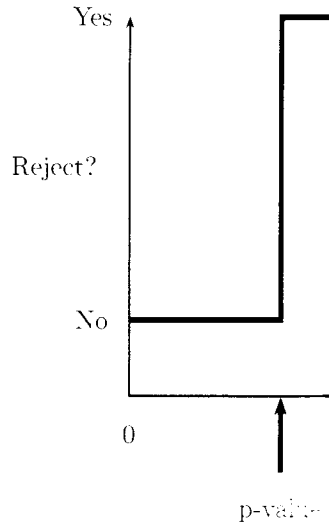


FIGURE 10.3. p-values explained. For
at level α? The p-value is the smalles
against H_0 is strong, the p-value wil

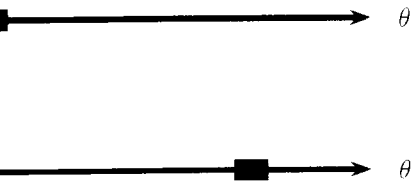
p-value	evidence
$< .01$	very
$.01 - .05$	strong
$.05 - .10$	weak
$> .1$	little

Warning! A large p-value is n
p-value can occur for two reasons:
has low power.

Warning! Do not confuse the p-v
not the probability that the null

The following result explains how

[2]We discuss quantities like $\mathbb{P}(H_0|\text{Data}$



statistical significance. A level α test
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$X^n \in R_\alpha \Big\}$.

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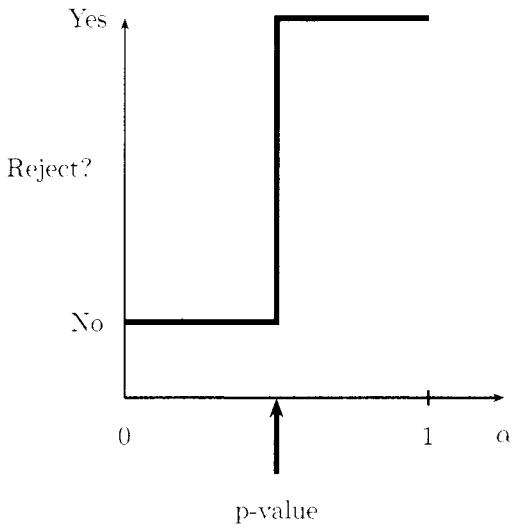

FIGURE 10.3. p-values explained. For each α we can ask: does our test reject H_0 at level α? The p-value is the smallest α at which we do reject H_0. If the evidence against H_0 is strong, the p-value will be small.

p-value	evidence
$< .01$	very strong evidence against H_0
$.01 - .05$	strong evidence against H_0
$.05 - .10$	weak evidence against H_0
$> .1$	little or no evidence against H_0

Warning! A large p-value is not strong evidence in favor of H_0. A large p-value can occur for two reasons: (i) H_0 is true or (ii) H_0 is false but the test has low power.

Warning! Do not confuse the p-value with $\mathbb{P}(H_0|\text{Data})$. [2] **The p-value is not the probability that the null hypothesis is true.**

The following result explains how to compute the p-value.

[2] We discuss quantities like $\mathbb{P}(H_0|\text{Data})$ in the chapter on Bayesian inference.

10.12 Theorem. *Suppose that the size α test is of the form*

$$\text{reject } H_0 \text{ if and only if } T(X^n) \geq c_\alpha.$$

Then,

$$\text{p-value} = \sup_{\theta \in \Theta_0} \mathbb{P}_\theta(T(X^n) \geq T(x^n))$$

where x^n is the observed value of X^n. If $\Theta_0 = \{\theta_0\}$ then

$$\text{p-value} = \mathbb{P}_{\theta_0}(T(X^n) \geq T(x^n)).$$

We can express Theorem 10.12 as follows:

> The p-value is the probability (under H_0) of observing a value of the test statistic the same as or more extreme than what was actually observed.

10.13 Theorem. *Let $w = (\widehat{\theta} - \theta_0)/\widehat{\mathsf{se}}$ denote the observed value of the Wald statistic W. The p-value is given by*

$$\text{p} - \text{value} = \mathbb{P}_{\theta_0}(|W| > |w|) \approx \mathbb{P}(|Z| > |w|) = 2\Phi(-|w|) \tag{10.7}$$

where $Z \sim N(0,1)$.

To understand this last theorem, look at Figure 10.4.
Here is an important property of p-values.

10.14 Theorem. *If the test statistic has a continuous distribution, then under $H_0 : \theta = \theta_0$, the p-value has a Uniform (0,1) distribution. Therefore, if we reject H_0 when the p-value is less than α, the probability of a type I error is α.*

In other words, if H_0 is true, the p-value is like a random draw from a Unif$(0,1)$ distribution. If H_1 is true, the distribution of the p-value will tend to concentrate closer to 0.

10.15 Example. Recall the cholesterol data from Example 7.15. To test if the means are different we compute

$$W = \frac{\widehat{\delta} - 0}{\widehat{\mathsf{se}}} = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} = \frac{216.2 - 195.3}{\sqrt{5^2 + 2.4^2}} = 3.78.$$

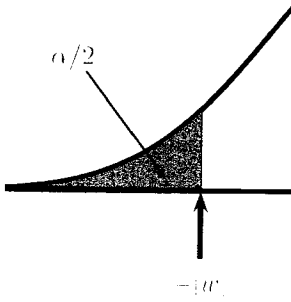

FIGURE 10.4. The p-value is
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$\sim N(0,1)$.

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10.3 The χ^2 Distribu

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