Expert Report on North Carolina's Enacted Congressional and General Assembly Districts

Christopher A. Cooper

December 23, 2021

Introduction

My name is Christopher A. Cooper. I have been asked to provide an analysis of the partisan characteristics of North Carolina's congressional and General Assembly maps, enacted on November 4, 2021. I am conducting this analysis as a private citizen and am not speaking for my employer, nor am I conducting this work on university time, or using university resources.

I am the Robert Lee Madison Distinguished Professor of Political Science and Public Affairs at Western Carolina University, where I have been a tenured or tenure-track professor since 2002. I hold a PhD and MA in Political Science from the University of Tennessee, Knoxville and a BA in Political Science and Sociology from Winthrop University. My academic research focuses on state politics and policy, elections, and southern politics—with particular application to North Carolina. To date, I have published over 50 academic journal articles and book chapters, co-edited one book focused on North Carolina (*The New Politics of The Old North State*), and co-authored one book related to politics in the South, including North Carolina (both books with the University of North Carolina Press). I teach courses on state and local politics, political parties, campaigns, and elections, southern politics, research methods, and election administration. In 2013, I was named the North Carolina Professor of the Year by the Carnegie Foundation for the Advancement of Teaching and I have received Western Carolina University's highest honors in teaching (Board of Governors Teaching Award), and scholarship (University Scholar). My current curriculum vitae is attached as Attachment A.

Much of my academic and applied research relates to North Carolina politics and policy and I am a frequent source for news media seeking comments about politics in the Old North State. My quotes have appeared in national and international outlets including *The New York Times*, *The Washington Post*, Politico, BBC, NPR's *All Things Considered*, and *The New Yorker*, as well as in North Carolina-based outlets including *The News and Observer*, *The Charlotte Observer*, *Asheville-Citizen Times*, *Carolina Journal*, *Spectrum News*, and NPR affiliates in Chapel Hill, Charlotte, and Asheville. I have written over 100 op-eds on North Carolina, southern and national elections and politics, including pieces in *The Atlanta Journal-Constitution*, NBC.com, *The News and Observer*, *The Charlotte Observer*, and *Asheville Citizen-Times*, and I regularly give talks about North Carolina politics, North Carolina elections, and the redistricting process to groups throughout the state. I previously served as an expert witness in *Common Cause v. Lewis*, 18-CVS-014001 (N.C. Super. Ct. Sep. 3, 2019).

I am being compensated at a rate of $300 per hour.

North Carolina is a state defined by competitive two-party politics in terms of its citizens and in its elections for statewide elective offices. Its congressional and state legislative delegations, by contrast, have defied this evidence of competitiveness and moderation and have leaned heavily towards the party in control of the General Assembly, despite the fact that Democrats and Republicans garner similar numbers of statewide votes.

This difference cannot be explained away as a result of where Democrats and Republicans happen to live. As Stanford political geographer Jonathan Rodden demonstrated, North Carolina does not show as much evidence of "natural clustering" as other states. "Due to the presence of a sprawling knowledge-economy corridor, a series of smaller automobile cities with relative low partisan gradients, and the distribution of rural African Americans, Democrats are relatively efficiently distributed in North Carolina at the scale of congressional districts."[1] Looking across all 50 states, Political Scientists Alex Keena, Michael Latner, Anthony J. McGann, and Charles Anthony Smith come to a similar conclusion at the state legislative level: "It is clear that geographical considerations such as the urban concentration of Democrats cannot explain away partisan gerrymandering. There is strong evidence that it is indeed possible to draw unbiased (or almost unbiased) districting plans, even in states with large and densely clustered city dwellers."[2]

As I demonstrate in the analysis that follows, the available evidence indicates that this gap in representation is due to partisan gerrymandering, drawing lines to benefit one party at the expense of the other. While a small deviation from established political patterns is not necessarily evidence of gerrymandering, the differences observed in North Carolina's political outcomes are large and sustained.

Gerrymandering is generally accepted as a threat to democracy in North Carolina and across the nation. This statement is true regardless of partisanship. For example, a 2018 Elon Poll found that just 10% of registered voters in North Carolina believe the current redistricting system is "mostly fair."[3] A more recent poll found that 72% of North Carolinians believe gerrymandering is "a very serious problem" or "a somewhat serious problem" while only 6% believe it is "not a problem." The same poll (which, it should be noted, includes question wording that references both Democratic and Republican gerrymandering) found that 74% of North Carolinians "support efforts by the courts to ensure maps are fair and constitutional."[4] Yet another recent poll found that 89% of North Carolina voters "oppose drawing voting districts to help one political party or certain politicians win an election."[5] A recent op-ed in *The News and Observer* by Republican Carter Wrenn and Democrat Gary Pearce illustrates bi-partisan agreement on the evils of gerrymandering in clear terms. They explain, "We agree that gerrymandering is a major problem that undermines the foundations of our democracy. We agree that districts shouldn't be drawn to help one political party,

---

[1] Rodden, Jonathan, *Why Cities Lose* (New York: Basic Books, 2019), 173.
[2] Keena, Alex, Michael Latner Anthony J. McGann, and Charles Anthony Smith, *Gerrymandering in the States: Partisanship, Race and the Transformation of American Federalism* (New York: Cambridge University Press, 2021), 86.
[3] Elon Poll, "The State of Political Knowledge in North Carolina," February 12-15, 2018, *available at* https://www.elon.edu/u/elon-poll/wp-content/uploads/sites/819/2019/02/Elon-Poll-Report-022318.pdf.
[4] Public Policy Polling, "North Carolina Survey Results," December 6-7, 2021, *available at* https://progressncaction.org/wp-content/uploads/2021/12/NorthCarolinaResults.pdf.
[5] RepresentUs, "North Carolina Polling: Voters See Gerrymandering as a Major Problem, Want Reform," August 9, 2021, *available at* https://represent.us/wp-content/uploads/2021/08/Rep-US-Polling-Memo-North-Carolina-0821.pdf.

no more than college basketball games should be rigged to favor one team."[6] The preference for fair maps—those not gerrymandered to achieve a partisan advantage—is not a partisan one.

Summary of Key Findings

- North Carolina is, by virtually any measure, a "purple state" with healthy two-party competition at the statewide level. The North Carolina Governor is a Democrat, while the U.S. Senators are Republicans. There are more registered Democrats than Republicans in the state, and in the 2020 election, the two-party vote share difference between Donald Trump and Joe Biden was the smallest of any state that Trump won.

- North Carolina has a history of gerrymandering for partisan gain.[7] North Carolina's maps since 2011, in particular, have demonstrated clear partisan bias[8] that has implications for democracy. Immediately after the 2011 redistricting cycle, North Carolina's democracy weakened considerably, according to one scholar, moving from a democracy score that placed the Old North State roughly in the middle of the pack to one near the bottom of the country.[9]

- As a result of the 2020 census, North Carolina earned an additional congressional seat because of population growth that occurred mostly in urban areas, which tend to favor Democrats: according to an analysis of U.S. census data by *The News and Observer*, more than 78% of North Carolina's population growth over the last decade came from the Triangle area and the Charlotte metro area.[10] Despite that fact, the number of anticipated Democratic seats actually *decreases* in the current congressional map, as compared to the last map enacted in late 2019 and used in the 2020 elections. The last map produced 5 Democratic wins and 8 Republican wins; this map is expected to produce 3 Democratic wins, 10 Republican wins and 1 competitive seat.

- In the congressional map, Democratic strongholds Mecklenburg, Guilford, and Wake counties are each divided across three districts, despite the fact that there is no population-based reason to divide them this many times. In the previous congressional map, Mecklenburg was divided into two districts, Wake into two districts, and Guilford fell completely in one district. The strategic splits in the enacted map ensure that large numbers of voters will have no chance of being represented by a member of their own party. These splits will also lead to voter confusion and fractured representational linkages.

---

[6] Gary Pearce and Carter Wrenn, "We're usually on opposite sides of political battles. But we agree on NC voting maps." *The News and Observer*, October 21, 2021, available at https://www.newsobserver.com/opinion/article255145572.html.
[7] Bitzer, J. Michael, *Redistricting and Gerrymandering in North Carolina: Battlelines in the Tar Heel State* (Palgrave Macmillan, 2021).
[8] *See, e.g.*, Keena, Alex, Michael Latner Anthony J. McGann, and Charles Anthony Smith, *Gerrymandering in the States: Partisanship, Race and the Transformation of American Federalism* (New York: Cambridge University Press, 2021), 86.
[9] Grumbach, Jacob M. "Laboratories of Democratic Backsliding." (Unpublished Manuscript: University of Washington, 2021), *available at* https://sites.google.com/view/jakegrumbach/working-papers. Insights from this manuscript are forthcoming in *Laboratories Against Democracy*, Princeton University Press (https://press.princeton.edu/books/hardcover/9780691218458/laboratories-against-democracy).
[10] David Raynor, Tyler Dukes, and Gavin Off, "From population to diversity, see for yourself how NC changed over 10 years." *The News and Observer*, October 18, 2021, *available at* https://www.newsobserver.com/news/local/article253546964.html.

- The enacted congressional map produces geographic contortions that combine counties in ways that, in some circumstances, have never existed before.

- The double-bunking that occurs in the enacted congressional map advantages the Republican Party. A Republican (Virginia Foxx) and a Democrat (Kathy Manning) are both drawn into in an overwhelmingly Republican district (congressional district 11), thus virtually guaranteeing that the Democrat (Manning) will lose her seat. There are no cases where two Republican incumbents seeking re-election are double-bunked. The map also produces at least one district with no incumbents, but that district (congressional district 4) overwhelmingly favors the Republican Party.

- Despite the application of the *Stephenson v. Bartlett* county clustering rule, the mapmakers had considerable leeway in drawing the vast majority of North Carolina House and Senate districts. The enacted district lines "pack" Democratic leaning voters into a small number of districts, thus producing a few Democratic districts with large electoral margins. The district lines "crack" the remaining Democratic voters across the remaining districts, so that Democratic voters cannot comprise a majority of any of those districts. Conversely, the maps distribute Republican VTDs more efficiently, to translate those Republican votes into a greater number of anticipated seats. These practices ultimately result in large Republican seat advantages in the General Assembly—advantages that far outweigh the Republicans' share of the aggregate vote between the two parties. These maps are likely to lead to a General Assembly that will not represent the will of the people of the state.

- Neutral, third-party observers have been uniform in their negative assessment of the enacted maps. For example, The Princeton Gerrymandering Project assessed a grade of "F" in partisan fairness and "C" in competitiveness for all three maps. Dave's Redistricting App (DRA) assesses the congressional map as "very bad" in proportionality and "bad" in terms of competitiveness. While the House and Senate maps fare slightly better in terms of proportionality according to DRA, DRA assesses both maps to be "bad" in terms of competitiveness. Both The Princeton Gerrymandering Project and DRA are nonpartisan and have given similar grades to Democratic gerrymanders in other states.

North Carolina's Partisan Competitiveness

North Carolina has long been known for political moderation and competitive two-party politics. In 1960, Political Scientist V.O. Key noted North Carolina's distinctiveness from the rest of the South, owing to its comparatively competitive two-party politics.[11] North Carolina journalist Rob Christensen and Wake Forest University Political Scientist Jack Fleer noted more recently that the state enjoys "two strong and competitive parties."[12] Work by contemporary observers reinforces the notion that North Carolina is a competitive two-party state where statewide offices are winnable for either major political party.[13]

Two-Party Competition in Election Results

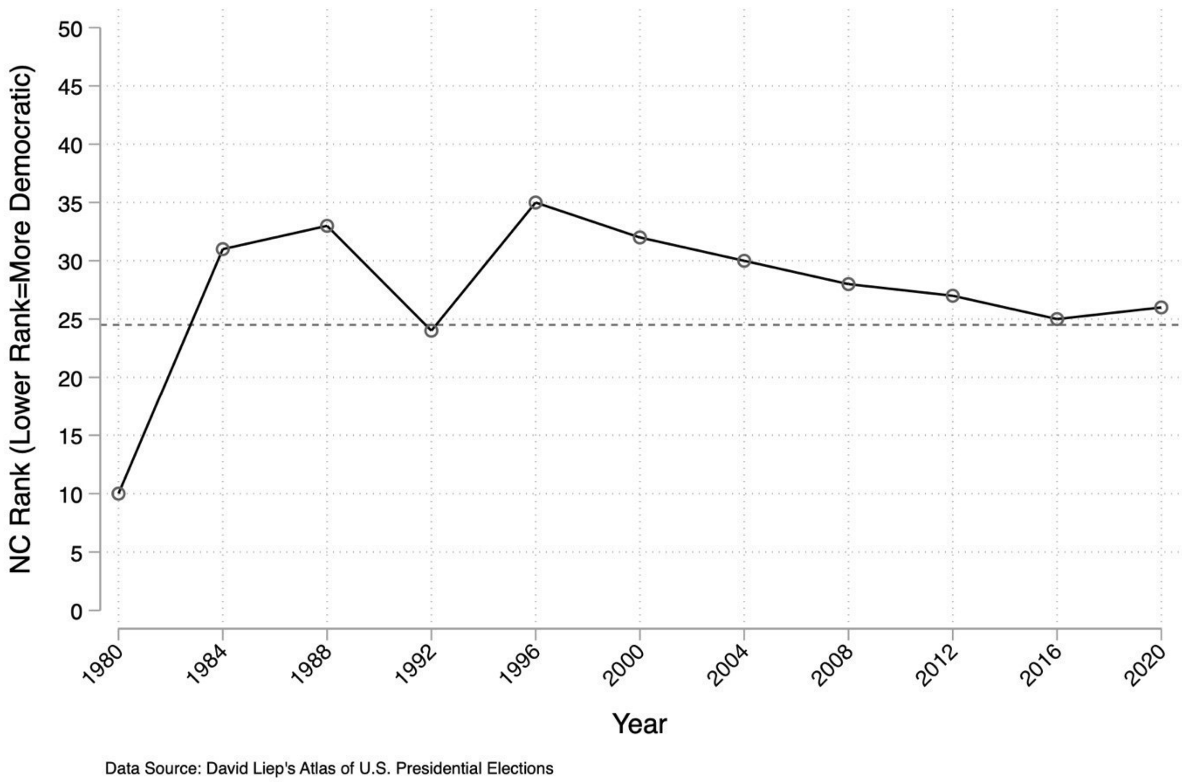
As I have written previously, one way to gauge the state's relative moderation and two-party competitiveness is simply to look at electoral results from races where gerrymandering is not possible—races where people are elected at the state level, rather than by districts that are subject to gerrymandering. The most prominent example of such an election, of course, is the U.S. presidential election.

The figure below plots North Carolina's presidential election results as ranked alongside those from other states, ranging from the state where the Democratic candidate received the largest vote share (1) to the state where the Democratic candidate receive the smallest vote share (50). Here, we see that North Carolina is best described as a competitive two-party state that sits roughly in the middle of the country in terms of partisan voting patterns. In 2000, North Carolina had the 32nd highest vote share for the Democratic candidate for president. In 2004, Democratic presidential candidate John Kerry received his 30th highest vote share in North Carolina. In 2008, then-presidential candidate Barack Obama's vote share in North Carlina was 28th highest in the country. In 2012, incumbent President Obama's vote share in North Carolina was 27th highest in the country. In 2016, North Carolina had the 26th highest Democratic vote share in the country and in 2020, it was the 27th highest.

---

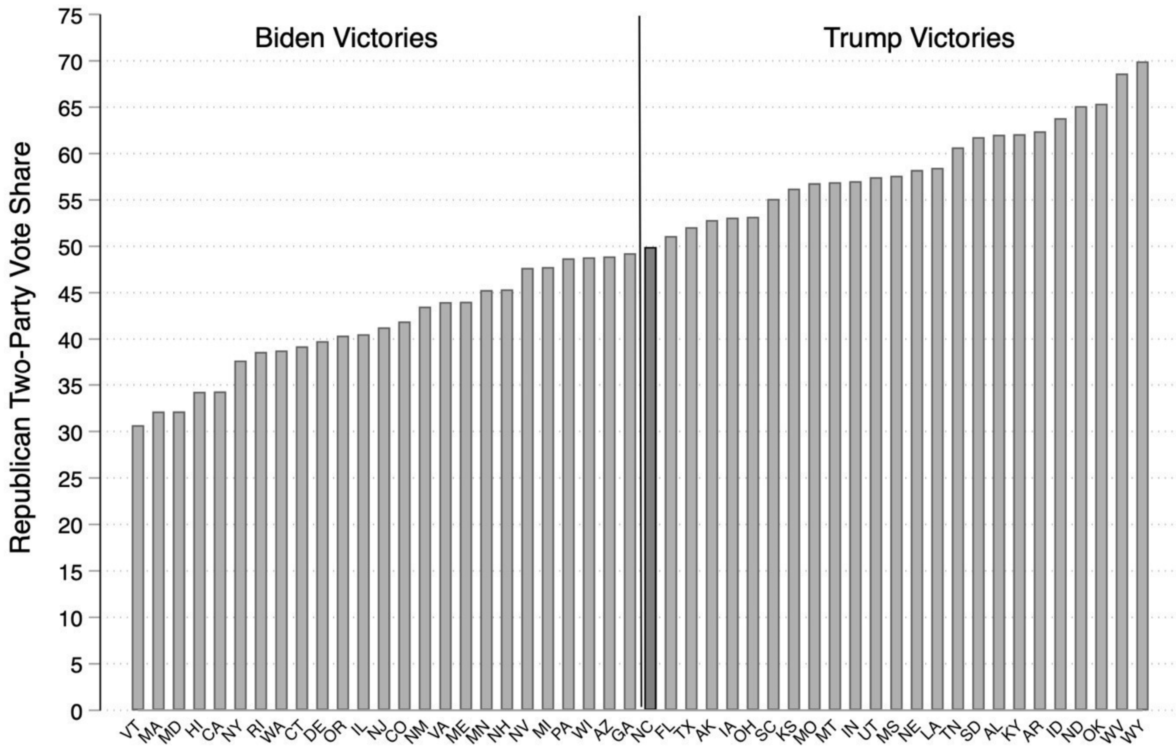
[11] *See* Key, V.O., Jr., *Southern Politics in State and Nation* (Knoxville: University of Tennessee Press, 1960).
[12] Christensen, Rob, and Jack D. Fleer, "North Carolina: Between Helms and Hunt No Majority Emerges," in Alexander P. Lamis, ed. *Southern Politics in the 1990s* (Baton Rouge: Louisiana State University Press, 1999), 106.
[13] Bitzer, J. Michael, and Charles Prysby, "North Carolina," in Charles S. Bullock III, and Mark J. Rozell, eds., *The New Politics of the Old South*, 7th Edition (Rowman and Littlefield, 2021).

Figure 1. North Carolina Rank in Democratic Vote Share for President Among the 50 States



In the 2020 election, North Carolina was perched on the razor's edge between Republican and Democrat—Donald Trump's two-party vote share was the smallest in North Carolina of any state he won in 2020. If any state can be described as "purple" or "competitive" in modern American politics, it is North Carolina.

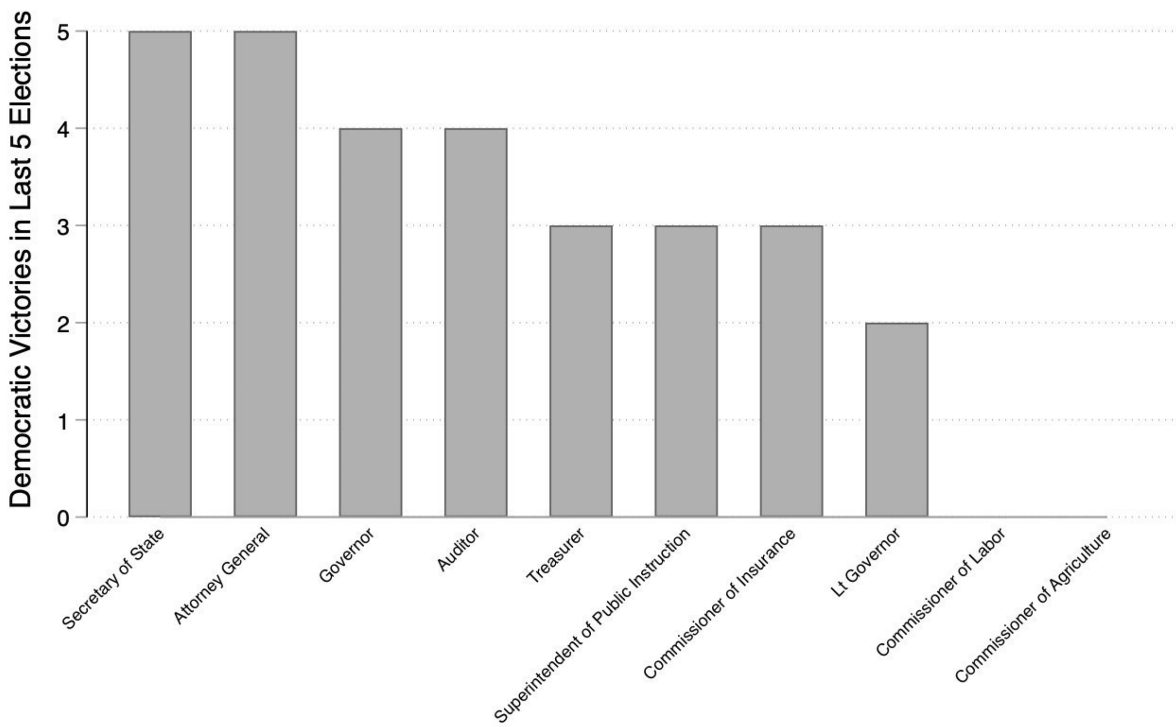
Figure 2. Two-Party Vote Share in the 2020 Presidential Election



Another way to understand North Carolina's competitiveness is to examine election results at the Council of State—ten members of the Executive branch who vary in prominence but are all elected in partisan quadrennial elections. These include the Governor, Lieutenant Governor, Secretary of State, State Auditor, State Treasurer, Superintendent of Public Instruction, Attorney General, Commissioner of Agriculture, Commissioner of Labor, and Commissioner of Insurance.

The result of these elections over the past five election cycles demonstrates once again that North Carolina enjoys significant partisan competition. Democrats have won 29 out of 50 Council of State elections since 2004.

Figure 3. Results of The Last Five Council of State Elections

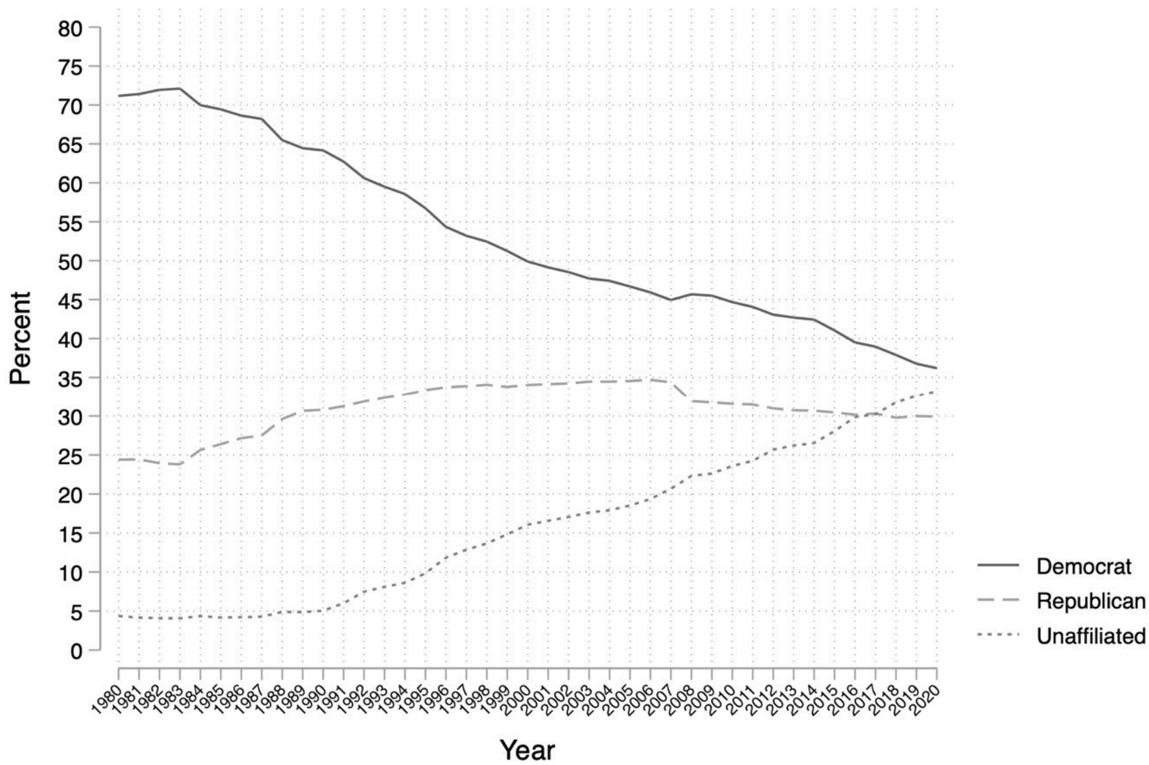


Note: Calculated from NC State Board of Eletions data. Council of State elections take place every four years.

Two-Party Competition and Moderation in the Electorate

North Carolina has considerable two-party competition in terms of voter registration. As the figure below indicates, Republican Party identification has never exceeded Democratic Party identification in the history of the state. While this is certainly not a sign of a liberal, Democratic state, it is similarly belies any contention that North Carolina is a conservative, Republican state.

Figure 4. Voter Registration in North Carolina



Partisan identification is, of course, just one indicator of the political lean of a state's citizens. And, given the rise in Unaffiliated voters in North Carolina, it is an increasingly noisy indicator.[14] Existing measures of statewide public opinion, however, come to the same conclusion: North Carolina does not lean heavily towards one party or ideology. One measure of state-level public opinion finds that North Carolina falls near the middle of the distribution of state-level political ideology as the 24th most liberal state in the country.[15] Another widely accepted measure finds that North Carolina is the 25th most liberal state in the country.[16]

Legislative Votes and Seats in the Aggregate

Historically, North Carolina's legislative delegation has not reflected these patterns of two-party competition and moderation. As the following three graphs demonstrate, North Carolinians consistently give about half of their two-party vote share to each party, yet the Republicans dominate in terms of legislative representation. This suggests that the representational linkage between voters and North Carolina's legislative representatives is weaker than between the voters and various other elected offices.

---

[14] Although using partisan identification as an indicator of voter preference can be problematic given that people generally change their voting pattern before changing partisan identification, North Carolina's party registration data is consistent with its moderate statewide voting patterns, as illustrated by the other measures included in this report.
[15] Berry, William D., Evan J. Ringquist, Richard C. Fording, and Russell L. Hanson, "Measuring Citizen and Government Ideology in the American States, 1960-93." *American Journal of Political Science* 42(1998): 327-48. Raw data are available at https://rcfording.com/state-ideology-data/.
[16] Tausanovitch, Chris, and Christopher Warshaw, "Measuring Constituent Policy Preference in Congress, State Legislatures, and Cities." *The Journal of Politics* 75(2013): 330-342. *See* http://www.americanideologyproject.com for data.

Figure 5. Comparing Votes and Seats in North Carolina's Congressional Delegation, 2012-2020

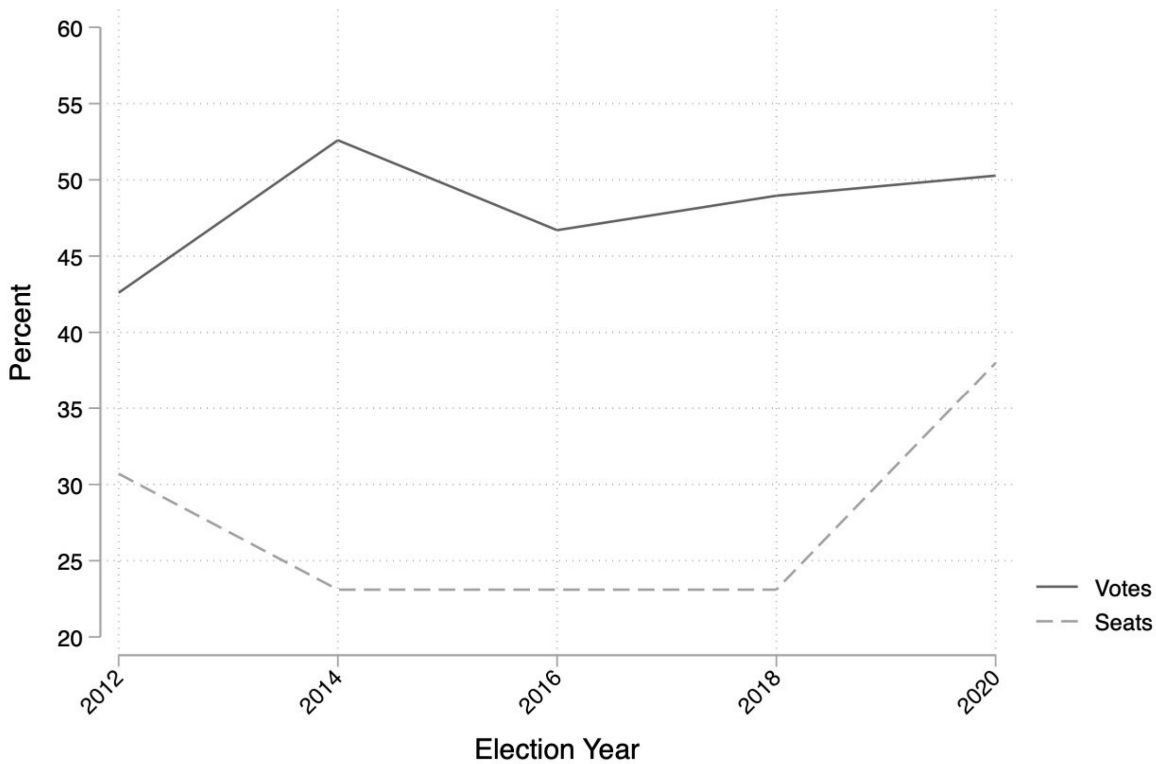


Figure 6. Comparing Votes and Seats in the North Carolina Senate, 2012-2020

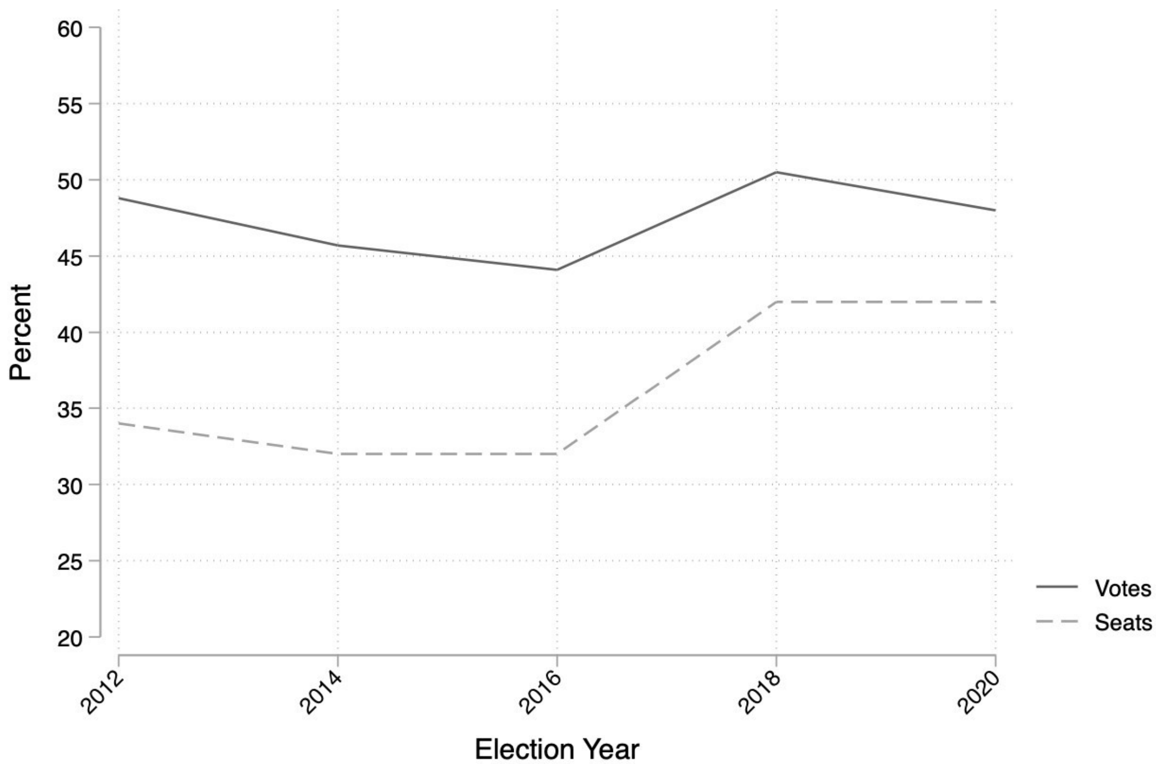
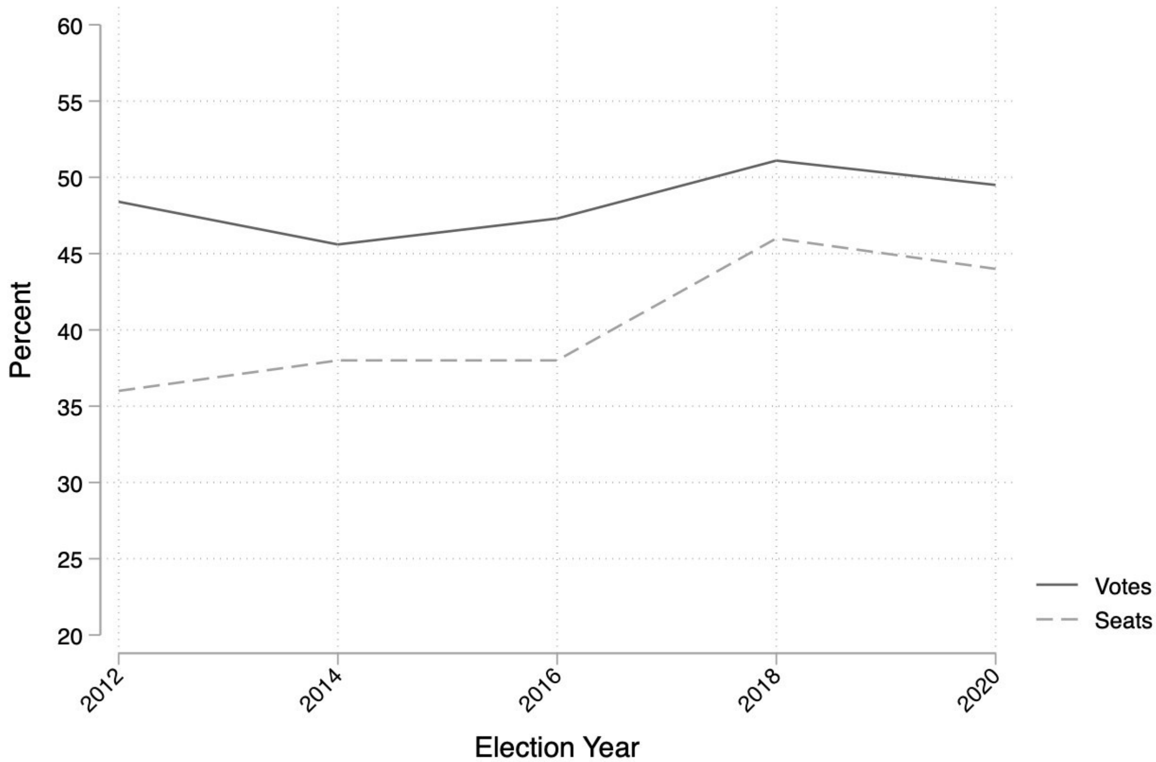


Figure 7. Comparing Votes and Seats in the North Carolina House, 2012-2020



Policy Outcomes

While North Carolina's statewide electoral outcomes, public opinion estimates, and party registration data all suggest a state that falls near the middle of the ideological and partisan spectrum in terms of citizen policy preferences, the partisanship of North Carolina's congressional and General Assembly delegations run counter to these measures. Further, available evidence suggests that the policy behavior and ideology of state legislators and members of Congress in North Carolina are at odds with statewide measures of two-party competition and ideological moderation. Estimates of voting patterns at the General Assembly[17] and congressional[18] levels reinforce that both delegations have moved in an increasingly conservative direction, while the aggregate public opinion of the citizenry has remained relatively constant. *See* figures 8 and 9 below.

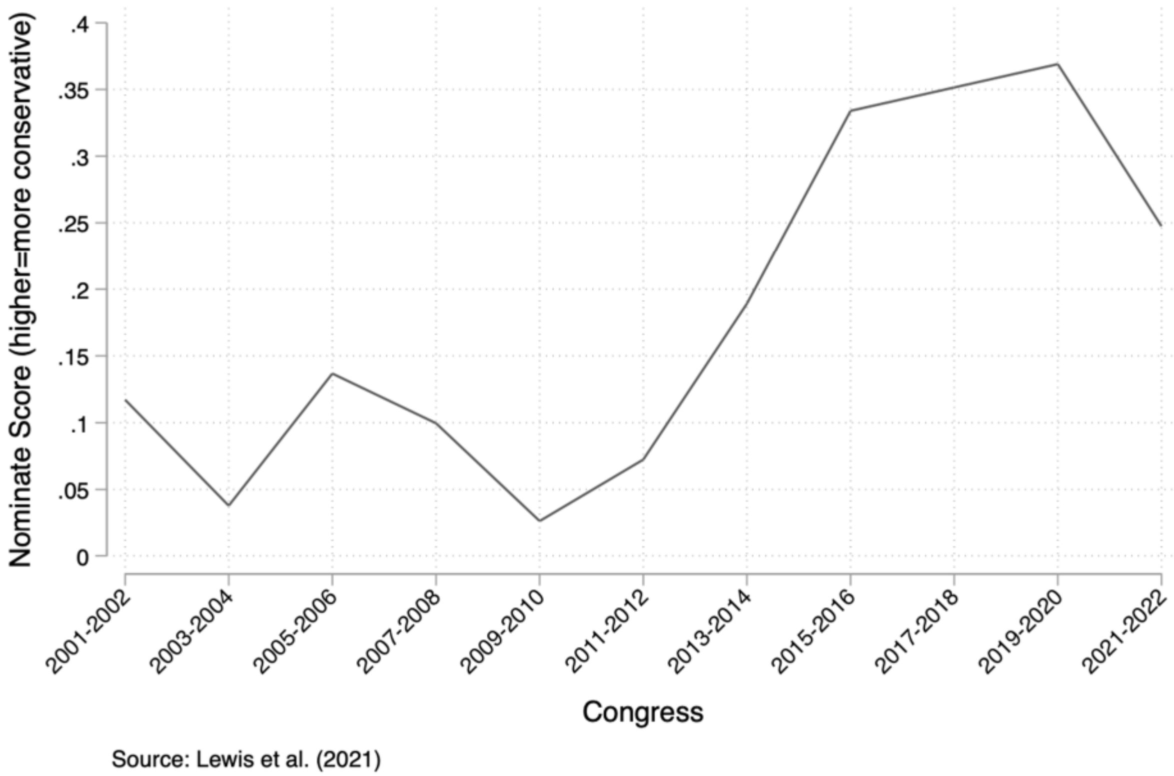
[17] Data are from Schor, Boris, and Nolan McCarty. 2020. American Legislatures Project, *available at* https://americanlegislatures.com.

[18] Lewis, Jeffrey B., Keith Poole, Howard Rosenthal, Adam Boche, Aaron Rudkin, and Luke Sonnet (2021). *Voteview: Congressional Roll-Call Votes Database*. https://voteview.com/.

Figure 8. Chamber Estimates of North Carolina General Assembly Ideology, 1995-2018

Source: American Legislatures Project (Schor and McCarty 2020)

Figure 9. Nominate scores of North Carolina's congressional delegation, 2001-2002 Congress through 2021-2022 Congress



In a forthcoming book, Political Scientist Jacob Grumbach finds that North Carolina experienced significant democratic backsliding in recent years—"among the most democratic states in the year 2000, but by 2018, they are close to the bottom."[19] It is important to note that Grumbach's measure is one of "small d" democratic backsliding—he does not measure partisanship, but rather a state's propensity to adhere to basic norms of democracy.

Taken together, these complementary measures of North Carolina voters' behaviors, ideological preferences, and partisanship indicate that North Carolina is a politically moderate state that enjoys two-party competition for the vast majority of elected offices. Beginning in 2011, however, North Carolina's congressional and General Assembly delegations have run counter to this trend, both in terms of partisanship and expressed policy preferences.

[19] Grumbach, Jacob M., "Laboratories of Democratic Backsliding," (Unpublished Manuscript: University of Washington, 2021), *available at* https://sites.google.com/view/jakegrumbach/working-papers. *See* a graph focusing on North Carolina's democratic backsliding on pg. 13. Insights from this manuscript are forthcoming in *Laboratories Against Democracy*, Princeton University Press (https://press.princeton.edu/books/hardcover/9780691218458/laboratories-against-democracy).

District Analysis

The remainder of this report is devoted to examinations of specific districts (in the case of Congress) and county "clusters" (in the case of the General Assembly). In the text that follows, I refer to the "current" maps as the maps that were used in the 2020 election and the "enacted" maps as the maps that have been approved by the North Carolina General Assembly for use in the 2022 elections. While I conducted all of the analysis that follows and wrote all of the verbiage, the shaded red-and-blue maps were produced by John Holden, a geographic information system (GIS) expert, using a "CCSC" measure of partisanship that I selected and describe below. Mr. Holden also produced the other maps in the following pages that show the effect of the district lines on certain municipalities.

I use a few different metrics in the analysis that follows. The first is the Cook Political Report's Partisan Voter Index (PVI), a standard metric of the expected "lean" of a congressional district using a composite of past elections. The second is the Civitas Political Index (CPI), a measure of partisan district lean for state legislative districts derived from prior Council of State votes. The CPI places each district on a scale from D+1 (a district that has a slight Democratic tilt) to D+36 (a district with an overwhelming Democratic tilt), with mirrored results on the Republican side indicated with an "R" instead of a "D." The third is a metric created for this analysis that combines the results of the 2020 Secretary of Labor and Attorney General races, the two closest Council of State races in North Carolina that year, into one measure, which I term the Competitive Council of State Composite (CCSC).[20] This measure allows for the use of relatively low-profile elections to get a sense of the "true partisanship" of the district. It is presented below as the raw difference in votes and is used in the shaded red-and-blue maps that follow. From time to time, I mention the percent of the electorate that voted for Donald Trump in the 2020 election to give yet another sense of the partisan lean of the district, county, or cluster.

Congressional District Analysis

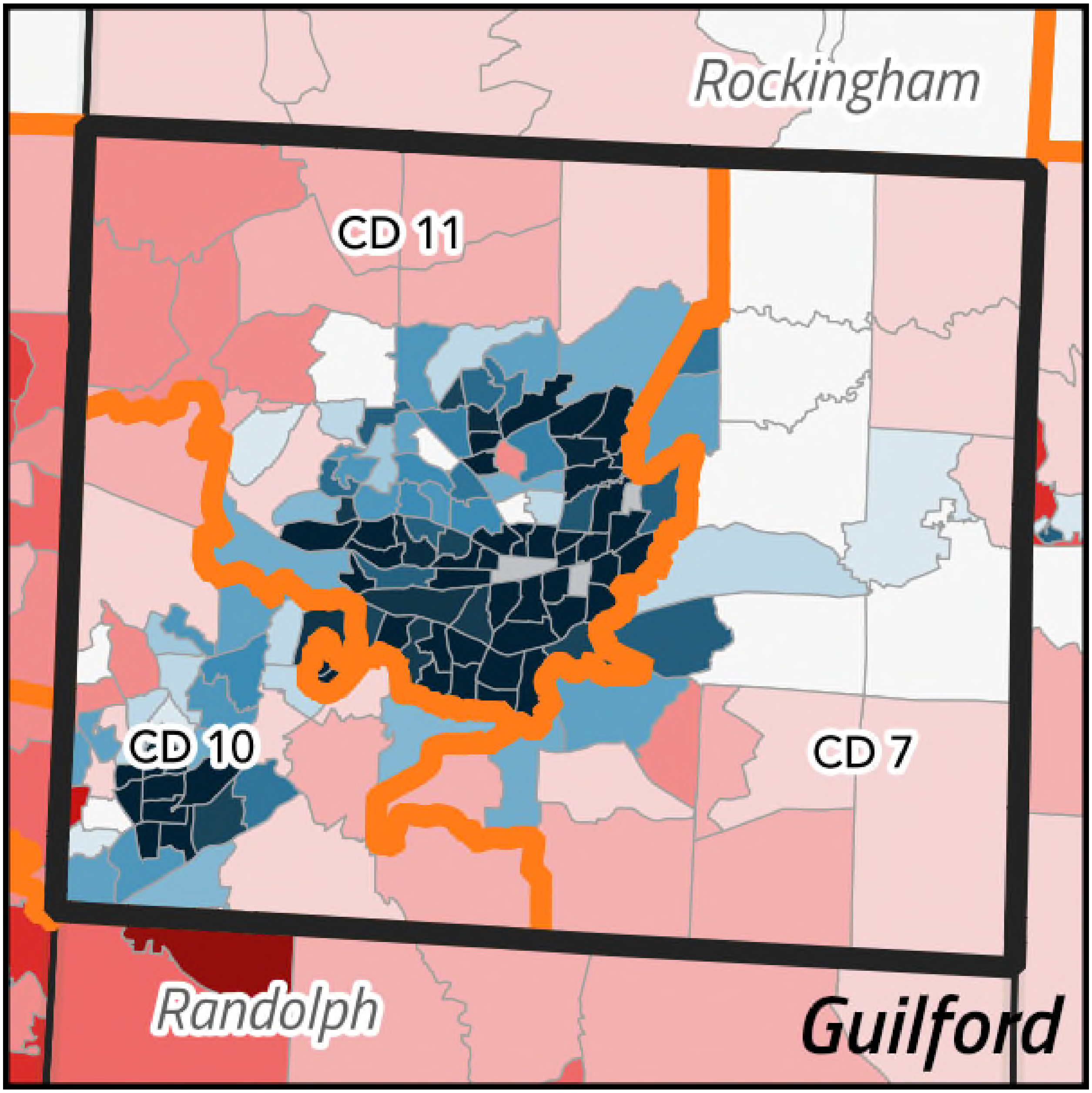
I begin by showing shaded red-and-blue maps demonstrating the trisection of Wake County, Mecklenburg County, and Guilford County by the congressional district lines (maps 1, 2, and 3 below). These maps show county lines in black, VTD lines in gray, and district lines in orange. The red-and-blue shading represents the relative vote margin using my CCSC—the composite results of the Secretary of Labor and Attorney General races in 2020—in each VTD, with darker blue shading representing larger Democratic vote margins and darker red shading indicating larger Republican vote margins (both normalized by acreage).

While district-by-district analysis is important, the congressional map is best understood as a single organism, rather than 14 separate entities—as one district moves in one direction, another must respond. This means that the unnecessary division of Mecklenburg, Guilford, and Wake counties across multiple congressional districts, achieved by the cracking and packing of Democratic voters in those counties, has ripple effects throughout the map. Map 4 shows the entirety of the congressional map with red-and-blue CCSC shading.

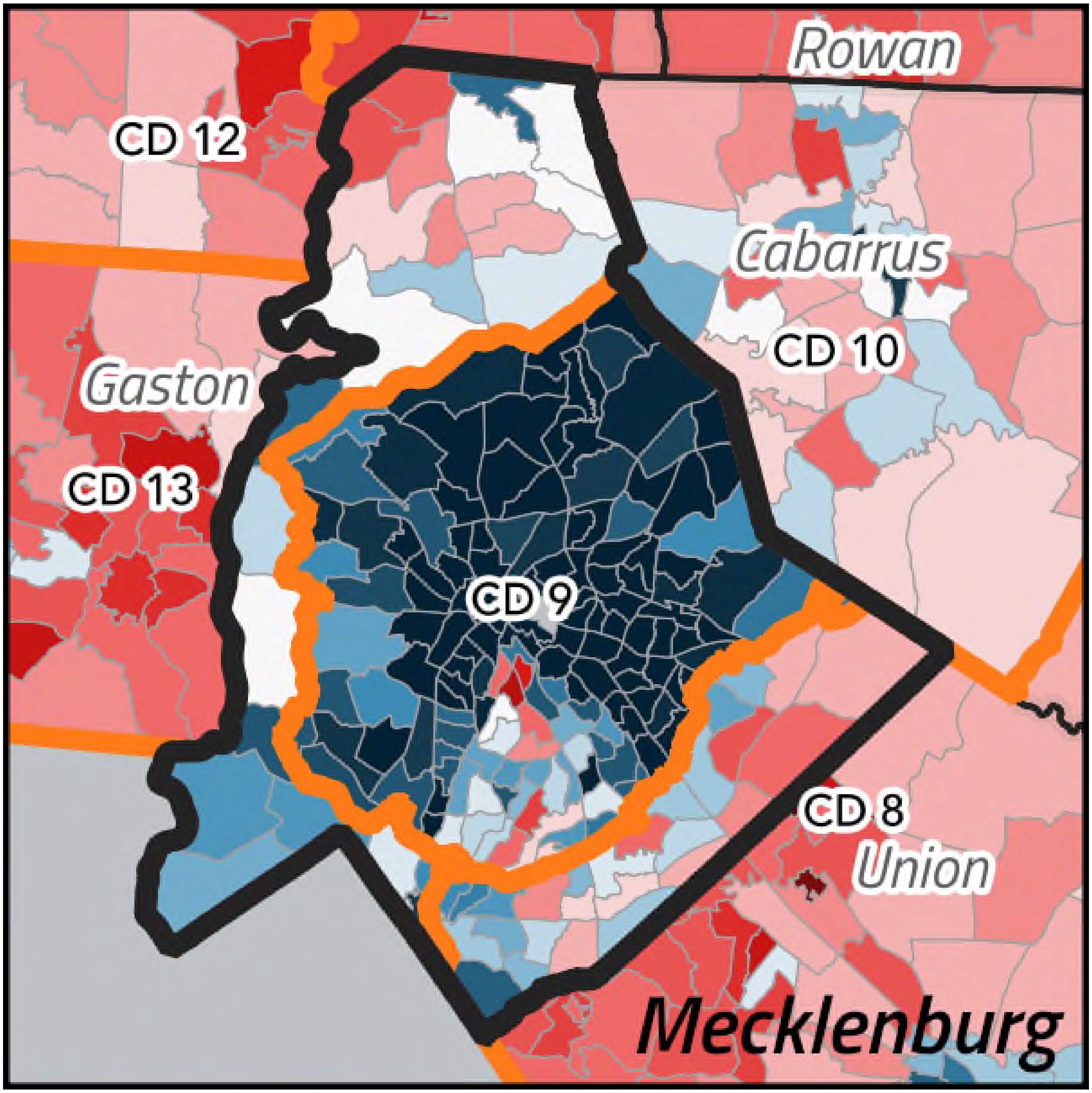
---

[20] The election data utilized for the CCSC metric, including to generate the red-and-blue shading on the maps that follow, was obtained from the North Carolina State Board of Elections website. *See* https://www.ncsbe.gov/results-data/election-results/historical-election-results-data.

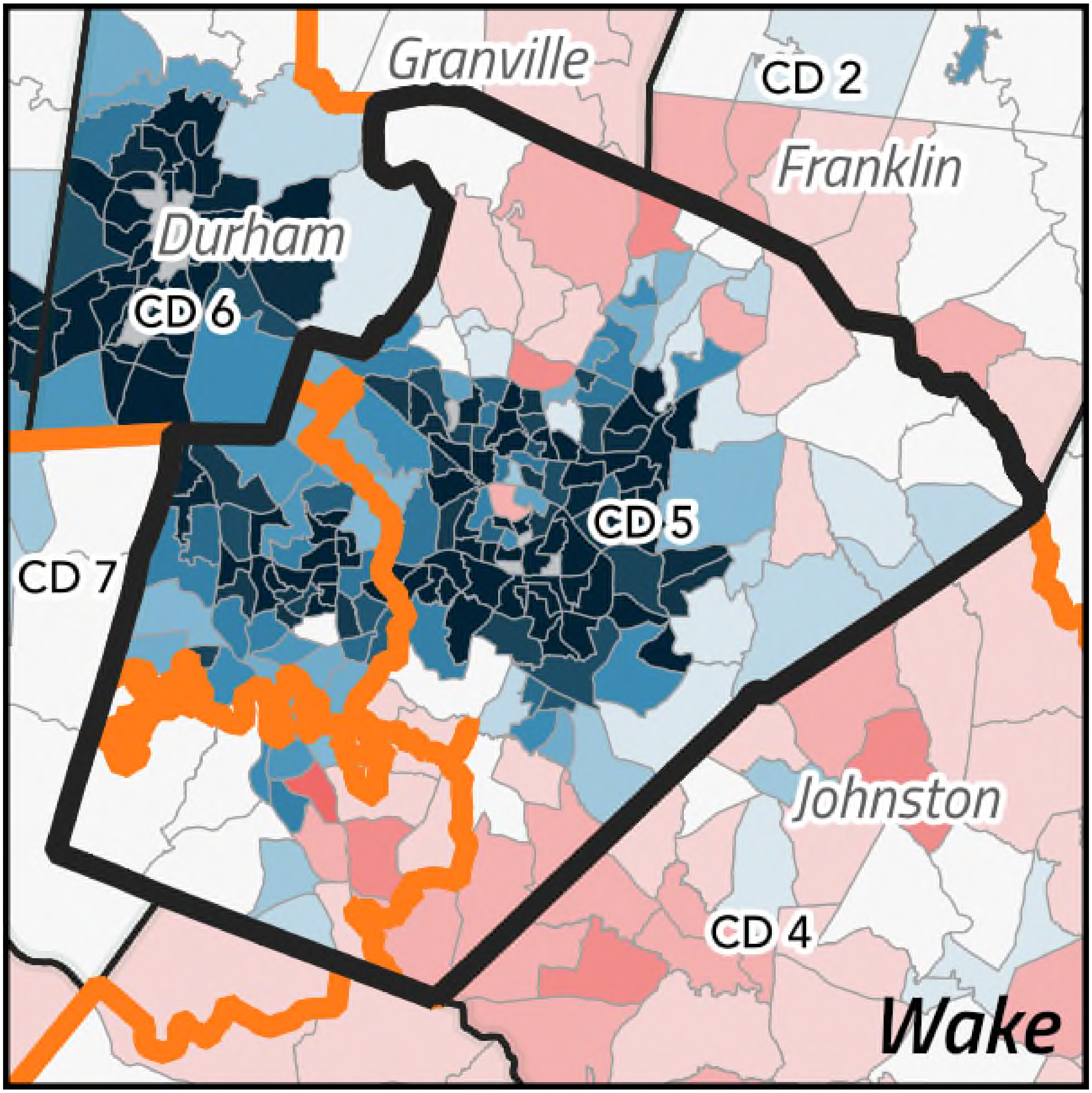
Map 1. Close-up of Guilford County VTD CCSC, split across three districts



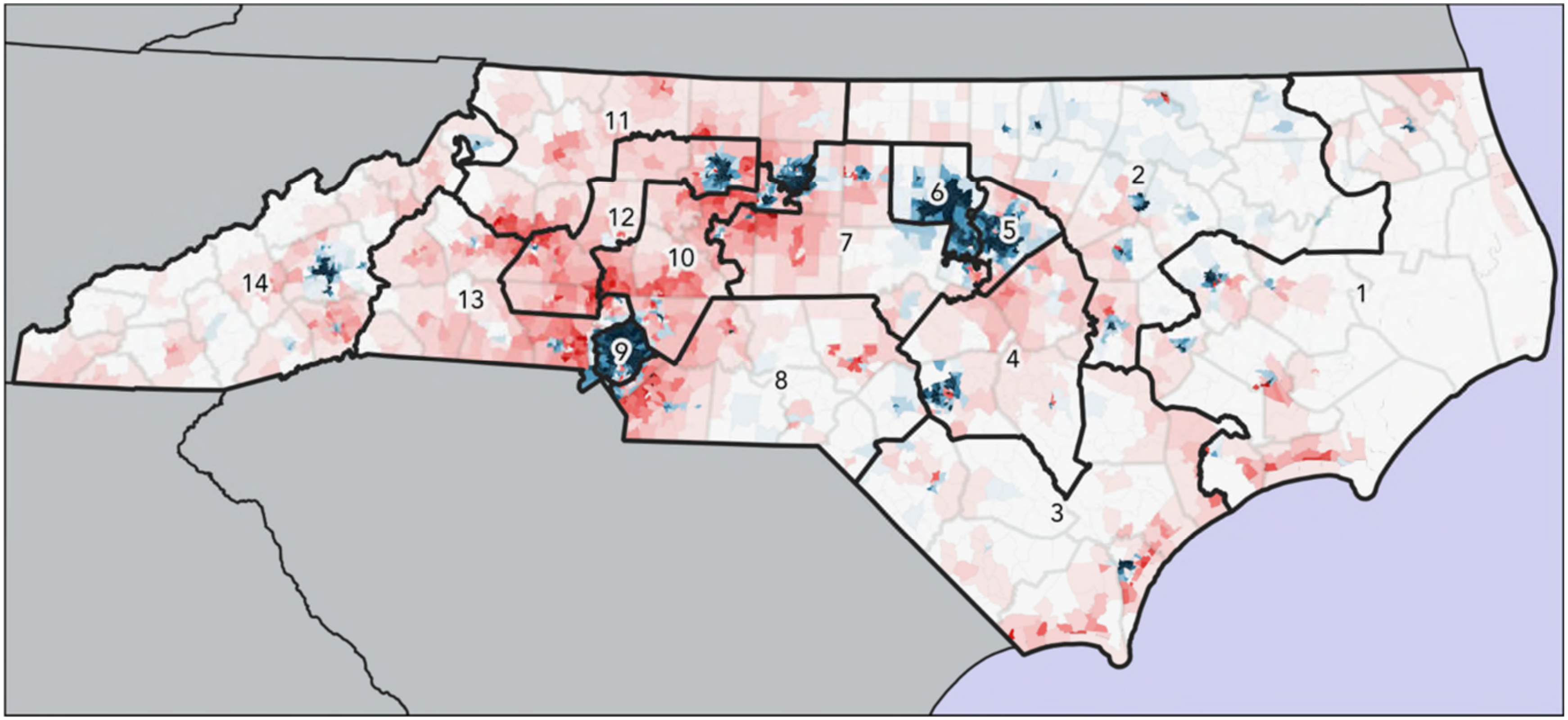
Map 2. Close-up of Mecklenburg County VTD CCSC, split across three districts



Map 3. Close-up of Wake County VTD CCSC, split across three districts



Map 4. Statewide overview of the enacted congressional map



As the table below shows, the PVI, CCSC, and Trump Percentage all tell a similar story: the enacted map will produce 10 Republican seats, 3 Democratic seats, and 1 competitive seat. At most, the enacted map could be expected to elect four Democrats to office in 2022—fewer than in the current map and far below what one would expect based on Democratic representation statewide or the results of other recent statewide elections.

Table 1. Summary Data for Each Enacted Congressional District

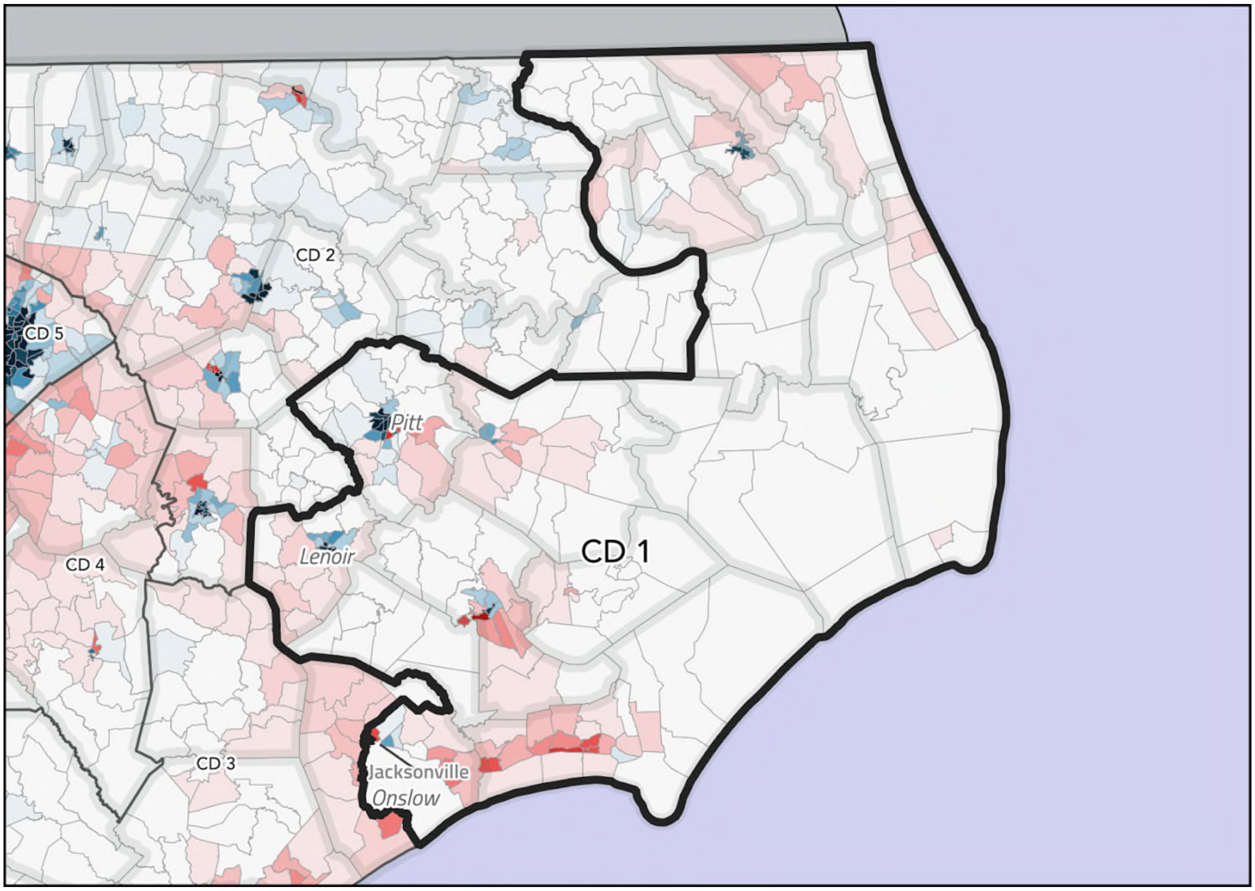
District	PVI	CCSC	Trump Perc
1	R+10	R +98,969	57%
2	Even	D +40,396	48%
3	R+10	R +111,451	58%
4	R+5	R +28,045	53%
5	D+12	D +227,327	34%
6	D+22	D +374,786	25%
7	R+11	R +115,682	57%
8	R+11	R +125,842	57%
9	D+23	D +325,717	25%
10	R+14	R +156,833	60%
11	R+9	R +94,407	57%
12	R+9	R +102,404	56%
13	R+13	R +150,187	60%
14	R+7	R +58,387	53%

NC-1

The enacted 1st congressional district is mostly comprised of the current NC-3, but also includes part of the current NC-1. Most potential congressional districts in this part of North Carolina would likely lean towards the Republican Party, but to create extra advantage for the Republican Party in other parts of the map, the current map brings the Democratic-leaning areas of Pitt County into NC-1, thus removing them from NC-2 and allowing NC-2 to become much more competitive for the Republican Party.

Despite moving the district line westward to include the Democratic portion of Pitt County, the enacted district remains virtually a guaranteed Republican victory with a PVI of R+10 (the current NC-3 is R+14). No Democratic member of Congress in the country represents a district that leans this far towards the Republican Party.

Map 5. VTD CCSC for NC-1



NC-2

The enacted 2nd congressional district includes the core of the current NC-1, along with portions of the current NC-4 and NC-13. The area that largely comprises the new NC-2 is currently represented by Democrat G.K. Butterfield and is considered a D+12 district by the Cook Political Report, making it a safe Democratic seat. Butterfield has the longest uninterrupted tenure of any member of North Carolina's congressional delegation. Under the enacted map, however, Butterfield's district changes radically, loses many of its Democratic strongholds (including the aforementioned loss of the Democratic areas in Pitt County) and now picks up enough Republican voters to move the district to "even," according to the Cook Political Report. For example, NC-2 picks up Caswell County, which does not include a single Democratic-leaning VTD, according to the 2020 Attorney General/Secretary of Labor CCSC in the map shown below. The 2020 Presidential vote share and CCSC score reinforce that this is an extremely competitive district. This is an enormous shift for what was formerly a Democratic stronghold.

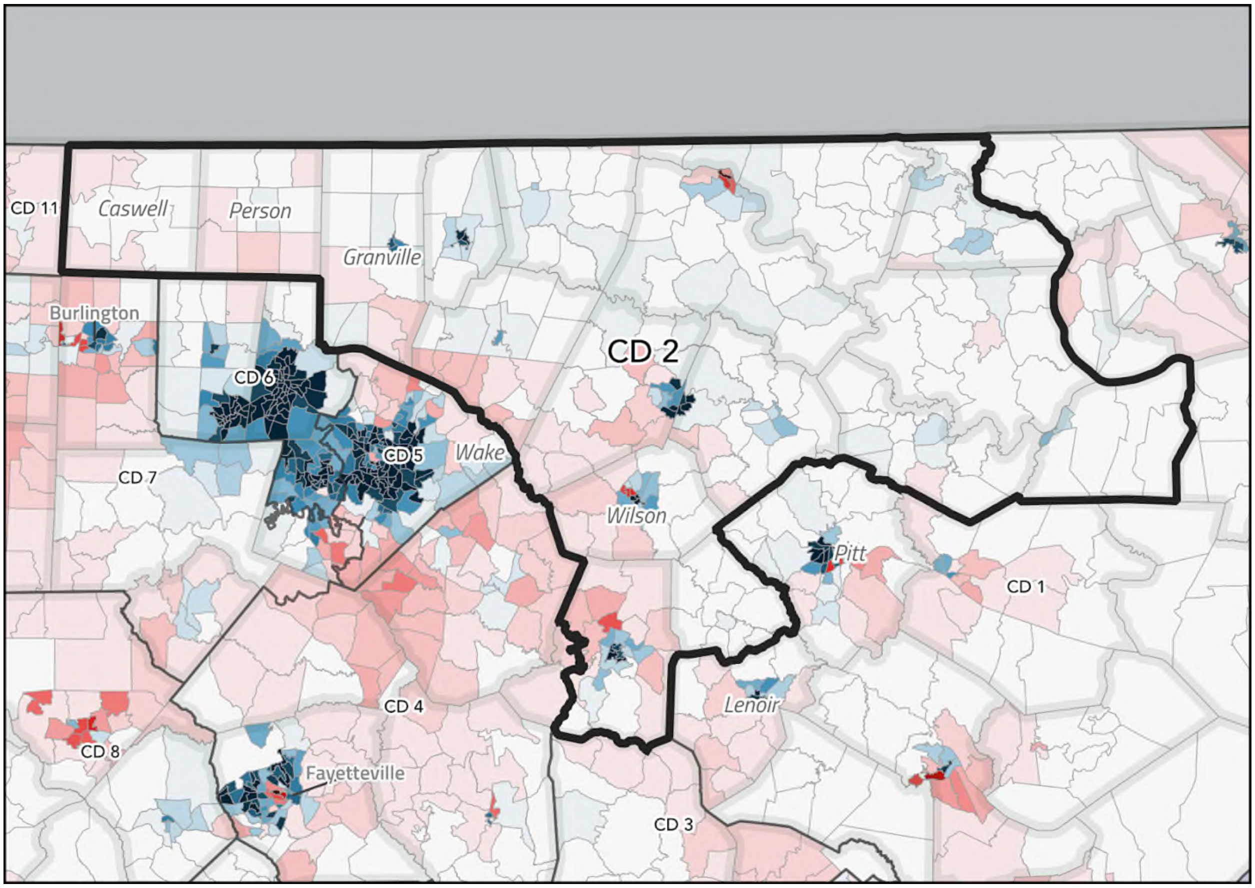
In addition to producing a clear partisan shift, the district is difficult to understand from a communities of interest perspective. The enacted district no longer includes any of Pitt County, nor does it include the campus of East Carolina University, which provided much of the economic engine of the district. The district now stretches from the Albemarle Sound to the Raleigh-Durham-Chapel Hill metropolitan area and eventually terminates in Caswell County, just northeast of Greensboro. Notably, Washington County and Caswell County have never been paired together in a congressional map in the history of North Carolina, further illustrating how little these counties have in common.

At a micro-level, the changes will split communities in important ways. For example, the cut-out in Wayne County, just west of Goldsboro, splits the students and families in Westwood Elementary School (which is located in NC-2) into two separate districts (NC-2 and NC-4). At one point, NC-2 passes through a narrow cut-off between the Neuse River to Old Smithfield Road that is less than one-third of a mile wide.

After the maps were enacted, G.K. Butterfield announced that he will not seek re-election,[21] making the district even more likely to shift to the Republican Party. If the Republicans take over this seat, it will be the first time that this part of North Carolina has been represented by a Republican since the late 19th Century.

[21] Bryan Anderson, "Democrat Rep. Butterfield to Retire, New District is a Toss-Up," *Associate Press News*, *available at* https://apnews.com/article/elections-voting-north-carolina-voting-rights-redistricting-e221c0732f457b2273f54ef102424eca.

Map 6. VTD CCSC for NC-2



NC-3

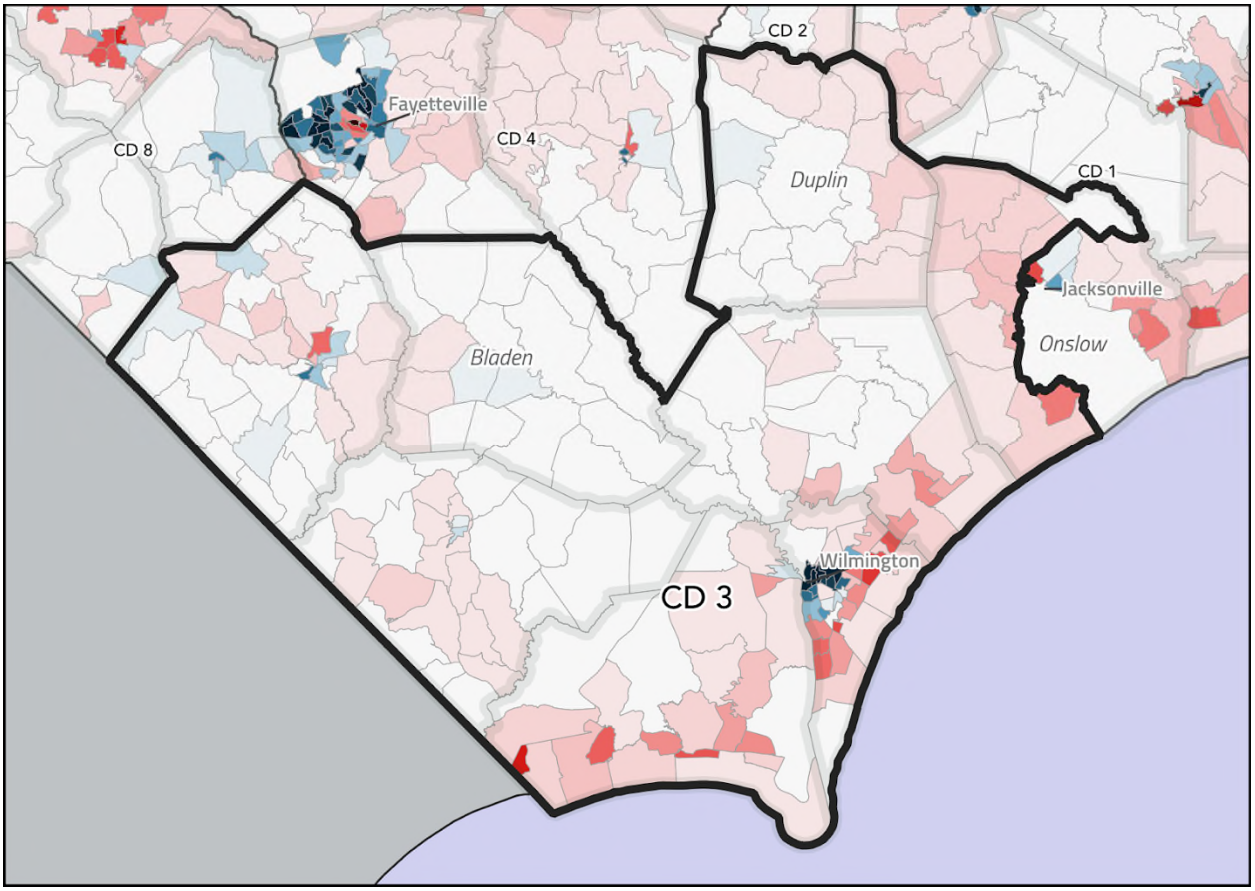
The enacted 3rd congressional district is mostly carved out of the current 7th congressional district, but also includes portions of the current 3rd and 9th districts. The current 7th district is considered R+11 by the Cook Political Report.

As enacted, this district once again denies North Carolina's Sandhills a consistent district of their own, despite repeated calls during the redistricting process,[22] and instead places portions of the Sandhills with the coastal enclave in and around Wilmington. The enacted map also creates an odd appendage in Onslow County that, as described in the section on NC-1, makes little sense from a communities of interest perspective.

The enacted district will almost certainly elect a Republican. It is slightly less Republican than the current NC-7 but still is considered R+10 by the Cook Political Report. It favored the Republicans by over 110,000 votes in the 2020 Attorney General/Secretary of Labor CCSC, and Donald Trump won the district with 58% of the vote. It is currently represented by Republican David Rouzer and is expected to remain in Republican hands.

[22] *See, e.g.*, Dreilinger, Danielle, "1 woman, 1 North Carolina address, 5 congressional districts. As North Carolina prepares to add a 14th congressional seat, Sandhills residents asked: why can't it be theirs? *Fayetteville Observer*. November 5, 2021.

Map 7. VTD CCSC for NC-3



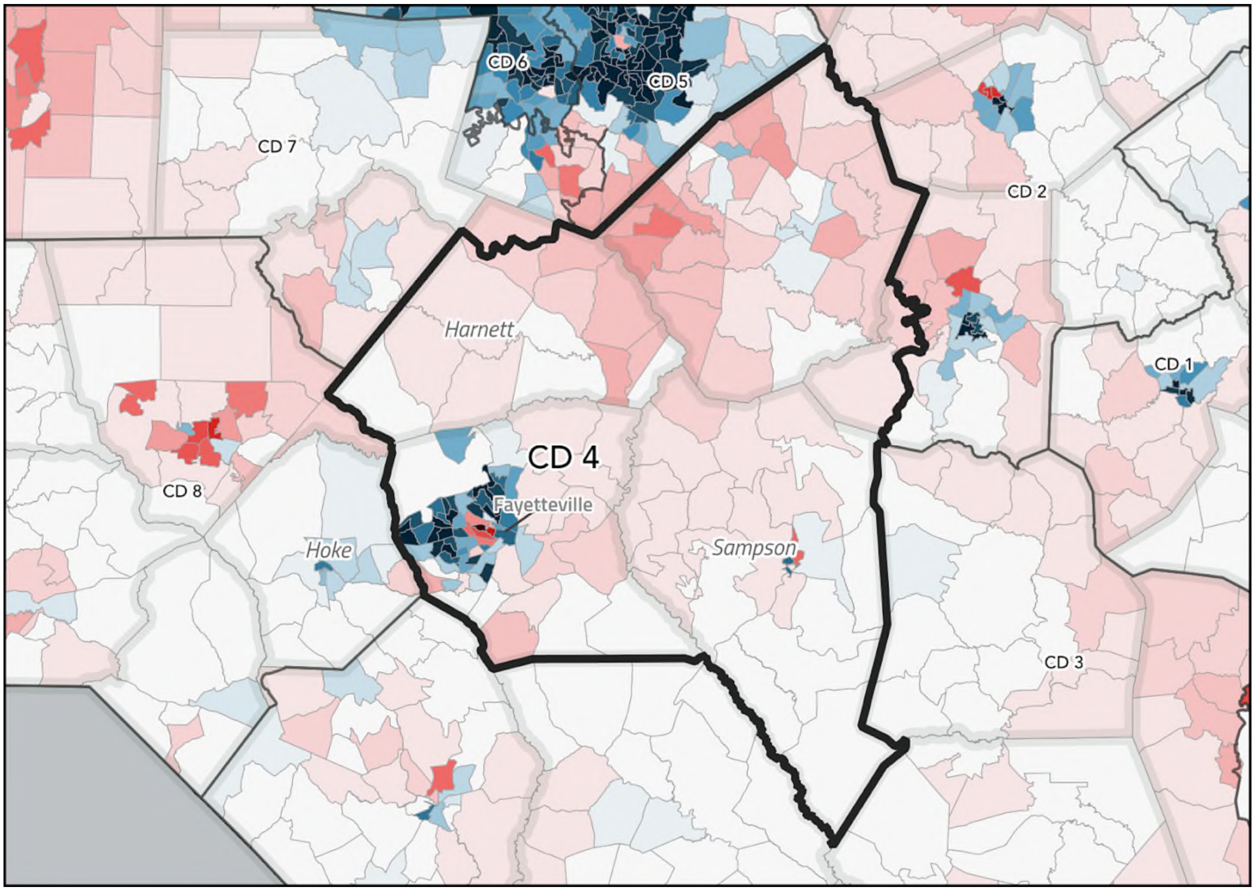
NC-4

The enacted 4th congressional district is carved out of a pocket of North Carolina that includes Johnston County and a portion of Harnett County, both of which are adjacent to Wake County, as well as portions of the Sandhills. The district is pieced together out of leftover portions from current districts 7 and 8, which were R+11 and R+6, respectively. It combines the Democratic-leaning area of Fayetteville with those areas to create a Republican-leaning district.

In addition to the carve out of Republican-leaning VTDs in Wayne County referenced above, this district takes a series of confusing jogs in the northwest part of Harnett County. A citizen driving southwest on Cokesbury Road would begin in NC-7, then rest on the line between NC-7 and NC-4, then into NC-4, then back on the line between the two, just before Cokesbury turns into Kipling Road whereupon the driver would move back into NC-4.

This district, which has no incumbent, is considered an R+5 district by the Cook Political Report, gave 53% of its vote share to Donald Trump in 2020, and gave an advantage to Republicans of about 28,000 votes in the 2020 Attorney General/Secretary of Labor CCSC.

Map 8. VTD CCSC for NC-4

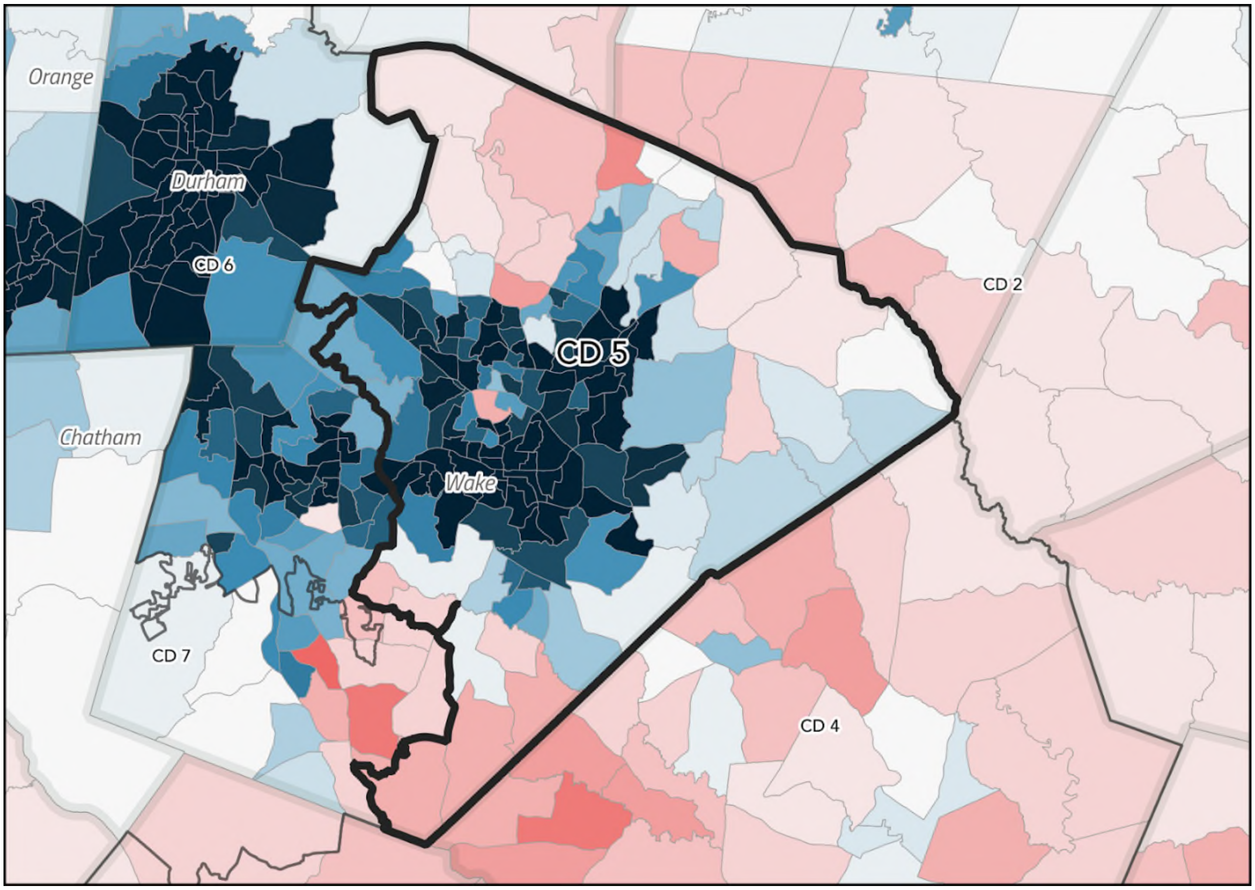


NC-5

The enacted map cracks Democrats in Wake County into three districts (NC-5, NC-6, and NC-7). Unlike NC-6 and NC-7, NC-5 is situated completely within Wake County and is made up of portions of current NC-2 and NC-4, districts that were D+12 and D+16. The effects of this are to pack Democratic voters into one district, thus increasing the probability that Republicans can win at least one of the adjacent districts. The enacted district is rated by the Cook Political Report as D+12, the CCSC shows a Democratic advantage of over 227,000 votes, and Donald Trump won just 34% of the vote.

This map clearly splits communities of interest. In one particularly egregious example, a small vein runs up Fayetteville Road by McCullers Crossroads in Fuquay-Varina, where the vein itself is in NC-7 and the areas on either side of it are in NC-5.

Map 9. VTD CCSC for NC-5

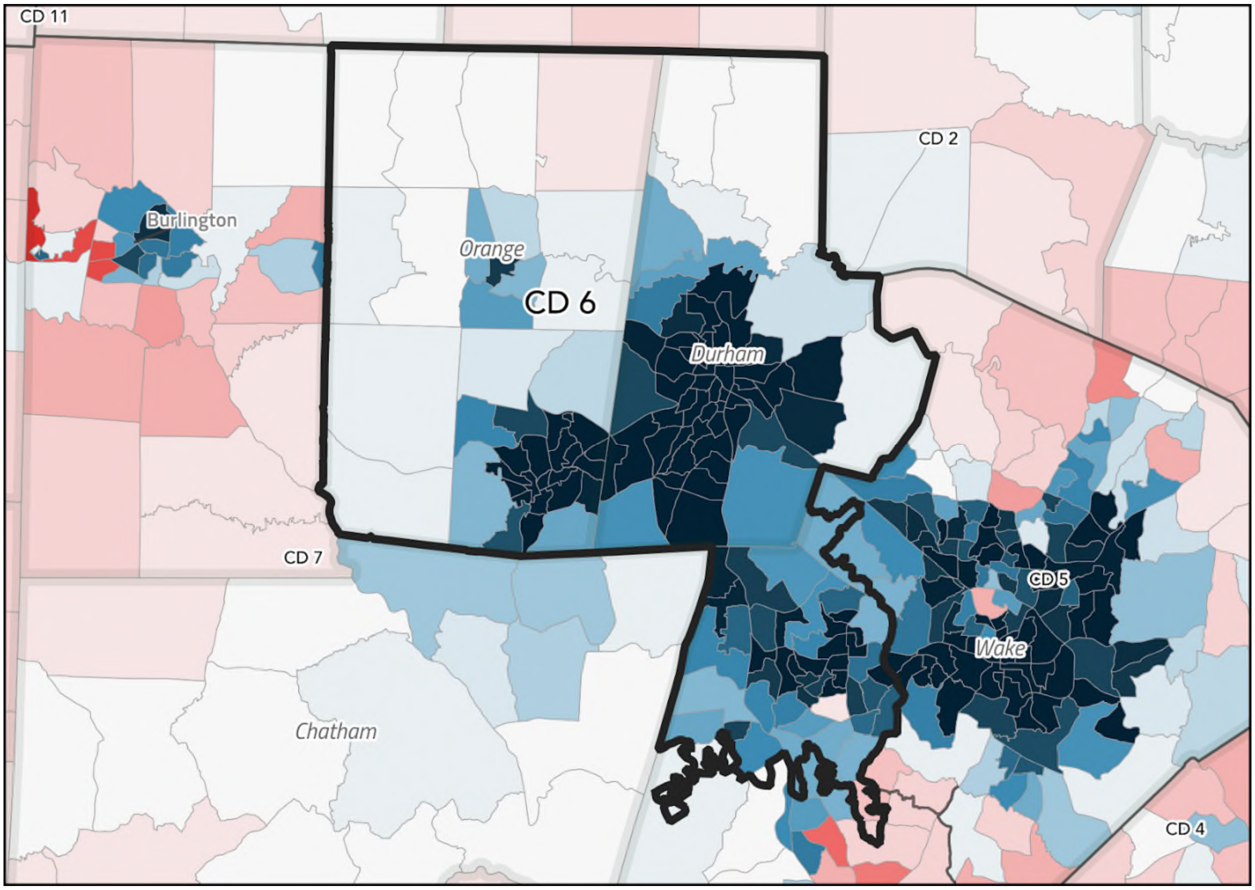


NC-6

The 6th district packs all of Orange and Durham counties and part of Wake County together into one overwhelmingly Democratic district, which is created out of portions of the current NC-4 and NC-2 (D+16 and D+12, respectively). As the map below demonstrates, the enacted NC-6 only includes four marginally Republican VTDs, according to the 2020 Attorney General/Secretary of Labor CCSC. Cook Political Report estimates this to be a D+22 district, Democrats had more than a 374,000 vote advantage in the CCSC and Donald Trump won only 25% of the vote in 2020. This district packs a greater proportion of Democratic voters in a single district than any district from the previous map. This district, like NC-5, includes Wake County, which is divided across three districts in the enacted map. The packing of Democrats in this district enables adjacent districts, in particular NC-7, to be drawn in ways that make it easier for Republican candidates to win.

The contours of this district bordering NC-7, on the southern end, split communities of interest in almost comical ways. In one example, a person traveling south on New Hill Olive Chapel Road would, in a matter of a few miles, move from NC-7 to the line between NC-6 and NC-7, back into NC-7, through NC-6, back into NC-7, back to the border between the two, back into NC-7, back to the border between the two, then back into NC-7. The contours of these lines are confusing to voters, and, as the map demonstrates, serve to pack as many Democratic precincts as possible into NC-6.

Map 10. VTD CCSC for NC-6

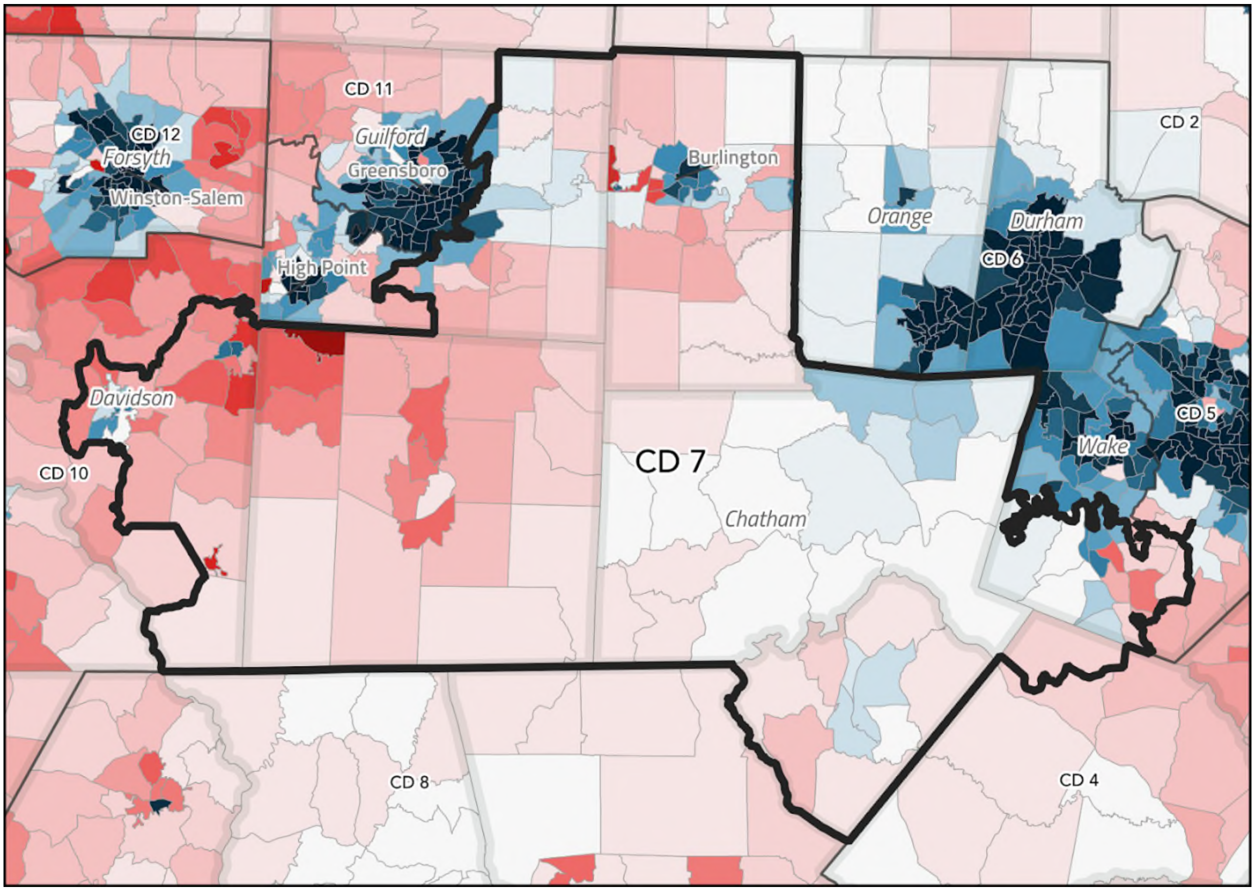


NC-7

The enacted 7th district includes the Republican-leaning Randolph, Alamance, Chatham, and Lee counties as well as portions of Guilford, Wake, and Davidson counties. It is carved out of current districts 13, 6, 4, and 2. As it is drawn, NC-7 splits both Guilford and Wake counties (each of which of is divided three times in the map as a whole). Despite including portions of two of the most Democratic counties in North Carolina, the district studiously avoids the Democratic-leaning areas of both counties. The eastern portion of the district in Wake County, near Apex, takes the unusual and confusing contours described in the description of NC-6 above.

The enacted NC-7 is considered R + 11 by the Cook Political Report, it gave Republicans a 115,682 vote advantage in the CCSC, and Donald Trump won 57% of the vote in this district. A Democratic candidate has virtually no chance of victory in the enacted 7th.

Map 11. VTD CCSC for NC-7

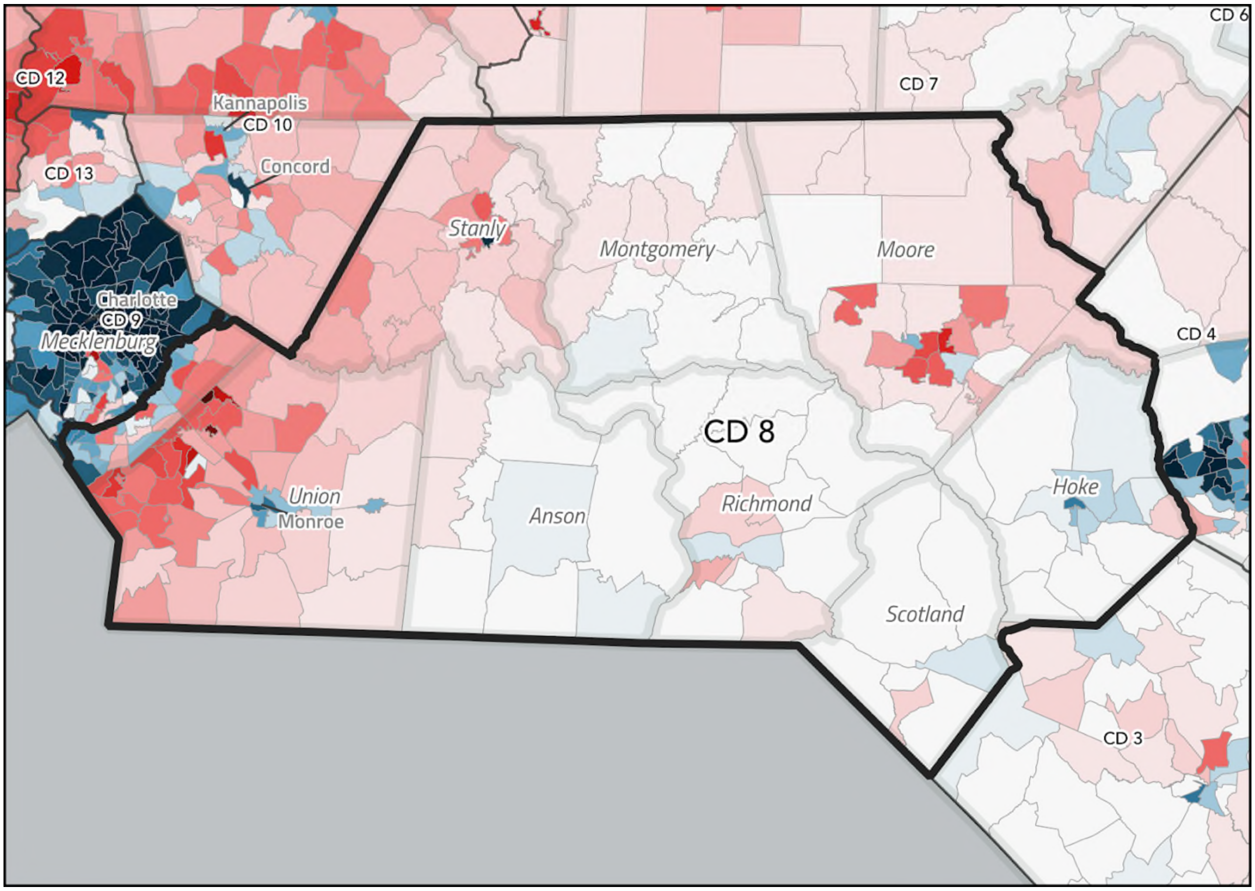


NC-8

The 8th district stretches from the Sandhills into Mecklenburg County and includes portions of the current 9th, 12th, and 8th districts. The core of the district comes from the current 9th district, which is R+6. The enacted NC-8 includes the entirety of Scotland, Hoke, Moore, Montgomery, Richmond, Anson, Union, and Stanley counties as well as the southern and eastern edge of Mecklenburg County. Although it includes portions of Mecklenburg County, one of the most Democratic-leaning areas in the state, as well as Democratic municipalities in Union, Anson, and Hoke, the 8th district is unlikely to elect a Democrat under any reasonable scenario. The enacted map stops just shy of the some of the darkest blue VTDs in Mecklenburg County.

The Cook Political Report calls the enacted NC-8 an R+11 district, the CCSC shows that the Republican candidate garnered over 115,000 more votes than the Democratic candidates for the two closest Council of State races, and Donald Trump won approximately 57% of the vote in the 2020 election.

Map 12. VTD CCSC for NC-8



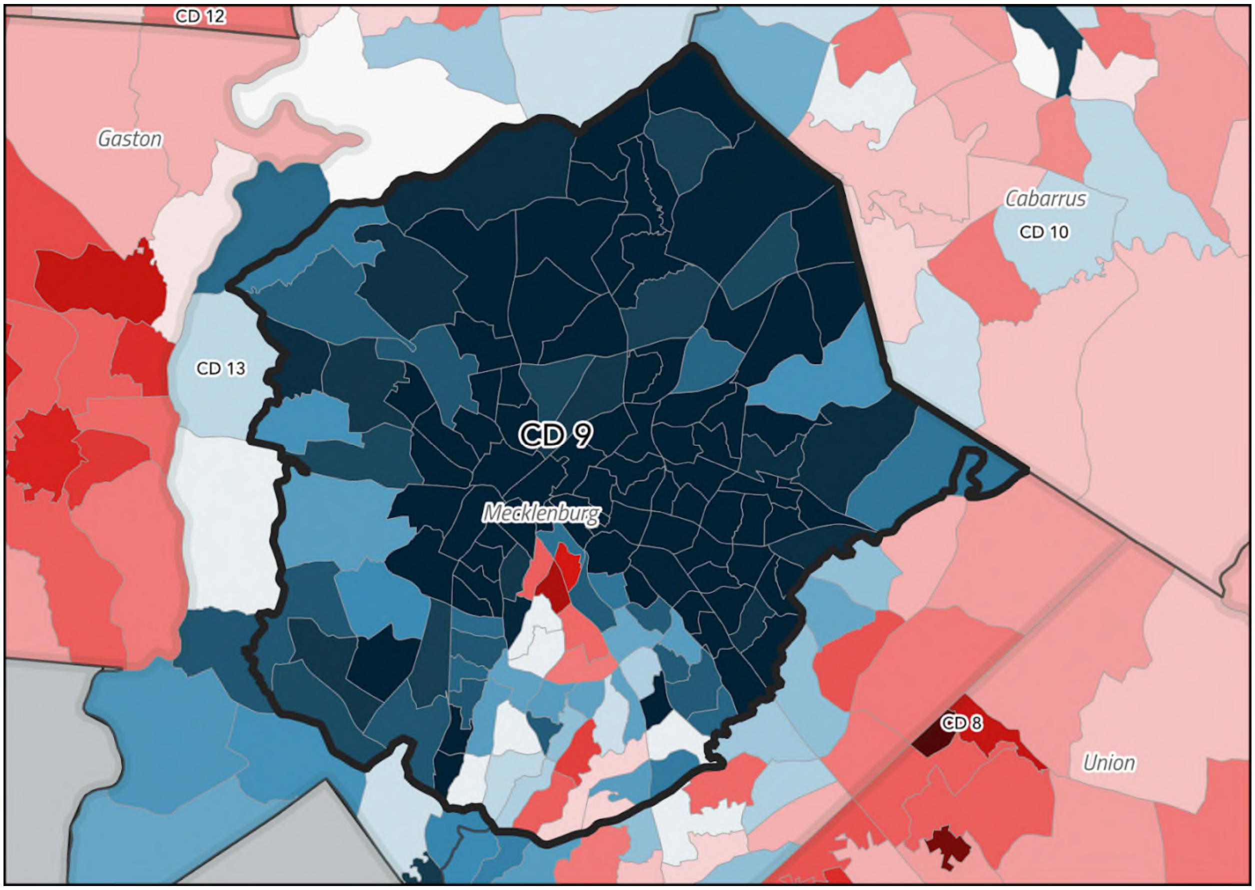
NC-9

The core of the enacted 9th congressional district comes from the current NC-12, but it also includes portions of the current NC-9. The result is the most packed district in the enacted map. The Cook Political Report rates the enacted NC-9 as a D+23 district, meaning that it leans more heavily towards the Democratic Party than any district in the last map. Donald Trump won just 25% of the vote in this district in the 2020 Presidential election and the CCSC indicates that the Democrats won over 325,000 more votes than the Republicans in the two closest Council of State races in 2020.

As with all examples of packing, the key to understanding this district is its effects on the surrounding districts. By ensuing that the Democratic candidate in NC-9 wins by an overwhelming margin, Republican voters will be more efficiently distributed across other districts, where they can have a greater affect on the outcome than they would otherwise. This ensures that neighboring NC-8, for example, will not be competitive. This also has the effect of ensuring that Republican voters in NC-9 have no chance of securing representation from a member of their own party.

The geographic contortions of this district are most apparent on its western edge, where a mere eight miles separates the western edge of NC-9 and the Mecklenburg County line.

Map 13. VTD CCSC for NC-9

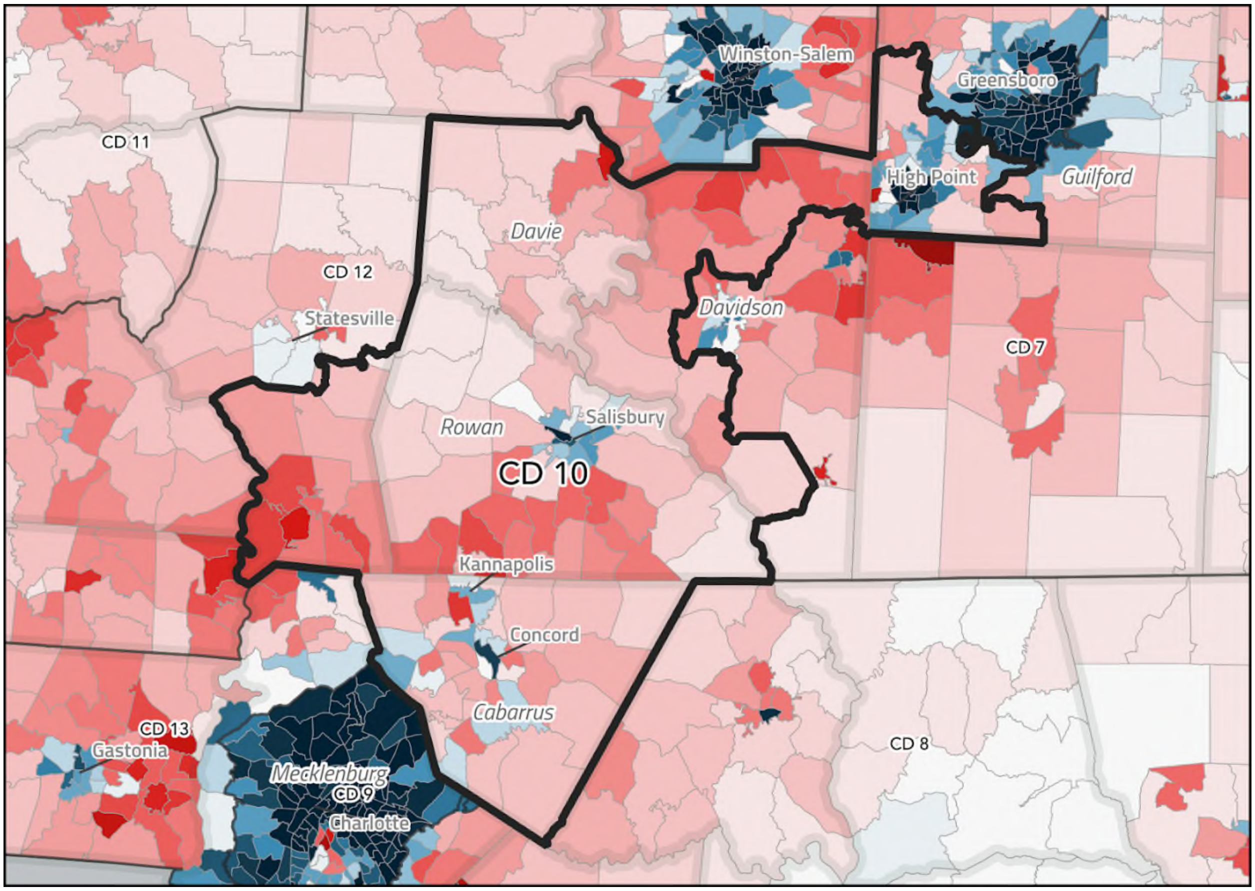


NC-10

The enacted NC-10 includes all of Rowan, Cabarrus, and Davie counties and parts of Iredell, Davidson, and Guilford counties. It is drawn out of portions of the current 10th, 9th, 6th, and 13th districts. Despite the inclusion of carefully curated portions of Democratic Guilford County, this district is a safe Republican seat and effectively removes any possibility that Democratic voters in High Point, Salisbury, Kannapolis, Concord, and elsewhere in Cabarrus can elect a member of their own political party. The Cook Political Report rates this district as R+14, the CCSC indicates that Republicans won more than 156,000 additional votes in the two key council of state races, and Donald Trump won over 60% of the Presidential vote in the enacted district.

NC-10 includes High Point, while NC-11 includes most of Greensboro and NC-12 contains Winston-Salem, meaning that the enacted map splits all three points of North Carolina's Piedmont Triad into separate congressional districts that favor Republicans. In the current map, this community of interest is together in NC-6, represented by Democrat Kathy Manning.

Map 14. VTD CCSC for NC-10



NC-11

The enacted 11th congressional district is carved out of the current 5th, 10th, and 6th districts. This map places a portion of Guilford County, including the City of Greensboro, in a district with Rockingham, Stokes, Surry, Alleghany, Ashe, Wilkes, Caldwell, and Alexander counties as well as a tiny boot-shaped sliver of Watauga County.

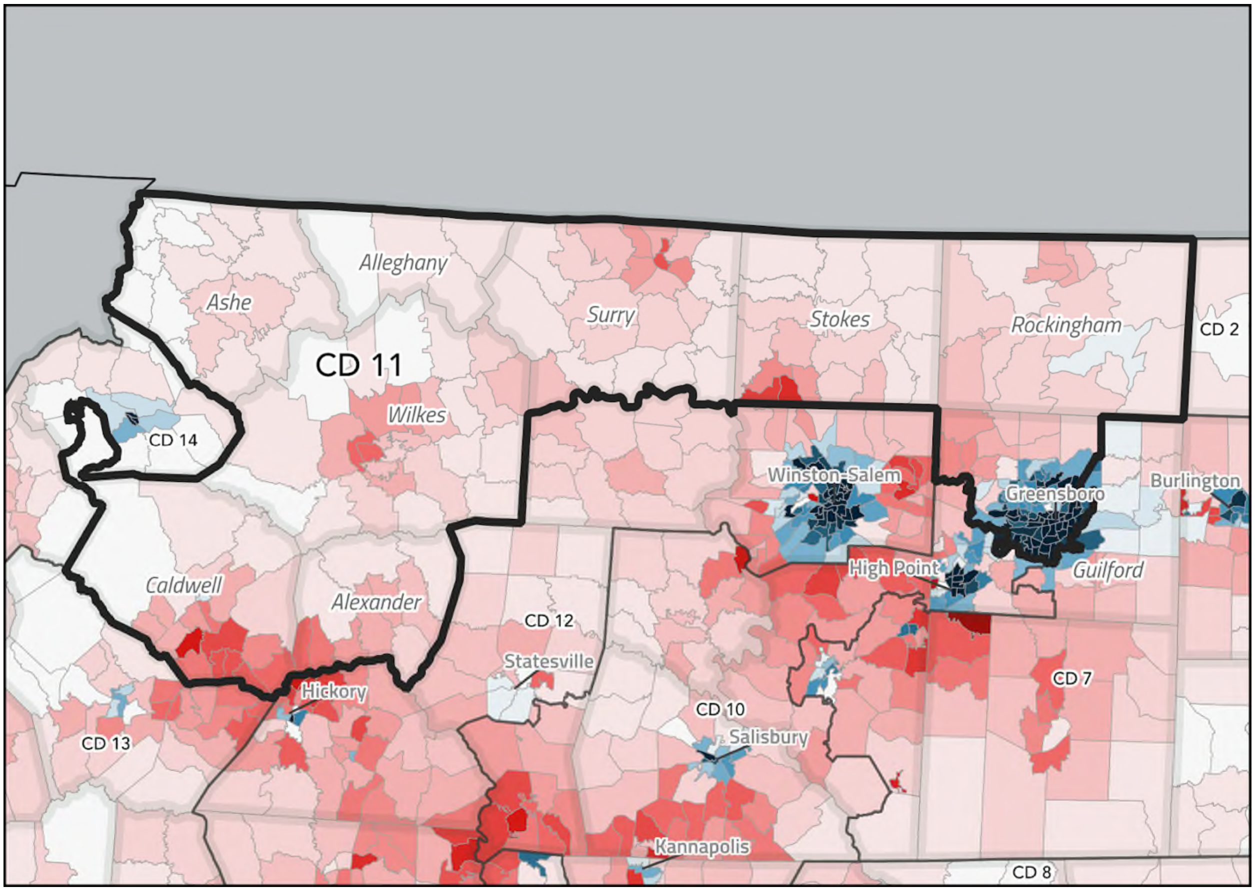
As discussed elsewhere, the enacted map splits Guilford County across three districts (the 10th, 11th, and 7th) and puts all three points of the Piedmont Triad in separate districts. By placing most of Greensboro in this overwhelmingly Republican district, the map ensures that the City of Greensboro, among the most Democratic and racially diverse cities in the state of North Carolina, will not be represented by a Democrat.

The enacted district is rated by Cook as R+9, 57% of the district voted for Donald Trump in the 2020 election, and Republicans held a 94,000 vote lead in the two closest Council of State elections. No Democrat in the current Congress represents a district that leans this heavily Republican.

It is difficult to imagine any sense in which some of the locations in this district have shared community interests. Geographically, NC-11 spans radically different parts of the state. Greensboro is firmly in the Piedmont, resting at under 900 feet elevation. Watauga and Ashe counties, by comparison, reside in the high country, with elevations that consistently run above 5,500 feet. The corners of the district have different area codes, are served by different media markets, and share virtually no characteristics in common other than the fact that they are both within North Carolina. In the history of North Carolina, Caldwell and Rockingham counties have never shared a congressional representative.

In addition to its geographic span, the enacted district stands out for its double-bunking of Republican Virginia Foxx and Democrat Kathy Manning. To shoe-horn Foxx into the new district, the mapmakers carved out a tiny sliver of Watauga County to allow her house to fall into the redrawn district. This passage is so narrow, in fact, that it is connected by a stretch of land that is roughly three miles wide and requires a traverse of the Daniel Boone Scout Trail.

Map 15. VTD CCSC for NC-11

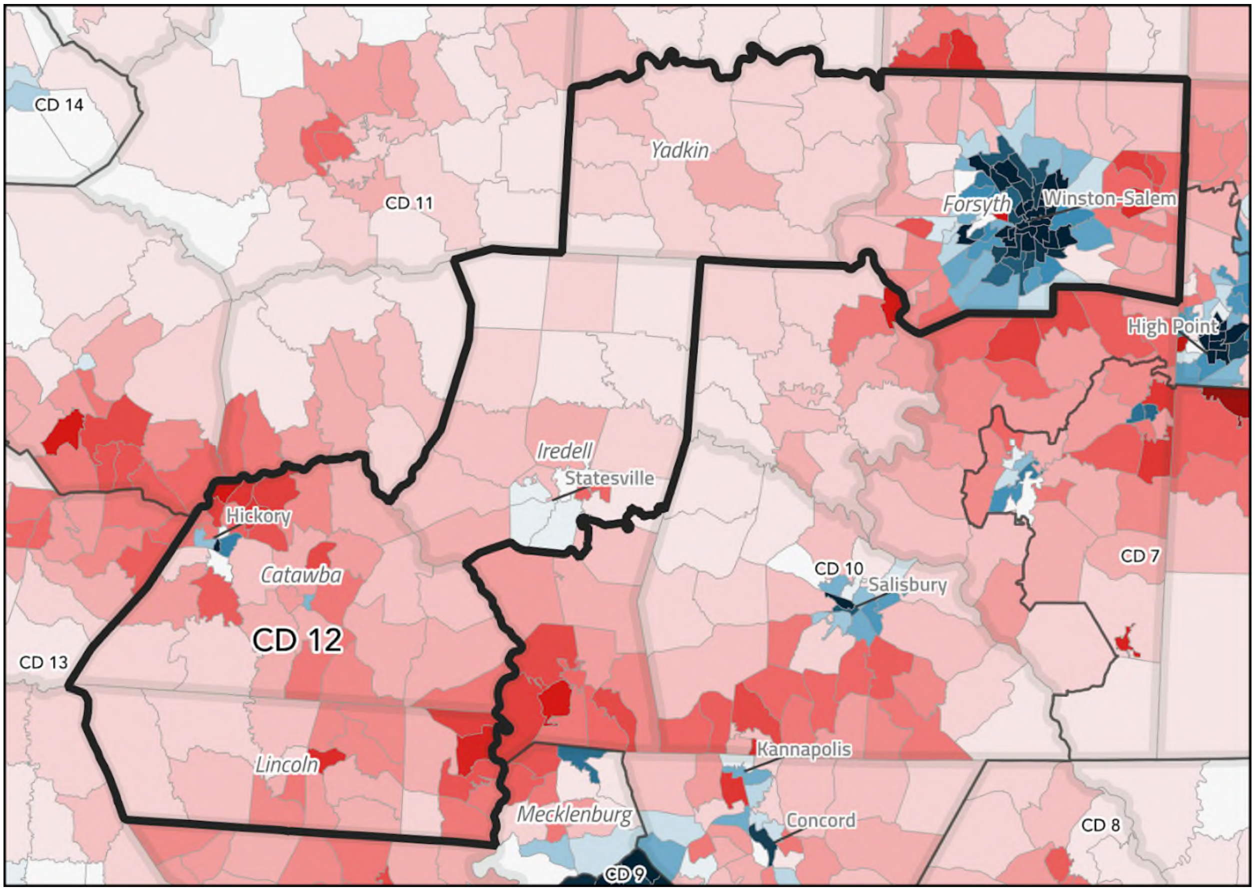


NC-12

The 12th congressional district stretches from Lincoln County at the southwestern corner, through Catawba, the northern part of Iredell, Yadkin, and Forsyth counties. As the map below makes clear, by including Winston-Salem with this overwhelmingly red swath of geography and walling it off from Democratic voters in High Point, the enacted map ensures that Republican Congressman Patrick McHenry, who lives at the southeast corner of this district, will maintain his seat and the Democratic voters in Winston-Salem will have virtually no chance to elect a member of their own party.

The Cook Political Report rates this district as R+9, Republicans had over a 100,000 vote margin in the two closest Council of State races, and Donald Trump won over 56% of the vote in this district.

Map 16. VTD CCSC for NC-12



NC-13

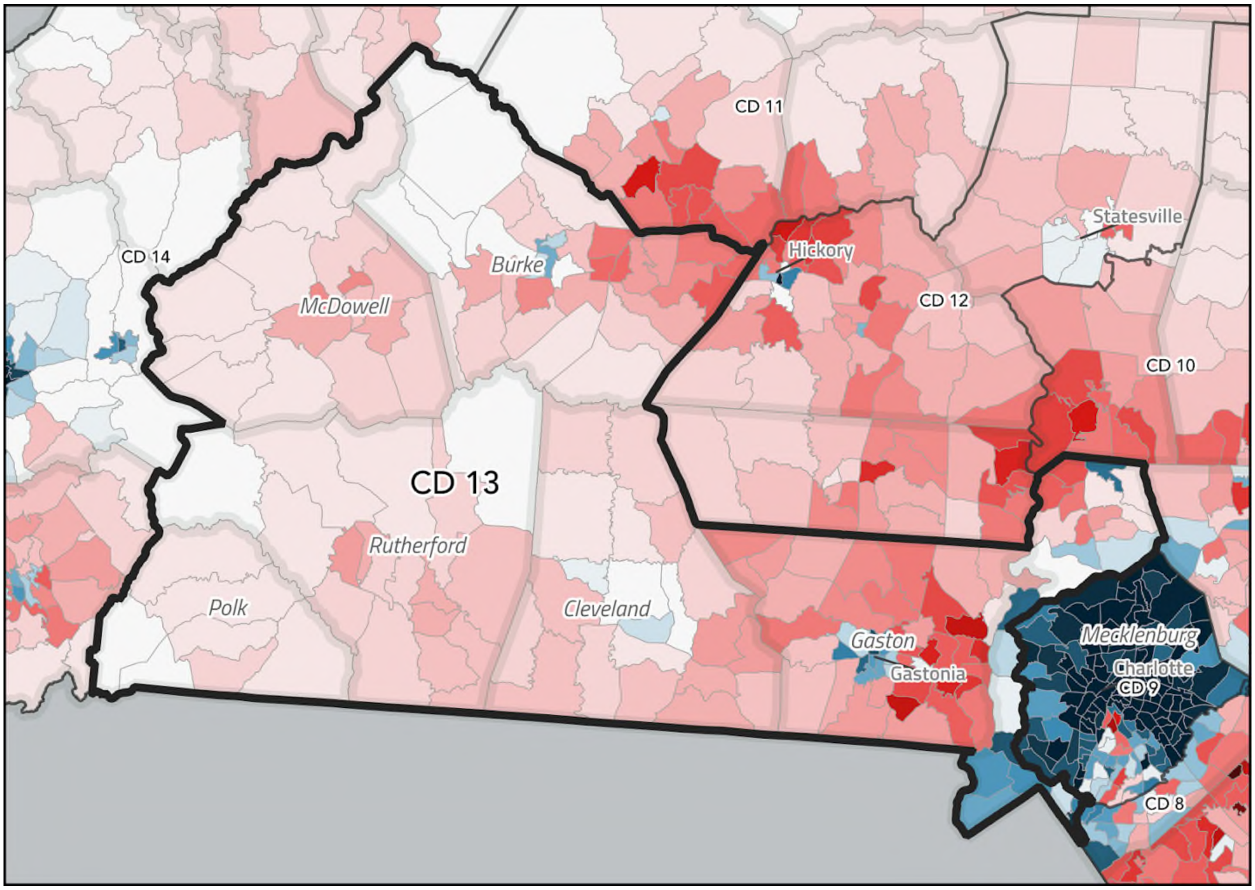
The 13th congressional district is carved out of portions of the current 11th, 5th, 12th, and 10th districts. As the map that follows demonstrates, the district includes Polk, Rutherford, McDowell, Burke, Cleveland, and Gaston counties, as well as part of Mecklenburg County.

The district was generally understood to be created for Republican Speaker of the House Tim Moore who lives in Cleveland County—*The Raleigh News and Observer and Charlotte Observer*'s editorial board even referred to it as "Moore's designer district."[23] Republican Madison Cawthorn recently announced that he will run in the 13th, and Moore soon noted that he would stay in the General Assembly. While the specifics of the candidates have changed, the fact that this is a Republican district that will elect a Republican candidate has not. This district was rated by the Cook Political Report as R+13, has a CCSC of R+150,187 votes, and gave 60% of its votes to Donald Trump in 2020.

As mentioned in the discussion of NC-9, the narrow passageway that is necessary to squeeze NC-13 into Mecklenburg County only consists of a few miles at one point—stretching from a Food Lion to the Mecklenburg County line. The enacted district also creates unusual pairings of counties that share little in common. For example, Polk and Mecklenburg counties have never resided in the same district.

[23] "Try not to Laugh at What Madison Cawthorn Just Did to NC Republicans," *Charlotte Observer*, November 13, 2021, https://www.charlotteobserver.com/opinion/article255769626.html.

Map 17. VTD CCSC for NC-13

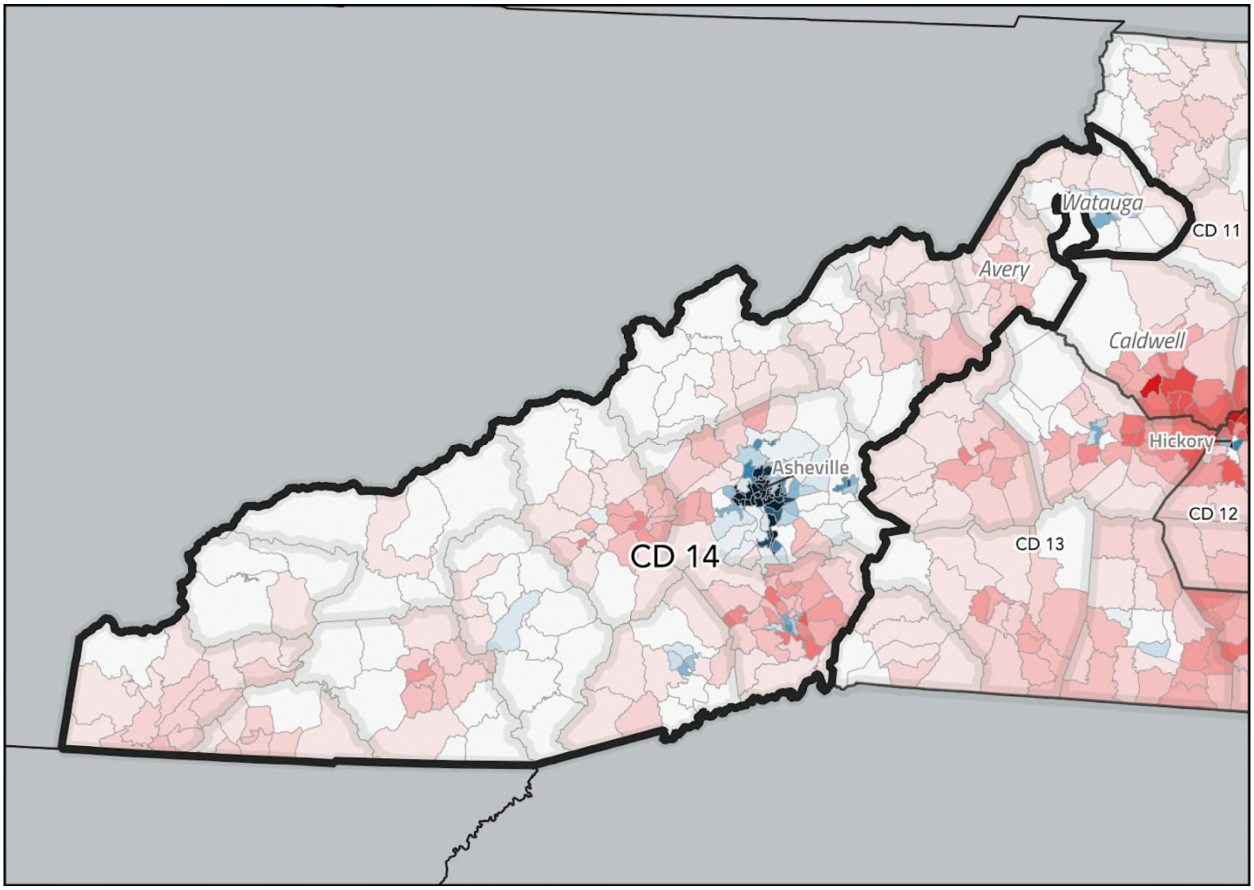


NC-14

The enacted 14th district includes most of the current 11th district as well as part of Watauga County, which previously sat in the 5th district. The current 11th district also lost the Republican strongholds of Polk and McDowell counties, as well as part of Rutherford County, which are now in the 13th district. These changes shifted the enacted NC-14 slightly in the Democratic direction (from a PVI of R+9 to R+7), although not enough to give a Democratic candidate a reasonable chance of victory. No Democrat in Congress represents a district that has a PVI score that leans this heavily towards the Republican Party. As a result, the 14th is expected to stay squarely in Republican hands.

Geographically, the 14th is a sprawling district that includes three media markets. Traversing the district from its western end in Murphy to its northeastern corner in Stony Fork would take approximately four hours. Perhaps because of the geographic incompatibility, Watauga County has not been in a district with the western end of the state since 1871—before Graham and Swain counties were even in existence. Adequately representing this massive swath of geography would be difficult for any member of Congress—Republican or Democrat.

Map 18. VTD CCSC for NC-14



General Assembly District Maps

Unlike the Congressional maps, the North Carolina House and Senate maps are minimally constrained by the *Stephenson* county clustering rule. This requires that in order to ensure relative population equality, "all counties get assigned to a distinct 'group' or 'cluster,' which can consist of either a single county or a number of adjacent counties."[24] Some districts, therefore, are contained in single district clusters that cannot be altered. For the remaining districts, however, mapmakers may have one or more types of discretion. There were four different groupings of counties where mapmakers were left to choose between more than one optimal cluster in the Senate map (yielding a total of 16 different potential county cluster maps) and three such county groupings in the House map (yielding a total of eight different potential county cluster maps).[25] And in all clusters where the population allowed for more than one district, the mapmakers had discretion over how to draw lines *within* the cluster.

In all, the General Assembly district maps benefit the Republican Party.

[24] Blake Esselstyn, "A 'Stephenson' explainer," September 2019, *available at* https://frontwater.maps.arcgis.com/apps/Cascade/index.html?appid=a408ed66ea0944308e85fe60e6e940aa.
[25] *See* Christopher Cooper, Blake Esselstyn, Gregory Herschlag, Jonathan Mattingly, and Rebecca Tippett, "NC General Assembly County Clusterings from the 2020 Census," *available at* https://sites.duke.edu/quantifyinggerrymandering/files/2021/08/countyClusters2020.pdf.

Senate Districts

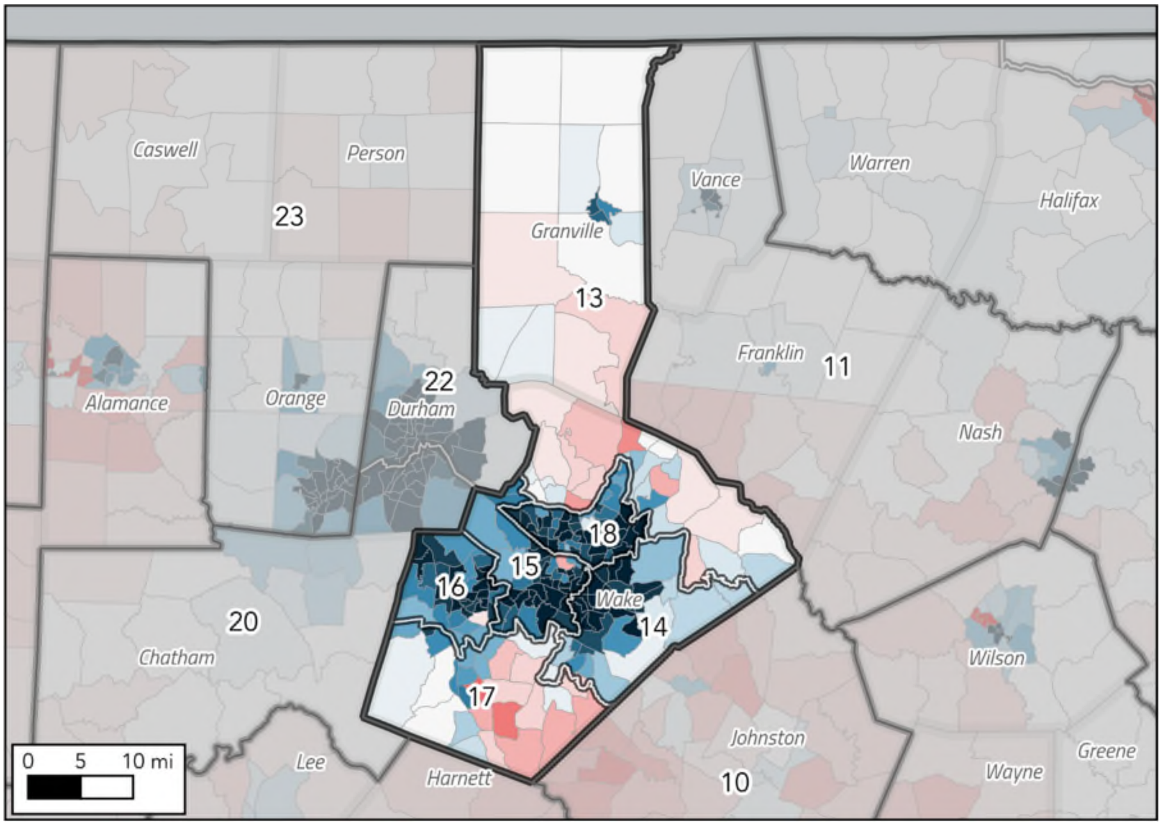
SDs 13, 14, 15, 16, 17, and 18: Granville and Wake County Cluster

Senate districts 13, 14, 15, 16, 17, and 18 are located in a cluster with Wake and Granville counties. Wake County gave 63.5% of its two-party vote share to Joe Biden in 2020. Wake County voters also supported the Democratic candidate for every statewide office and there are no Republicans on the Wake County Commission. On the other hand, Granville County is one of the most purple counties in North Carolina, supporting Donald Trump for President and Democrat Roy Cooper for Governor in 2020.

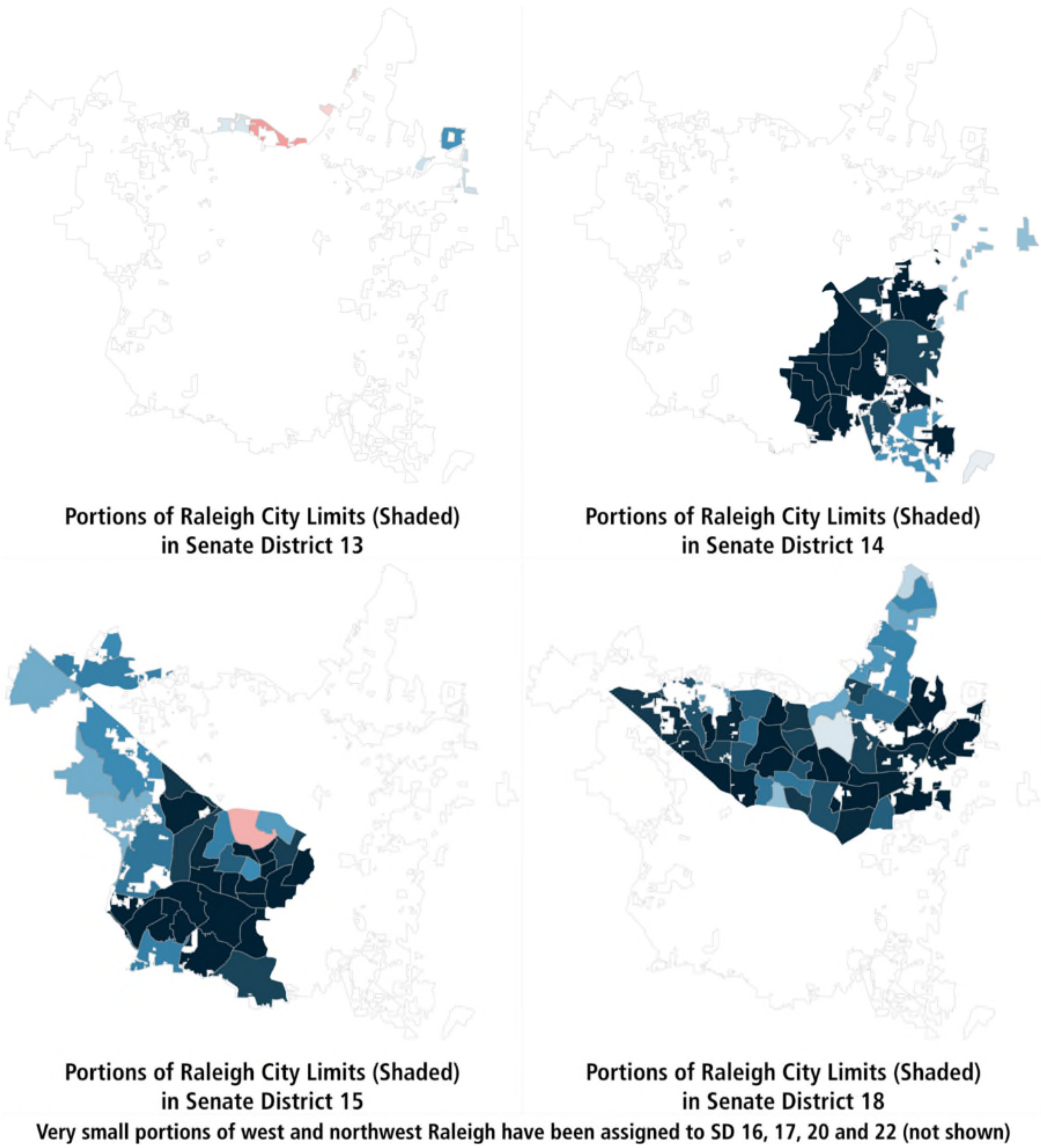
The enacted map packs Democratic VTDs in SDs 14, 15, 16, and 18 (according to the CPI, D+24, D+19, D+16, and D+15, with CCSC scores of D+93,699, D+81,915, D+59,594, and D+68,225, respectively), creating an artificially competitive SD-17 and SD-13 (both of which have a CPI score of 0, indicating no lean and a CCSC score of D+ 3,574 and R+3,686 votes, respectively). SD-13 is created by including all of Granville County and pairing it with Republican VTDs on the northern and northeastern portions of Wake County, avoiding the blue VTDs in North Raleigh, which are left in SD-18 by creating a horn-shaped section that juts up into SD-13.

The second map in this series (Map 20) demonstrates the ways in which the City of Raleigh is strategically divided across four Senate districts.

Map 19. VTD CCSC for the Granville and Wake County Cluster



Map 20. Map of Raleigh Municipal Splits

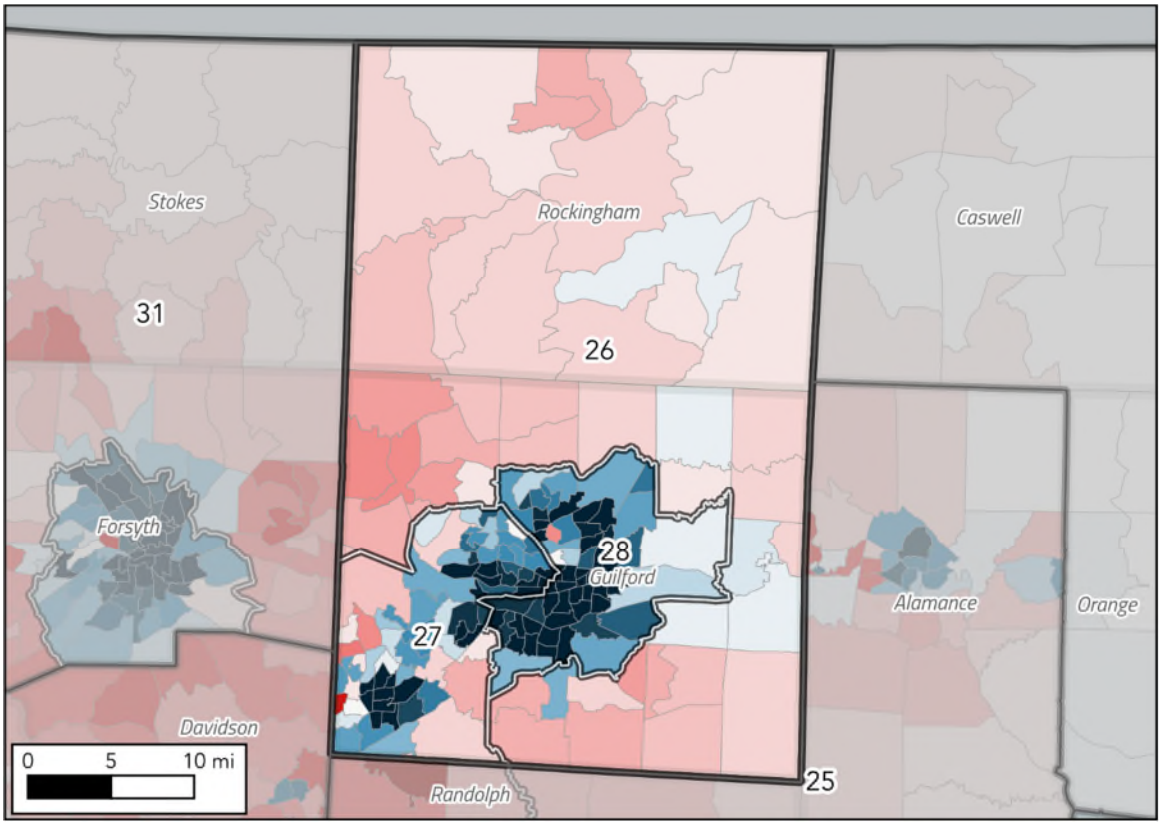


SDs 26, 27, and 28: Guilford and Rockingham County Cluster

Senate districts 26, 27, and 28 are located in a county cluster with Rockingham and Guilford counties. Rockingham County leans heavily towards the Republican Party whereas Guilford is among the most Democratic counties in North Carolina. In 2020, Guilford gave 61.7% of its vote share for President to Joe Biden, the 8th highest in the state. Guilford voters also voted for the Democratic candidate by overwhelming margins in every race decided at the county level in 2020.

The enacted map packs Democrats in SD-27 and SD-28. SD-27 is estimated to be D+12 by the CPI and has a D+50,846 CCSC score; whereas SD-28 is D+27 and has a D+104,632 advantage according to the CCSC. SD-26, on the other hand, includes all of Rockingham County and then extends southwest into Guilford County until it meets the Piedmont Triad International Airport, and east and south until it meets the eastern and southern borders of the county. SD-26's sprawling C-shape allows for a safe Republican (R+11, R+54,396) district by connecting the northern and southern portions of this cluster together.

Map 21. VTD CCSC for the Guilford and Rockingham County Cluster

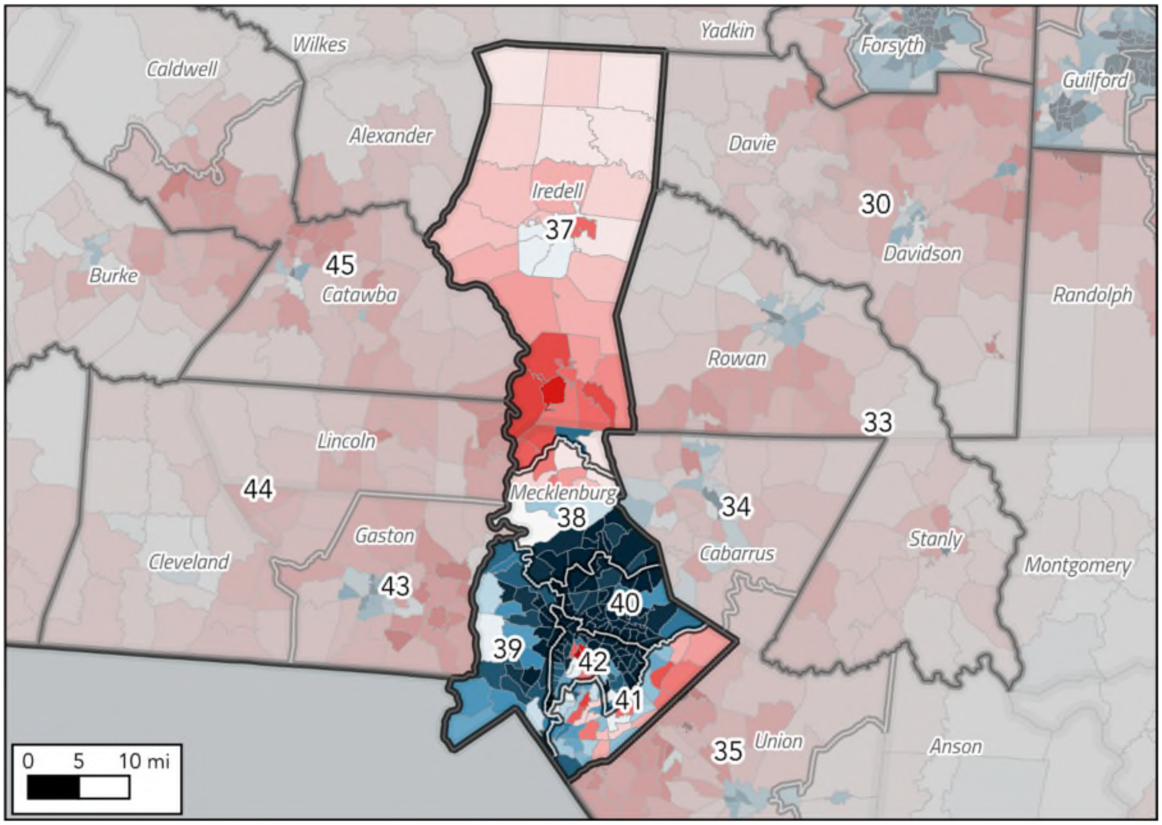


SDs 37, 38, 39, 40, 41, and 42: Iredell and Mecklenburg County Cluster

Senate districts 37, 38, 39, 40, 41, and 42 are located in a grouping that includes Iredell and Mecklenburg counties. Mecklenburg County is the second most populous and among the most Democratic counties in North Carolina. In the 2020 Presidential election, only two other North Carolina counties gave a larger proportion of their two-party vote share to Joe Biden. Every member of Mecklenburg's current state legislative delegation is a Democrat, all nine county commissioners are Democrats, and Democratic candidates received the plurality of the votes in every county-wide contest. It is clearly a Democratic stronghold, and is trending even more so in that direction.

As you can see below, the enacted map packs Democratic voters into SDs 39 and 40; neither includes a single Republican VTD and they are heavily Democratic based on CPI (D+23 and D+33, respectively) and the CCSC scores (D+71,497 and D+90,354, respectively). SDs 38 and 42 are also considered "Safe Democratic" seats (D+17, D+71,597 and D+15, D+65,179, respectively). SD-41, however, is considered a "Toss-up" seat (D+1, D+5,474) and SD-37 is a "Safe Republican" seat (R+13, 64,380). By packing Mecklenburg's Democratic voters in SDs 38, 39, 40, and 42, the mapmakers allowed for SD-41, in the south of Mecklenburg County, to be artificially competitive, while still ensuring that SD-37 remains a safely Republican district. SD-37 is also notable because it double-bunks Democrat Natasha Marcus and Republican Vickie Sawyer into the same district; Marcus' home rests approximately one mile from the border with SD-38.

Map 22. VTD CCSC for the Iredell and Mecklenburg County Cluster

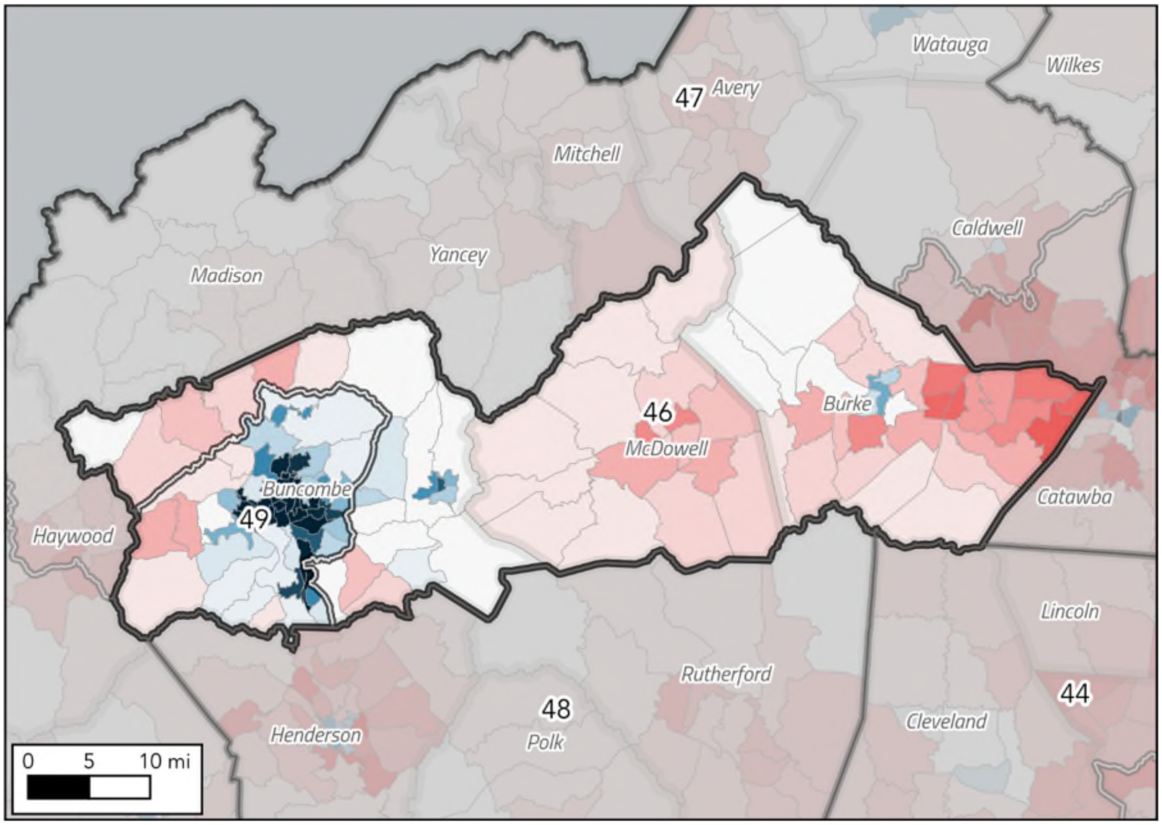


SDs 46 and 49: Buncombe, Burke, and McDowell County Cluster

Senate districts 46 and 49 are located in a county cluster with Buncombe, Burke, and McDowell counties. The map-drawers had considerable discretion here, however, as they could have instead paired Buncombe County with Henderson County, a much more natural fit since northern Henderson County, in particular, has become a bedroom community of Asheville (in Buncombe), and has considerable shared natural interests. Instead, Buncombe is paired with McDowell and Burke counties. It would take someone an hour and 45 minutes to pass from Sandy Mush on the west side this cluster to Hickory on the east side, and would almost certainly necessitate driving through both Senate districts. The enacted map also separates Asheville from the Asheville Watershed.

The effect of this choice is to pack Democratic voters in SD-49 (D+16), leaving the geographically expansive SD-46 to favor the Republican Party (R+13). By pairing Henderson with Polk and Rutherford counties in the cluster to the south, the map also creates a district heavily favored for the Republican Party in that cluster, SD-48. After the maps were enacted, incumbent Republican Chuck Edwards (currently in the Senate district covering Buncombe, Henderson, and Transylvania counties) announced he would be running for Congress and Republican State House Representative Tim Moffitt (whose current House district is in Henderson County) announced he would be running for Edwards' vacated Senate seat.

Map 23. VTD CCSC for the Buncombe, Burke, and McDowell County Cluster

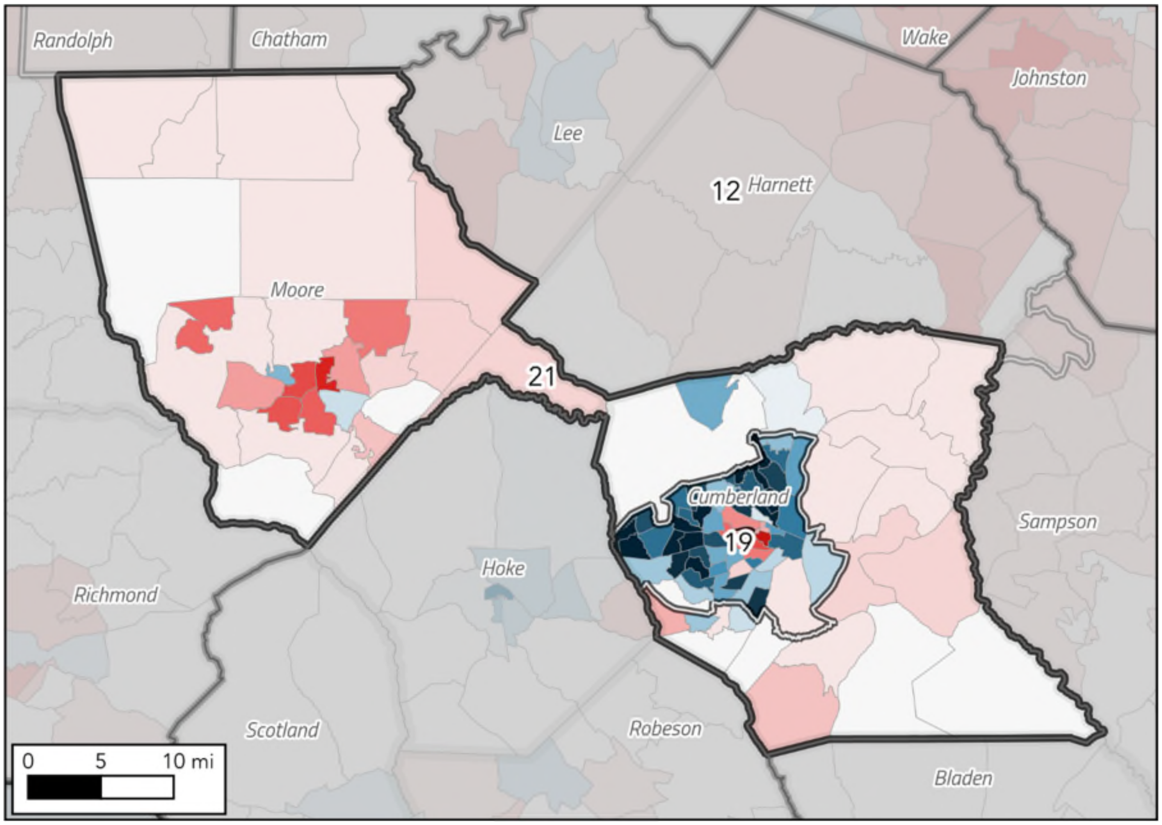


SDs 19 and 21: Cumberland and Moore County Cluster

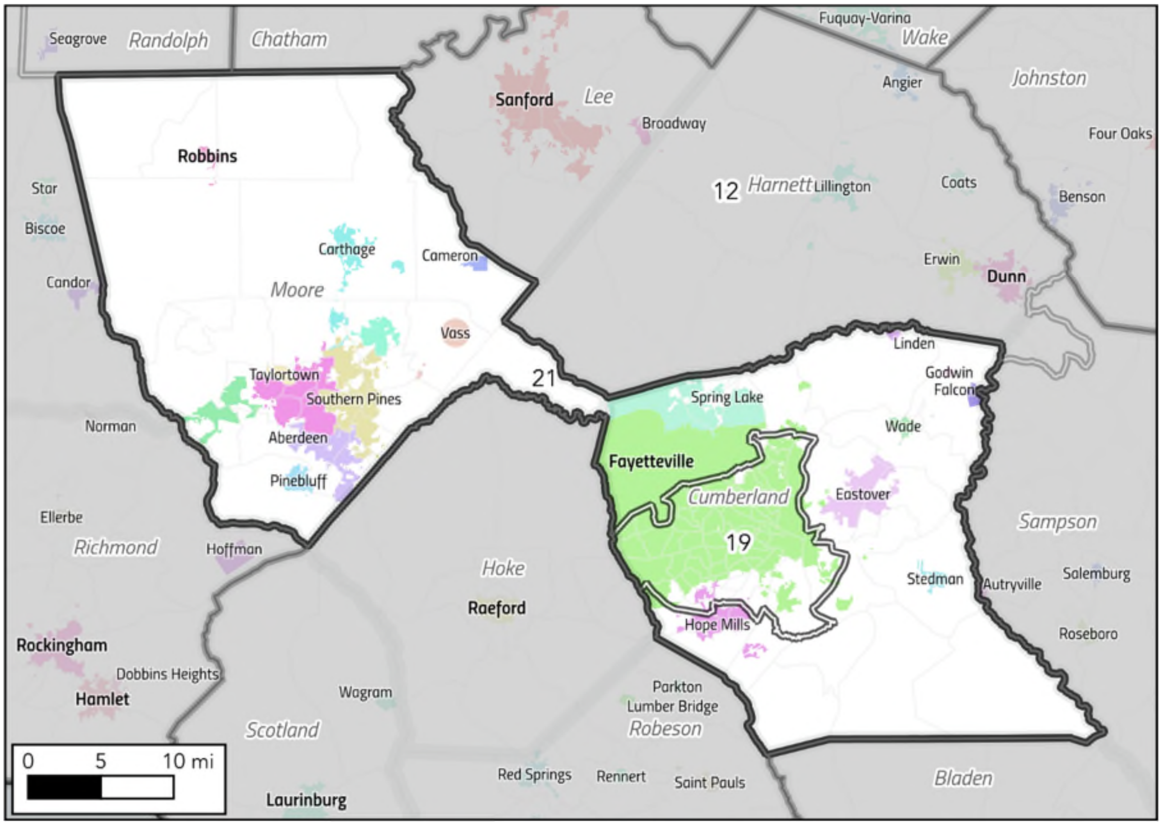
Senate districts 19 and 21 are located in a county cluster with Cumberland and Moore counties. The enacted map packs Democratic voters in and around Fayetteville into SD-19, a district that is rated D+17 by the CPI and advantaged the Democratic Party by 64,539 votes in the CCSC. SD-21 is then left to favor the Republican Party by R+9 and 41,391 votes.

As demonstrated in Map 25, the enacted map splits Fayetteville and Hope Mills across two districts and, as Map 24's red-and-blue shading displays, the district boundaries are careful to separate off Democratic voters and VTDs in SD-19 from adjacent Republican VTDs.

Map 24. VTD CCSC for the Cumberland and Moore County Cluster



Map 25. Municipal Splits for the Cumberland and Moore County Cluster



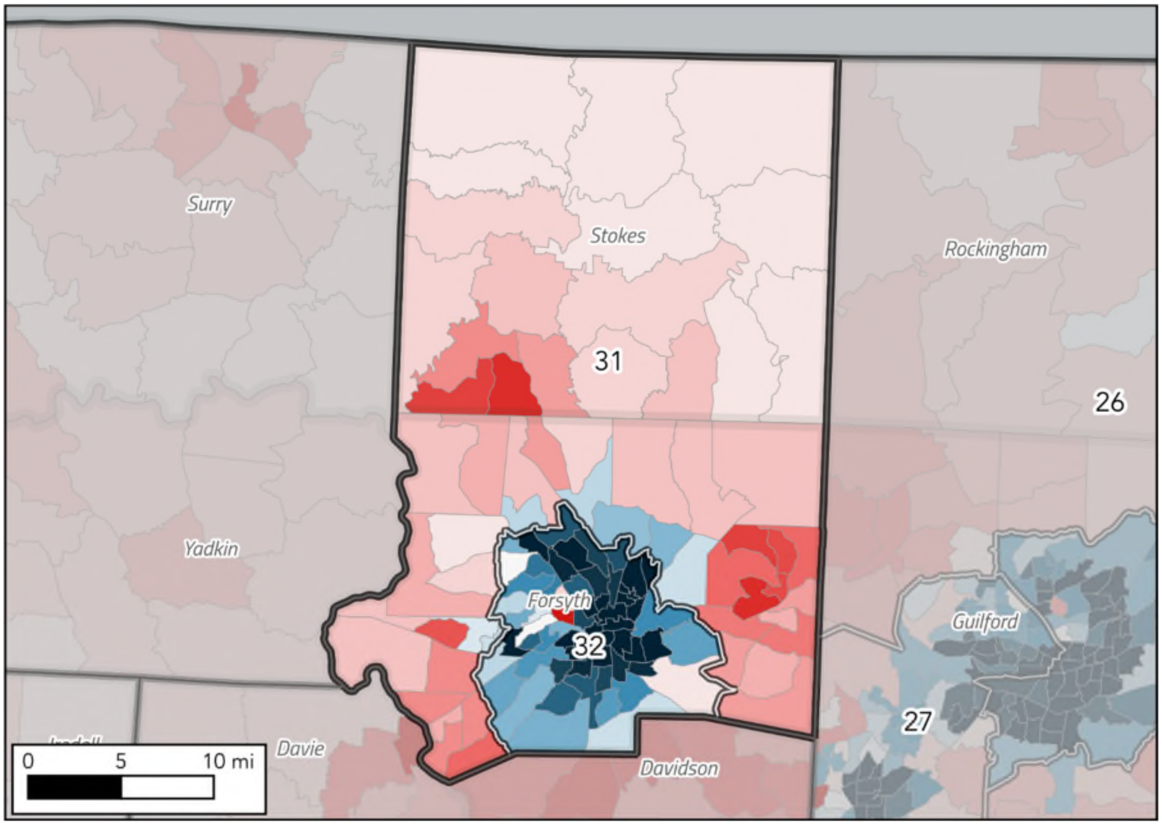
SDs 31 and 32: Forsyth and Stokes County Cluster

Senate districts 31 and 32 are located in a county cluster with Forsyth and Stokes counties. A few choices created the partisan effects of this cluster. First was the choice of the cluster, itself. The mapmakers had a choice about whether to pair Forsyth with Stokes or with Yadkin to the west. Yadkin has a lower Republican vote advantage per the CCSC. Therefore the decision to pair Forsyth with Stokes, instead, helped tip the scales towards a Republican advantage. The decisions made within the cluster reinforced that advantage.

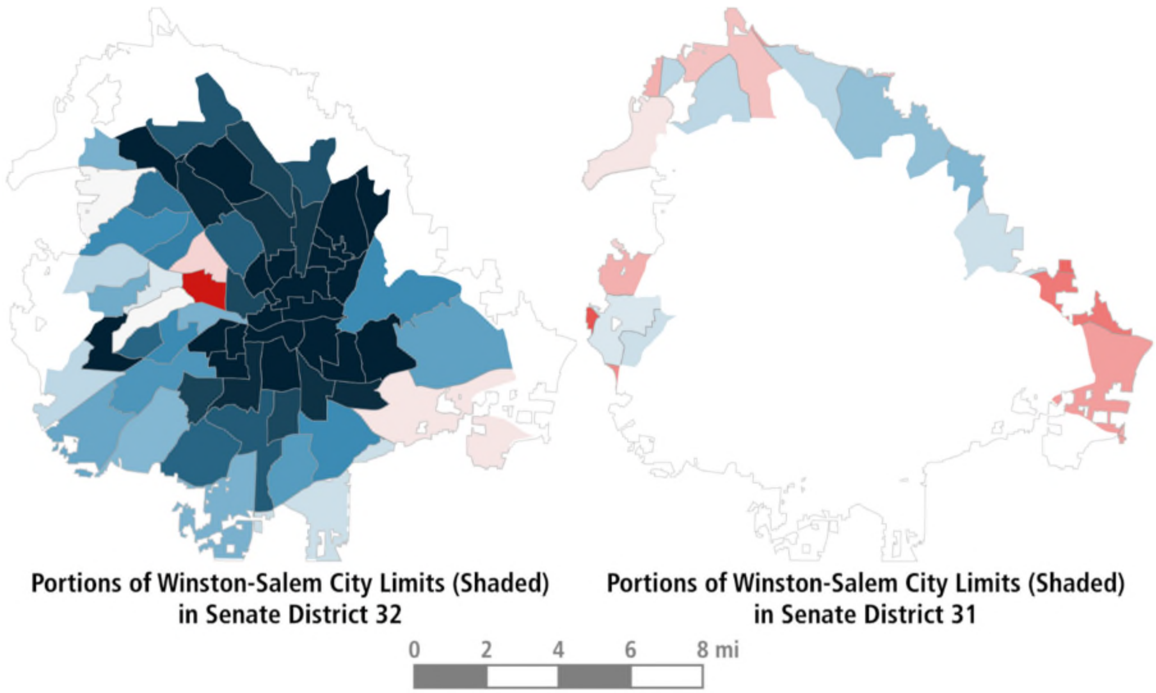
In a now familiar pattern, the enacted map packs Democratic voters in SD-32 (D+20, D+77,058) and leaves the remaining district in the cluster squarely in Republican hands. SD-31 favors the Republican Party by R+11; the CCSC favors the Republican Party by 58,073 votes.

Map 27 displays the strategic split in Winston-Salem with the most Democratic VTDs in that city packed into SD-32 while Republican SD-31 captures the more Republican VTDs on the city's edges.

Map 26. VTD CCSC for the Forsyth and Stokes County Cluster



Map 27. Map of Winston-Salem Municipal Splits

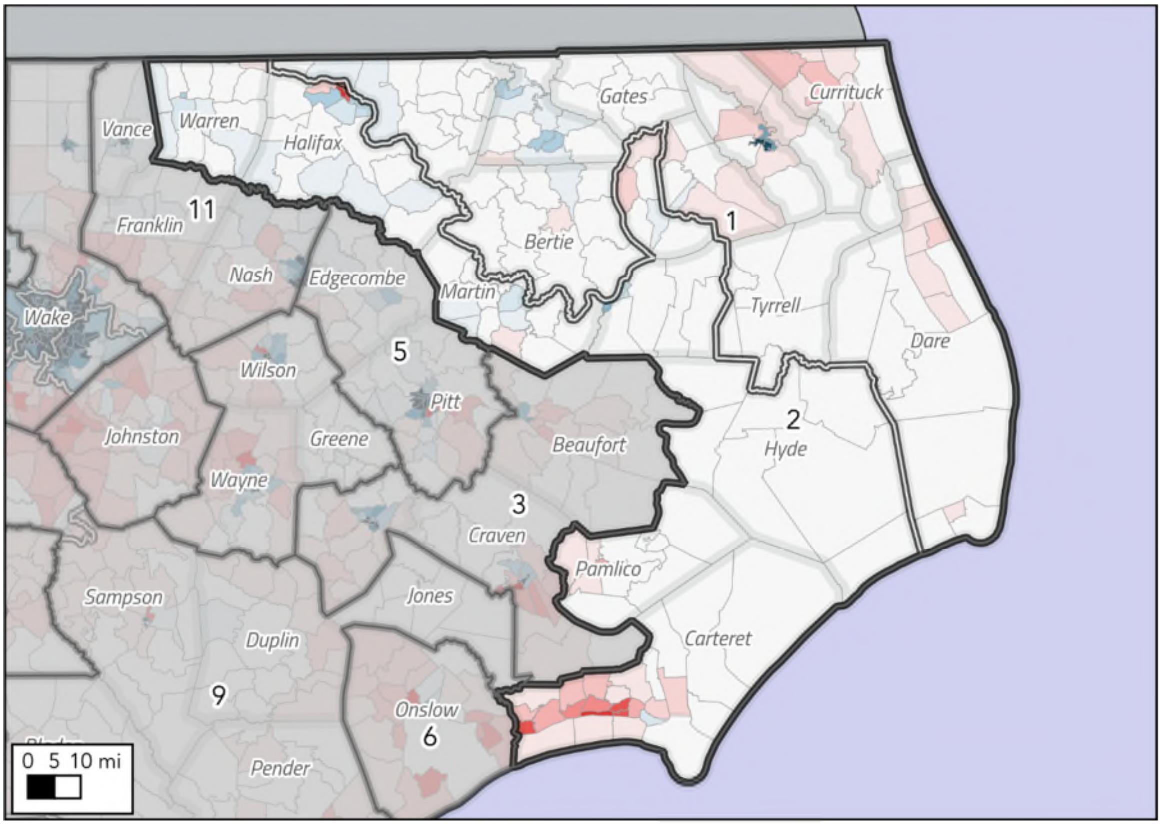


SDs 1 and 2: Northeastern County Clusters

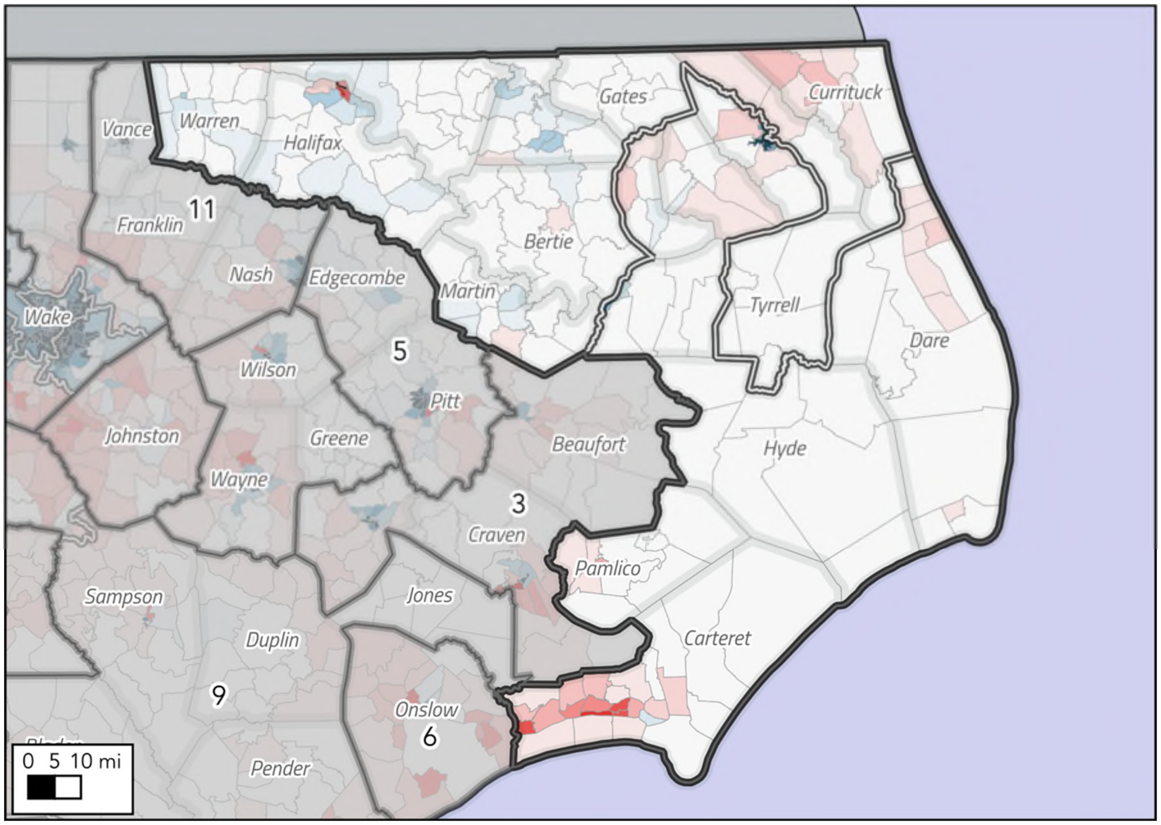
Senate districts 1 and 2 are located in two adjacent county clusters that contain Bertie, Halifax, Hertford, Northampton, and Warren counties. Many of these counties are among the most racially diverse in the state.

The mapmakers had one consequential choice to make here—the choice of which counties would be included within each cluster (the size of each cluster is such that the clusters can contain only one district, each). The choice of cluster helped tilt the scales in the direction of the Republican Party, as evidenced in Maps 28 and 29 below. If the map-drawers had chosen the alternative county cluster configuration (Map 29), the result would have been much more likely to favor the Democratic Party in one district (with a projected CCSC score of D+10,270) and the Republican Party in the other district (with a projected CCSC score of R+49,916). Instead, the enacted map pairs more Republican voters together resulting in two districts that lean towards the Republican Party (SD-1: R+2, R+16,350; SD-2: R+4, R+23,296), despite the competitiveness of most of the VTDs in this cluster.

Map 28. VTD CCSC for the Northeastern County Clusters



Map 29. Potential Northeastern County Clusters That Were Not Selected



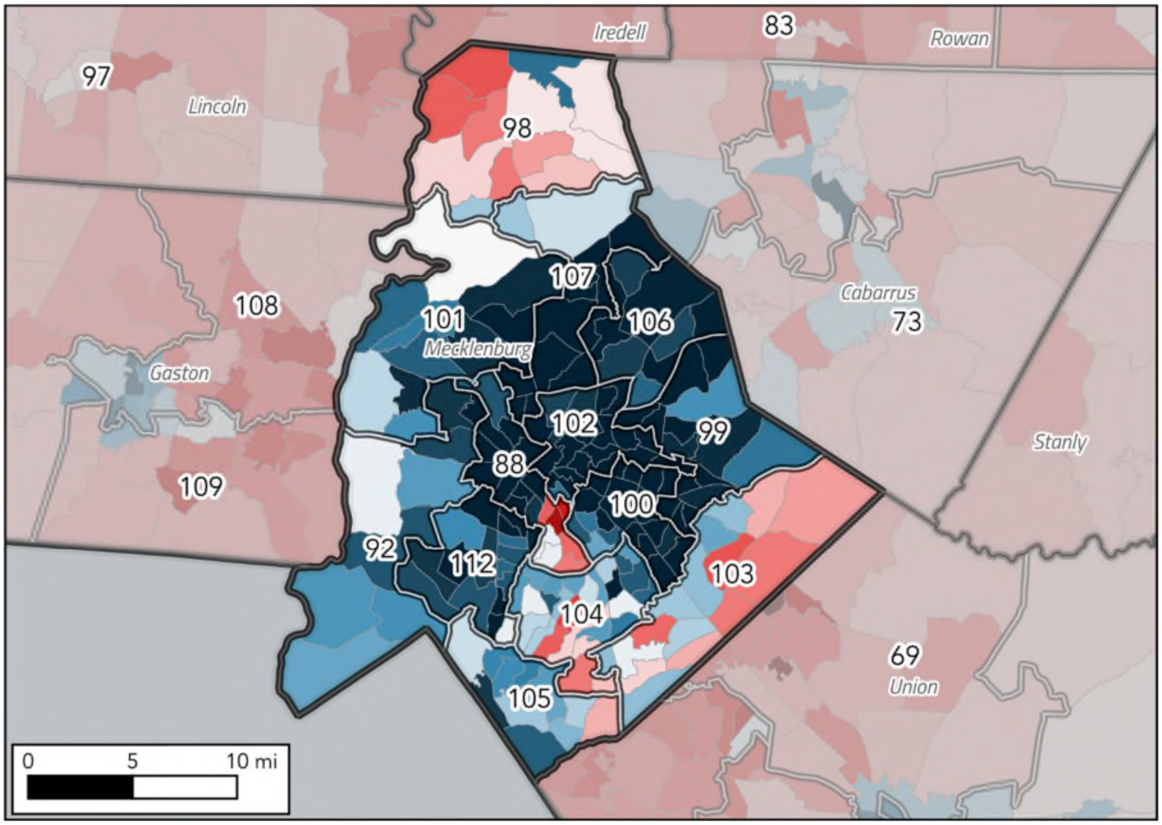
House Districts

HDs 88, 92, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, and 112: Mecklenburg County Cluster

Mecklenburg County is the home of Charlotte as well as six other municipalities. As noted above, Mecklenburg County is dominated by Democratic voters and is becoming even more so as the county continues to grow in population.

The enacted map places no Republican VTDs in HDs 92, 99, 100, 101, 102, 106, 107, and 112, leaving every Republican-leaning VTD in HDs 88, 103, 104, and 105. This arrangement provides Republican candidates the greatest probability of victory possible in this sea of blue. In particular, HDs 98 and 103 are carved out of the pockets of Republican voters in the north and southeast portions of the county so as to be particularly favorable to Republicans. HD-98 is rated by CPI as R+5 and HD-103 is rated as even, with CCSC scores of R+4,359 and R+2,645, respectively.

Map 30. VTD CCSC for the Mecklenburg County Cluster



HDs 11, 21, 33, 34, 35, 36, 37, 38, 39, 40, 41, and 49: Wake County Cluster

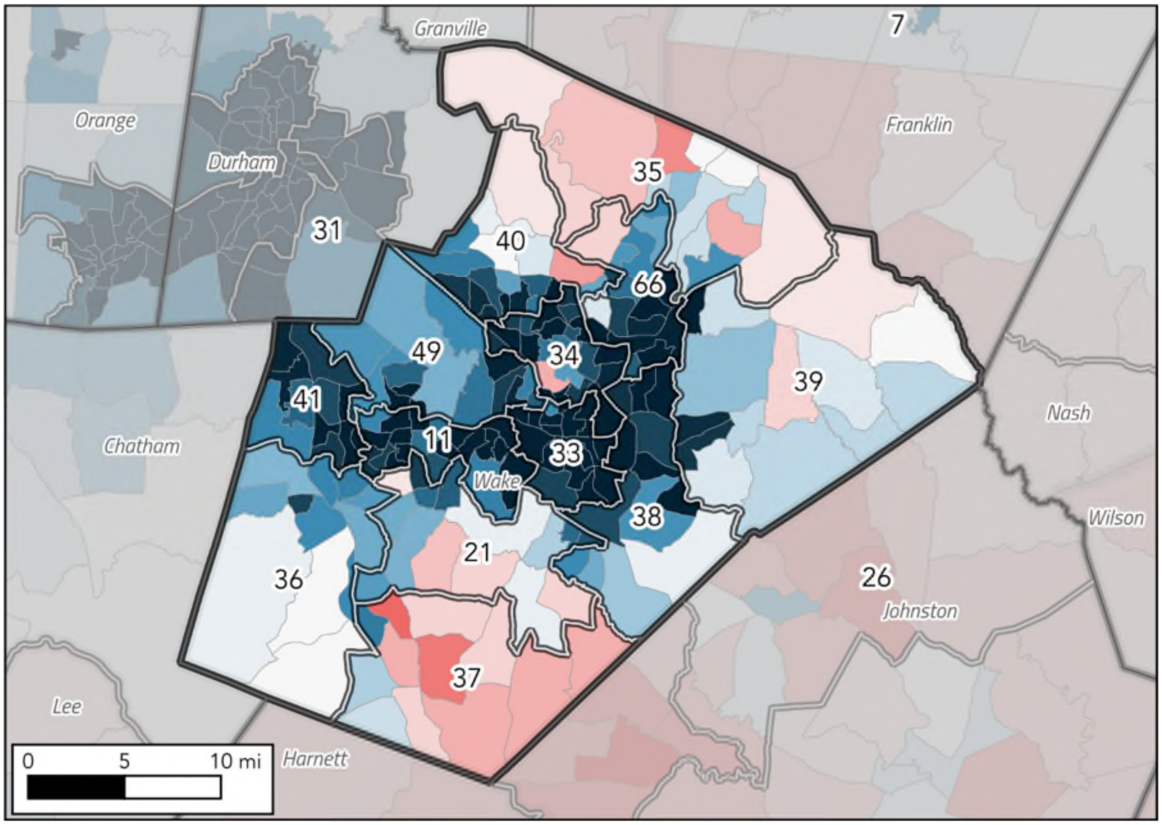
House districts 11, 21, 33, 34, 35, 36, 37, 38, 39, 40, 41, and 49 are located in the Democratic stronghold of Wake County, which includes Raleigh and 11 other municipalities. As noted above, Wake County gave 63.5% of its two-party vote share to Joe Biden in 2020 and supported Democratic candidates for every statewide office. There are no Republicans on the county commission.

The enacted map packs Democrats into as few districts as possible, creating contorted districts that, in the case of HDs 11, 33, 36, 38, 41, and 49, include no Republican VTDs. This leaves HD-37 as a Republican leaning district, which will benefit the Republican candidate Erin Pare, who narrowly defeated a Democrat in the last election. These district boundaries also increase the probability that a Republican can defeat the Democratic incumbent Terence Everitt in HD-35, in the northern portion of Wake County. HD-37 is rated as R+3 by the CPI and has a R+6,400 score; HD-35 is rate as R+1 by the CPI and has a R+2,264 CCSC score.

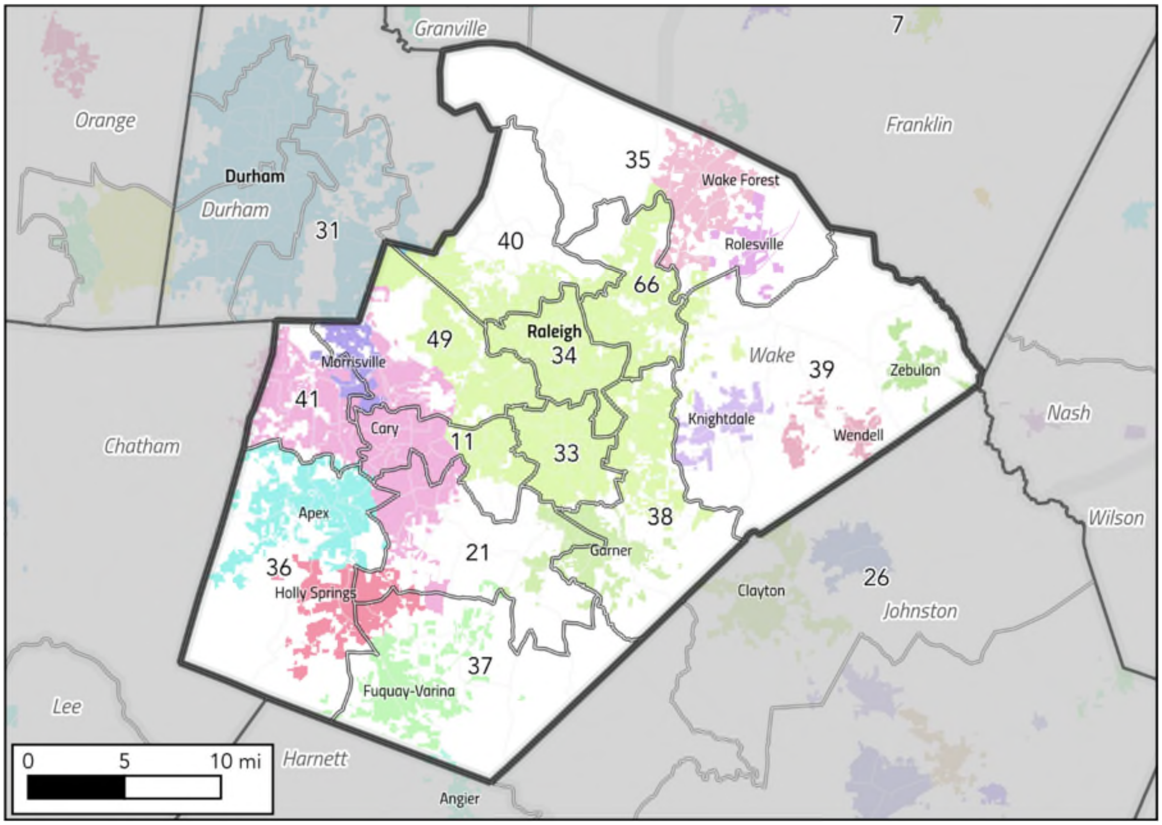
The partisan effects of small decisions are particularly apparent in the spike that juts up from HD-66 into HD-35, keeping the Democratic VTDs in that spike fenced off from the more Republican-leaning VTDs in HD-35. If the district lines took a slightly different jog here, it would increase the probability of Everitt securing re-election.

As Map 32 indicates, the enacted map also splits a number of cities both large (Raleigh, shaded in light green, split across nine districts; Cary, shaded in pink, split across four districts) and small (Garner, Fuquay-Varina, Apex, Holly Springs, and Morrisville). The district boundaries appear calculated to provide a partisan advantage for Republican candidates rather than adhere to any municipal boundaries.

Map 31. VTD CCSC for the Wake County Cluster



Map 32. Municipal Splits in the Wake County Cluster

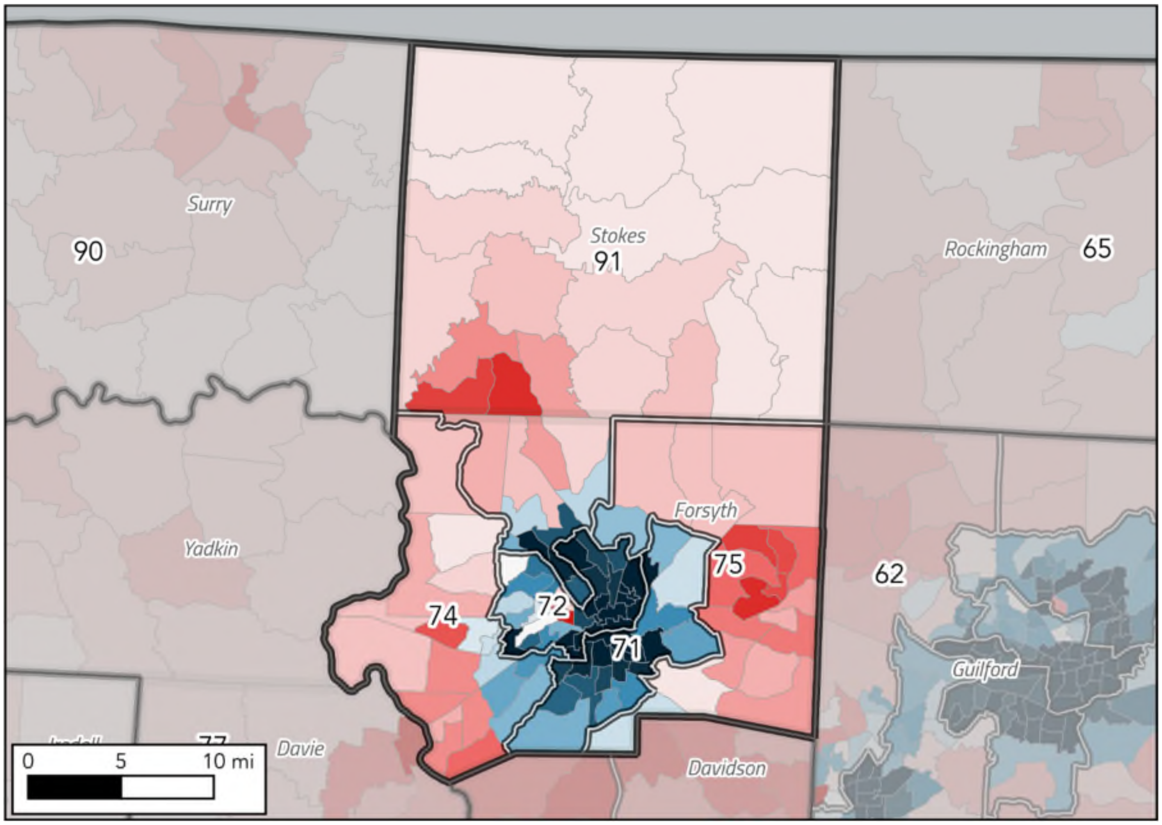


HDs 71, 72, 74, 75, and 91: Forsyth and Stokes County Cluster

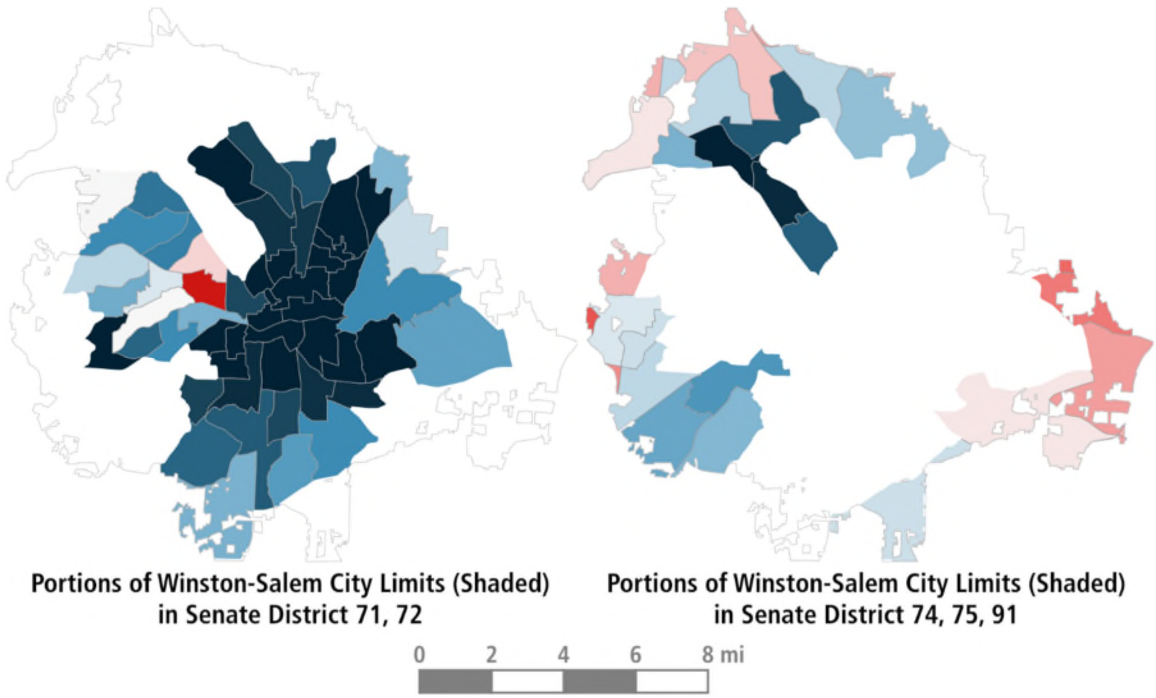
House districts 71, 72, 74, 75, and 91 are located in Forsyth and Stokes counties. The enacted map splits Winston-Salem across all five districts in this cluster and packs Democratic voters into HDs 71 and 72 (HD-71 does not include a single Republican VTD), leaving HD-75 and HD-91 almost certain to elect a Republican and HD-74 as a Republican leaning district (with a CPI score of R+3 and a CCSC score of R+7,846).

The splits of Winston-Salem do not make sense without reference to the anticipated voting behavior of the VTDs arranged into each district. For example, HD-91 includes all of Republican-leaning Stokes County, but instead of joining Stokes with a broader expanse of northern Forsyth County to create a more compact district, HD-91 juts down into the center of Winston-Salem, picking up some of the most Democratic VTDs in the cluster (which include Bethabara Moravian Church, Arts Council Theatre, and Mision Hispana VTDs—43.8% of the population in the latter VTD identifies as black and 29.5% identifies as Hispanic), ensuring that Democratic voters in the core of Winston-Salem have essentially no chance at electing a member of their own party, and dividing a major North Carolina city unnecessarily. But this arrangement does allow HD-74, to the west, and HD-75, to the east, to lean in favor of Republican candidates, despite their proximity to the deep pocket of Democratic voters in the city that those districts overlap with on their outer edges.

Map 33. VTD CCSC for the Forsyth and Stokes County Cluster



Map 34. Detail of Winston-Salem Splits



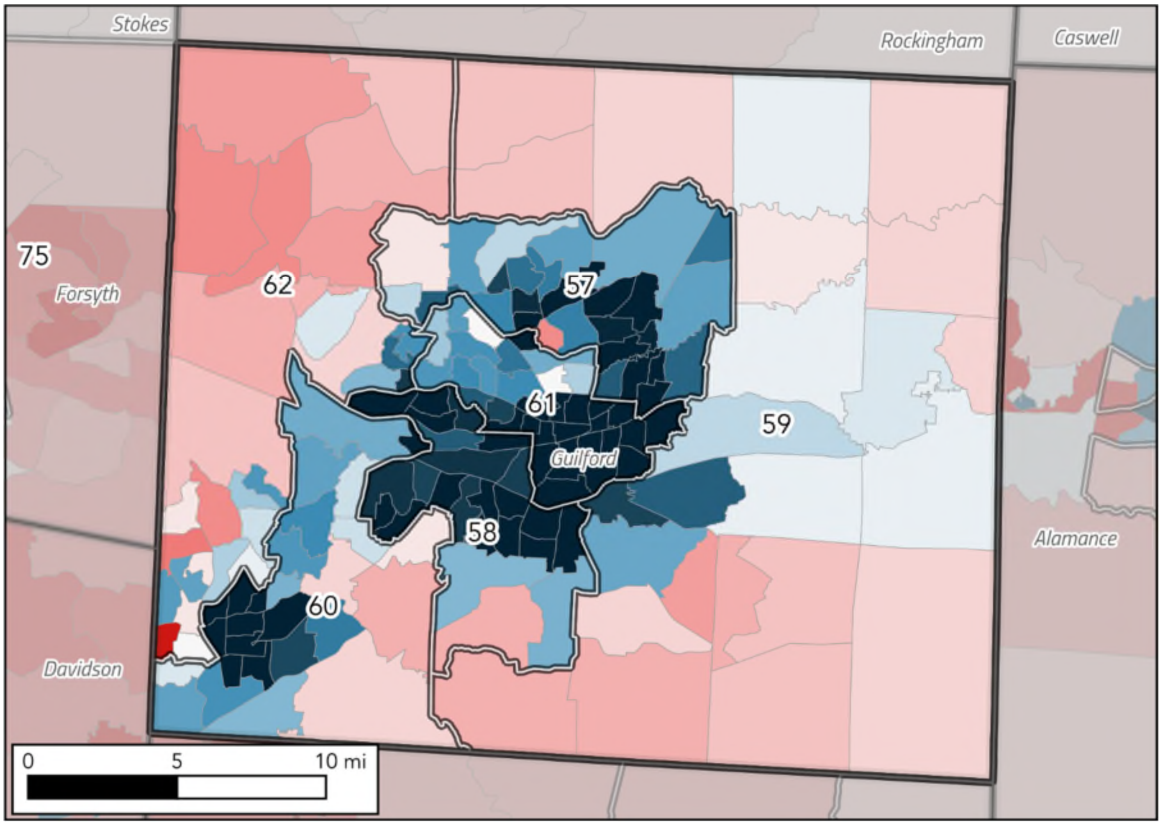
HDs 57, 58, 59, 60, 61, and 62: Guilford County Cluster

HDs 57, 58, 59, 60, 61, and 62 are all contained within the Democratic stronghold of Guilford County, which contains Greensboro and High Point. As noted above, Guilford County voters have provided Democratic candidates large margins of victory in recent state- and county-wide elections.

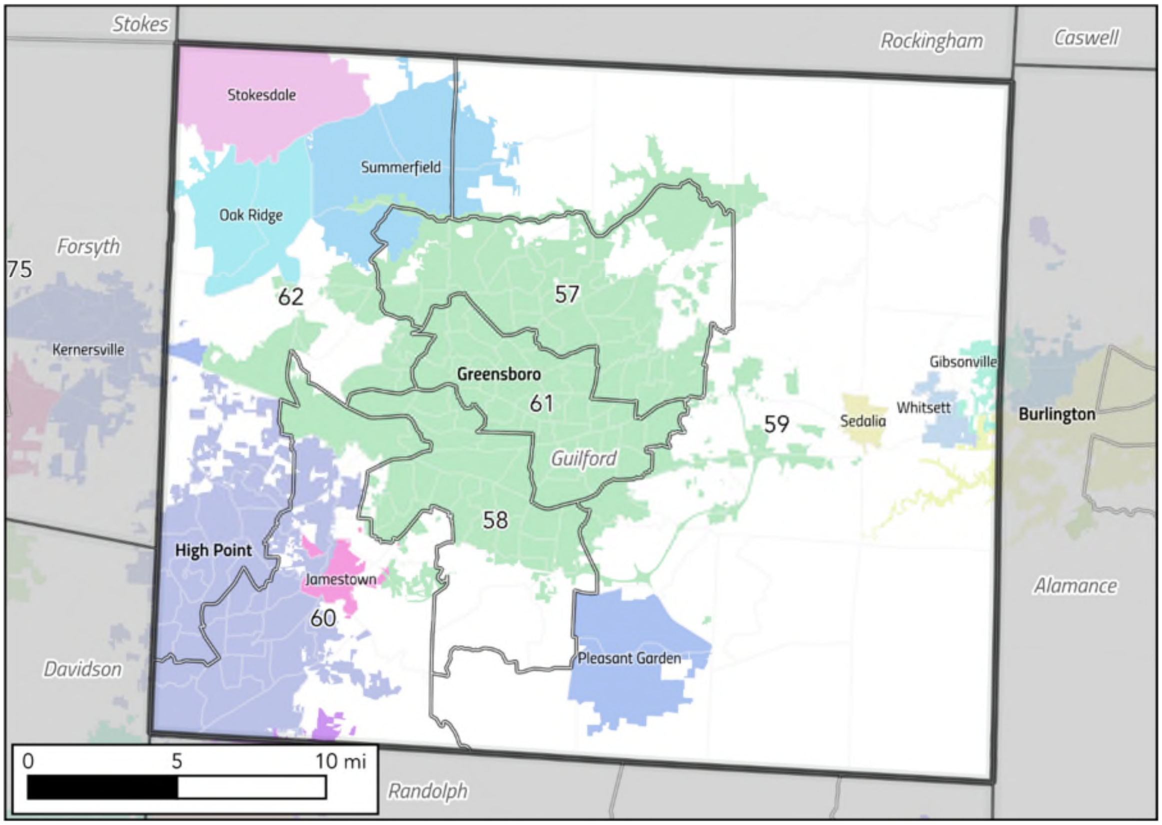
The enacted map packs Democratic voters into HDs 57, 58, 60, and 61. By studiously avoiding the Democratic leaning VTDs in the center of the county, HD-59 creates a reverse C shape that pieces together the southern and northern VTDs in an arrangement that creates district rated as R+2 by CPI, with a R+4,794 CCSC score. Meanwhile, HD-62 rests on the western edge of the county and includes pieces of both Greensboro and High Point, while avoiding the most Democratic areas of these cities. HD-62 is rated by the CPI as R+5 and has a CCSC score of R+11,030.

The enacted map splits Greensboro across all six districts and splits the city of High Point across two districts and Summerfield across three districts (*see* Map 36).

Map 35. VTD CCSC for the Guilford County Cluster



Map 36. Municipal Splits in the Guilford County Cluster



HDs 114, 115, and 116: Buncombe County Cluster

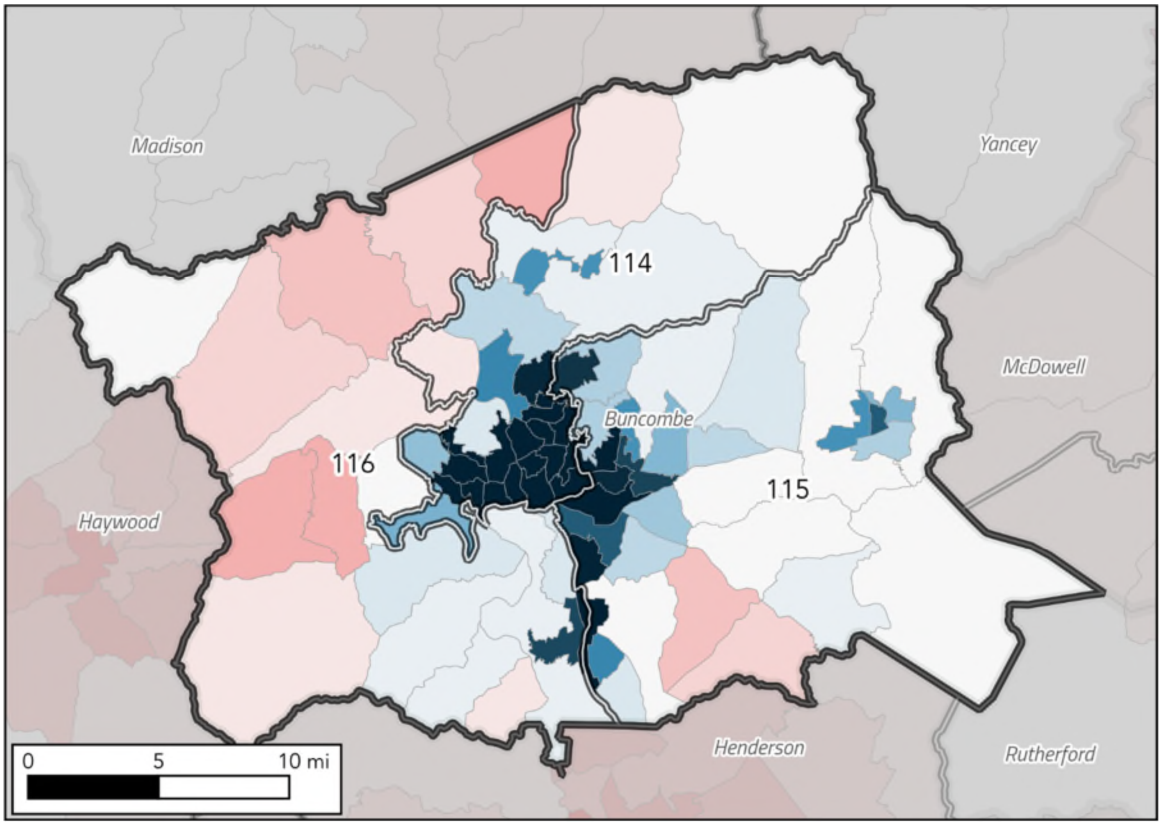
Buncombe County is located in Western North Carolina. It is anchored by Asheville, but also includes five other municipalities—Montreat, Biltmore Forest, Black Mountain, Woodfin, and Weaverville. Due to the *Stephenson* rule, Buncombe County is a single county cluster that must include three districts. Within the county, however, there were a number of choices the map-drawers had before them.

Buncombe is an overwhelmingly Democratic county and has been trending more Democratic each year. In 2020, 60.7% of the county's two-party vote share went to Joe Biden, the 10th highest in the state. Buncombe voters voted for the Democratic candidate in every county-wide contest in 2021 and Buncombe's county commission includes only one Republican.

In both the current map and the enacted map, Buncombe County includes HDs 114, 115, and 116. All three districts are currently represented by Democrats, with Susan Fisher in HD-114, John Ager in HD-115, and Brian Turner in HD-116. By shifting the current district lines where the districts meet in Asheville, however, the enacted map packs as many Democrats as possible into HD-114, while HD-115 stays relatively constant in terms of predicted vote share. The C-shaped HD-116 now includes most of the Republican-leaning VTDs in Buncombe, transforming it from a safely Democratic district into a district that leans towards the Republican Party (HD-116 is rated by CPI as R+3 and has a CCSC score of R+5,800).

The enacted map also places the pocket of overwhelmingly white voters of Biltmore Forest in the competitive HD-116, while the traditionally African American community of Shiloh to the east is left in HD-115. Soon after the maps were passed, all three Democratic incumbents announced that they would be retiring and not running for office in these newly drawn districts.

Map 37. VTD CCSC for the Buncombe County Cluster

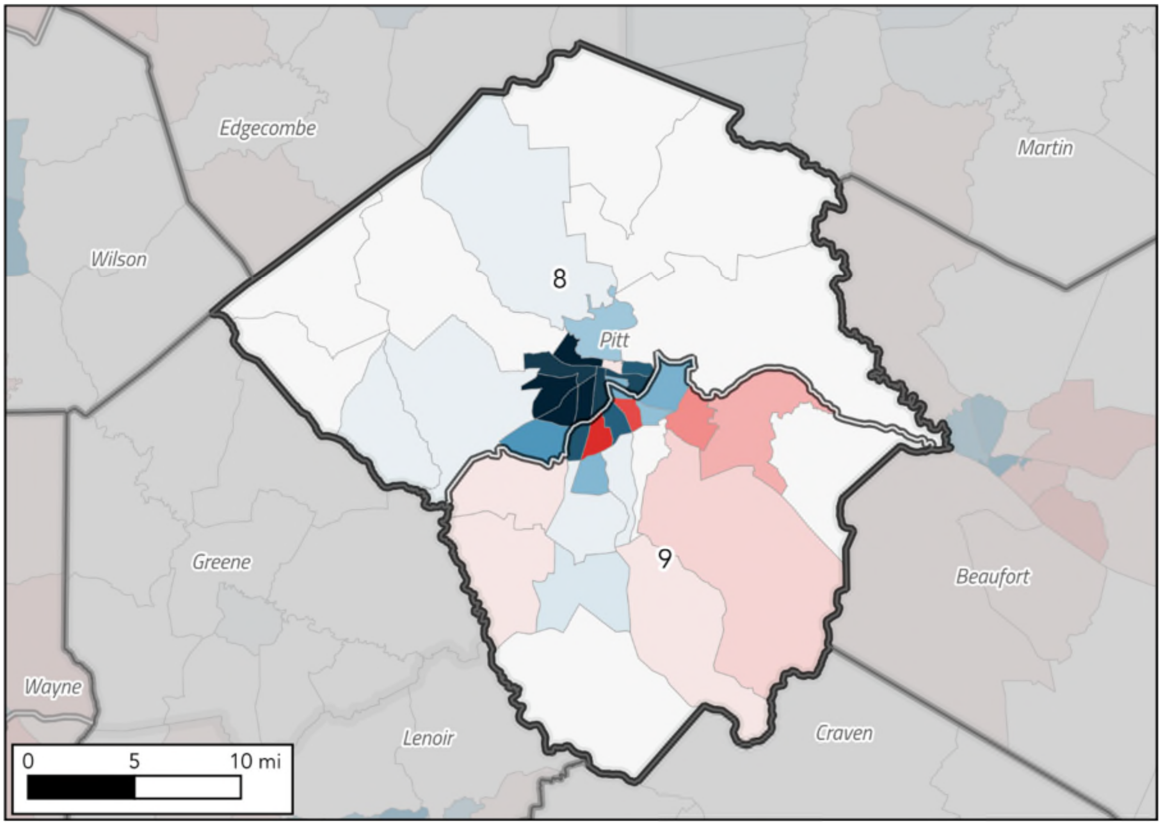


HDs 8 and 9: Pitt County Cluster

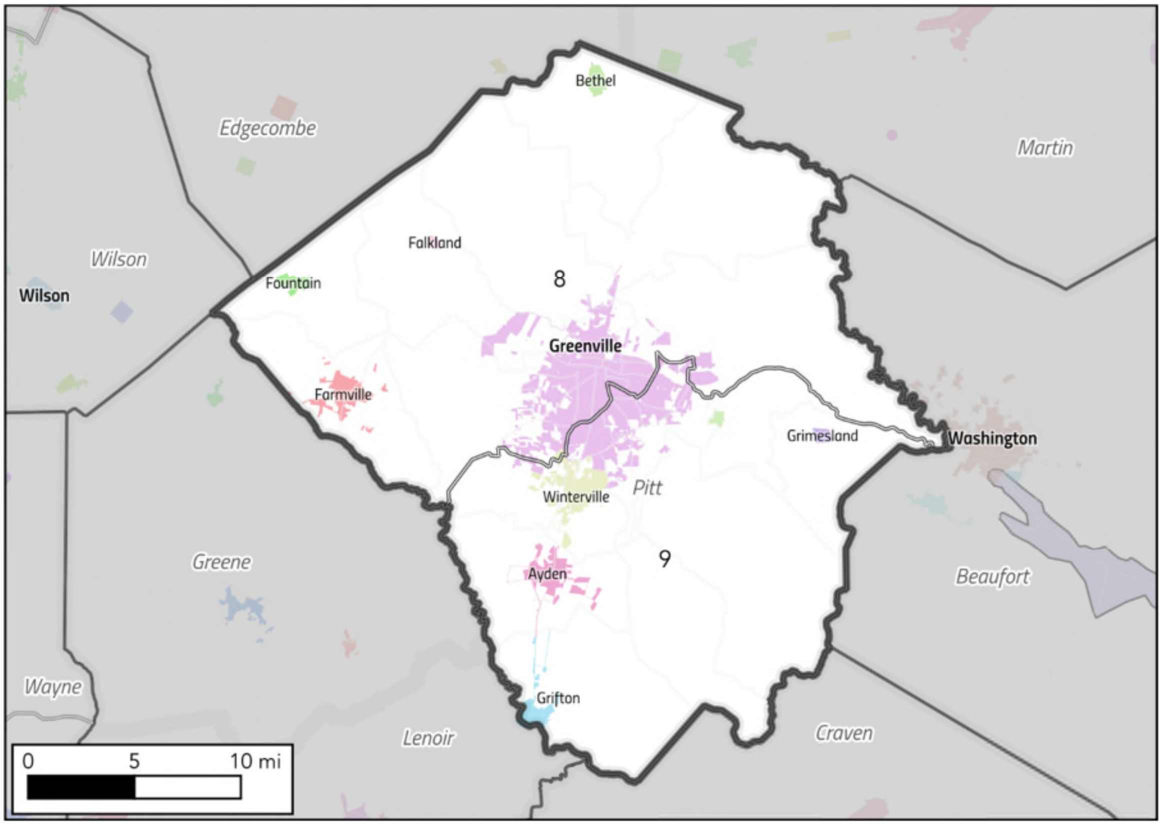
HD 8 and 9 are located in Pitt County, a county that gave 55% of its vote share to Joe Biden in the 2020 election, making it the 19th most Democratic county in the state according to this metric. The county is currently represented by two Democrats: Kandie Smith in HD-8 and Brian Farkas in HD-9.

By splitting Greenville at a particularly consequential location, the enacted map packs most Democrats in that city into HD-8 and fences them off from two Republican-leaning VTDs in HD-9. This particular division of Greenville makes HD-8 a much safer seat for Democrats and allows for a Republican-leaning district in Farkas' HD-9, which is rated by the CPI as R+3 and has a CCSC score of R+4,503. These district boundaries are difficult to explain with reference to communities of interest or natural geography. For example, students in East Carolina University's College of Health and Human Performance would take classes in HD-9, while their residence halls would be in HD-8. Similarly, as students walked from the ECU Hill District to Dowdy-Ficklen Stadium on Saturdays to watch the Pirates, they would be entering not only a sea of purple-clad football fans, but a different House district as well.

Map 38. VTD CCSC for the Pitt County Cluster



Map 39. Municipal Splits in the Pitt County Cluster

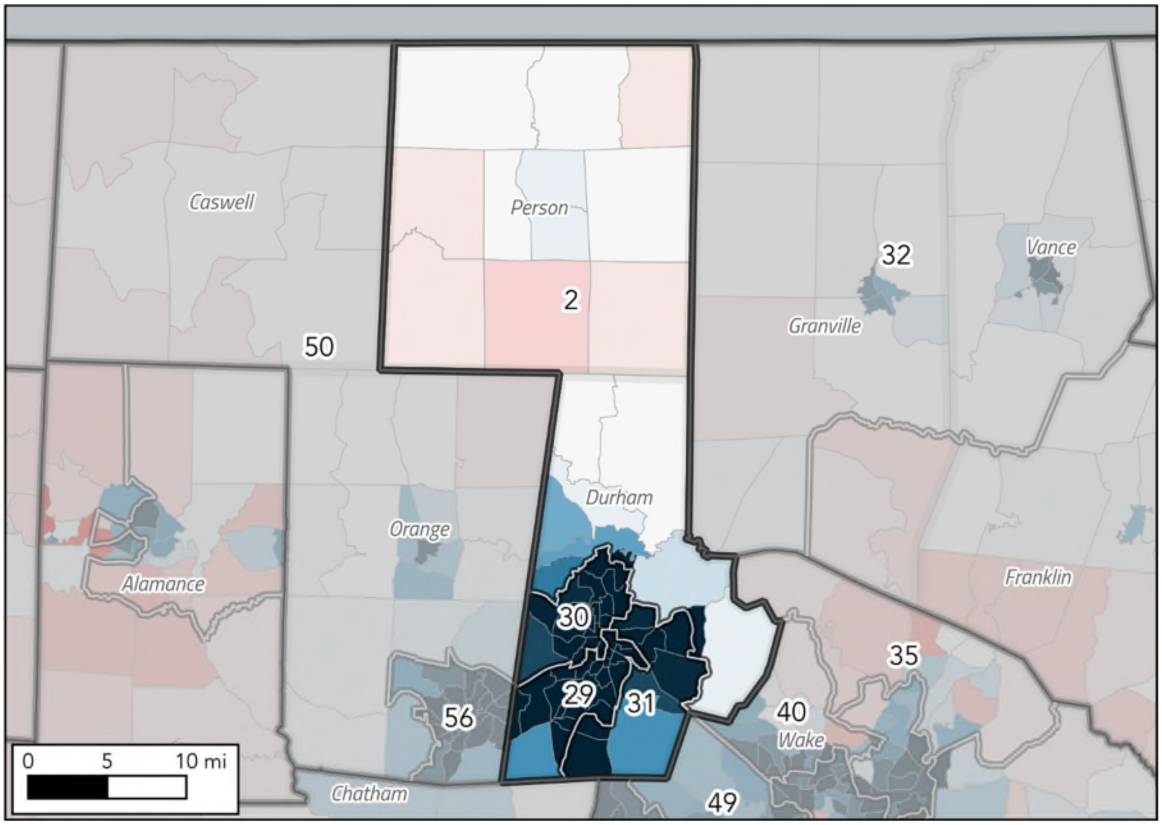


HDs 2, 29, 30, and 31: Durham and Person County Cluster

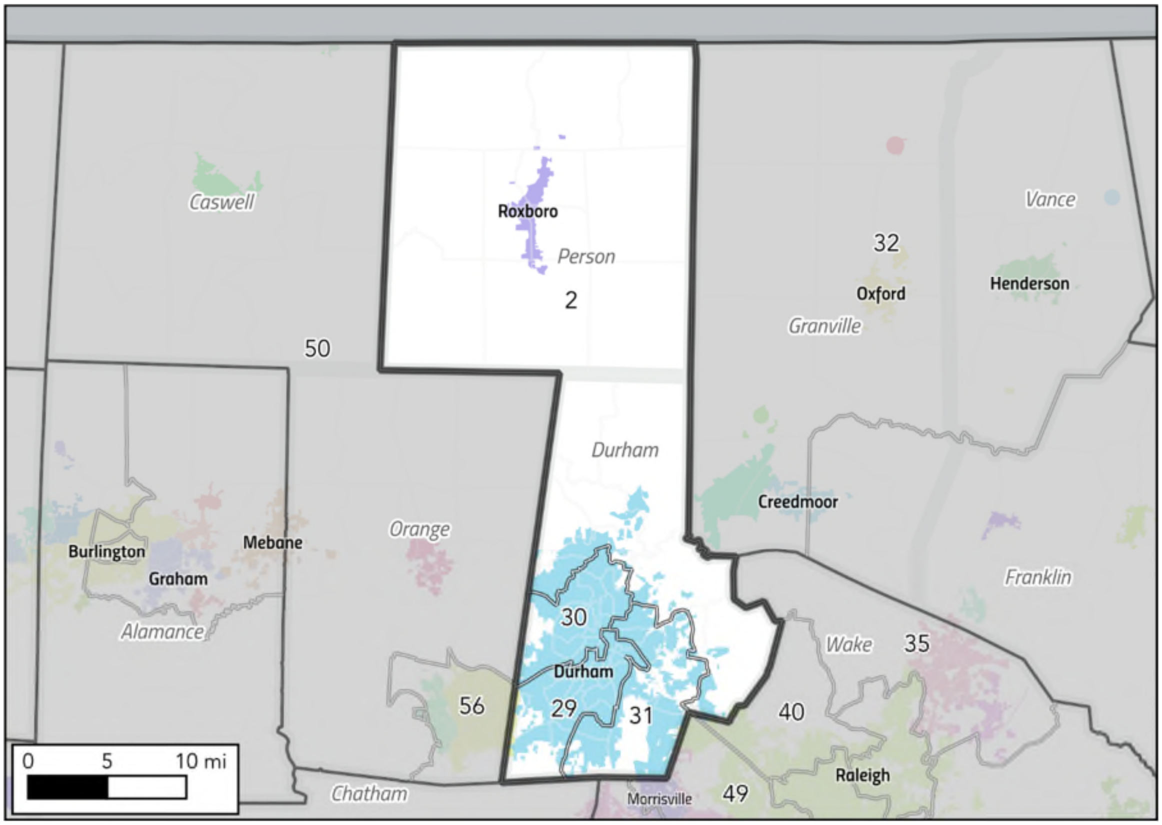
House districts 2, 29, 30, and 31 are located in a cluster with Durham and Person counties. While Person County leans towards the Republican Party, Durham County is the most Democratic county in the state, by almost any metric. Durham County gave 81.6% of its two-party vote share to Joe Biden in the 2020 election and voted overwhelmingly for Democratic candidates in every county-wide election.

The enacted map splits the City of Durham across all four districts but packs Democratic voters in HDs 29, 39, and 31; there is not a single Republican or competitive VTD in those districts. Meanwhile, HD-2 grabs all of the less Democratic and more competitive VTDs within Durham County, studiously avoiding the darkest blue VTDs in the northern end of the City of Durham. The result of these district boundaries that pack Democratic voters in the three districts in the south of Durham County is a claw-shaped appendage that allows HD-2 to be as competitive for the Republican Party as possible, giving the Republican incumbent a chance in this largely blue cluster.

Map 40. VTD CCSC for the Durham and Person County Cluster



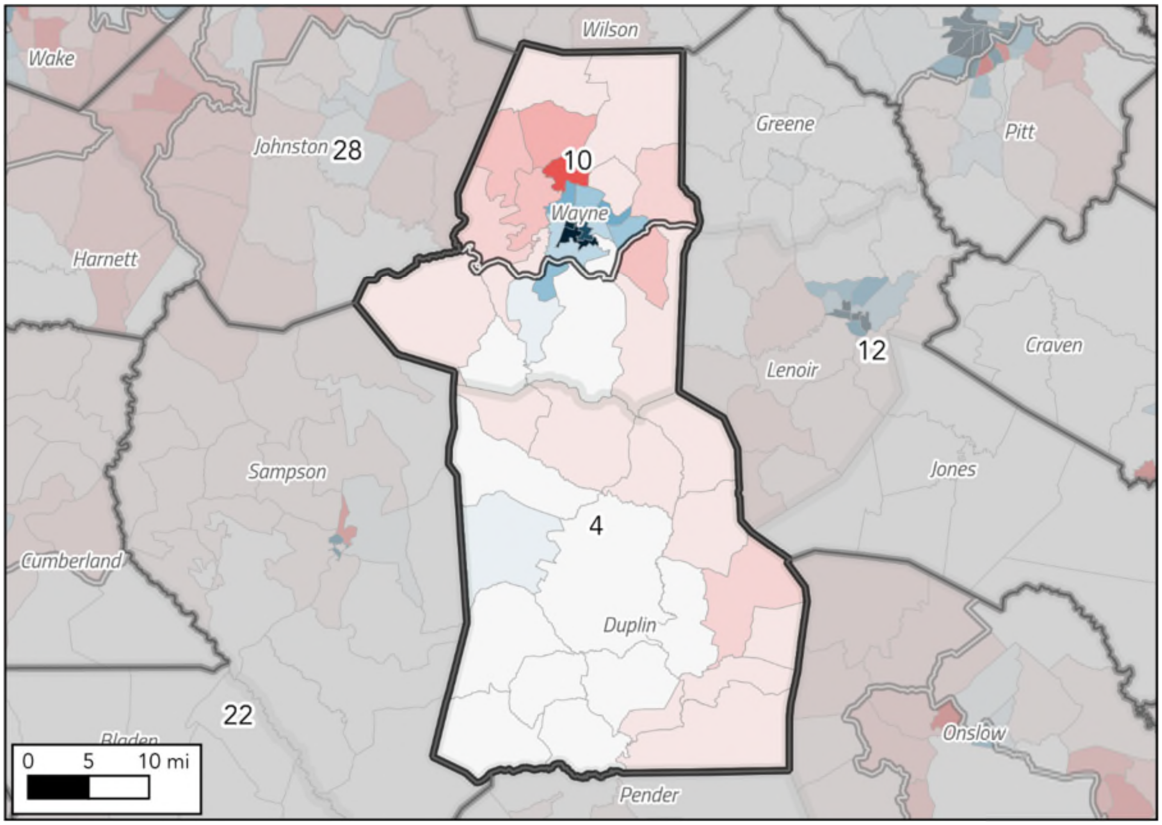
Map 41. Municipal Splits in the Durham and Person County Cluster



HDs 4 and 10: Duplin and Wayne County Cluster

House districts 4 and 10 are located in Duplin and Wayne counties, southeast of Wake County. The district boundary that runs through Wayne County ensures that there will be two Republican districts. HD-4 is rated R+8 by the CPI and advantages the Republican Party by 14,079 votes, according to the CCSC. HD-10 is rated R+3 by the CPI, with a R+4,951 CCSC advantage.

Map 42. VTD CCSC for the Duplin and Wayne County Cluster

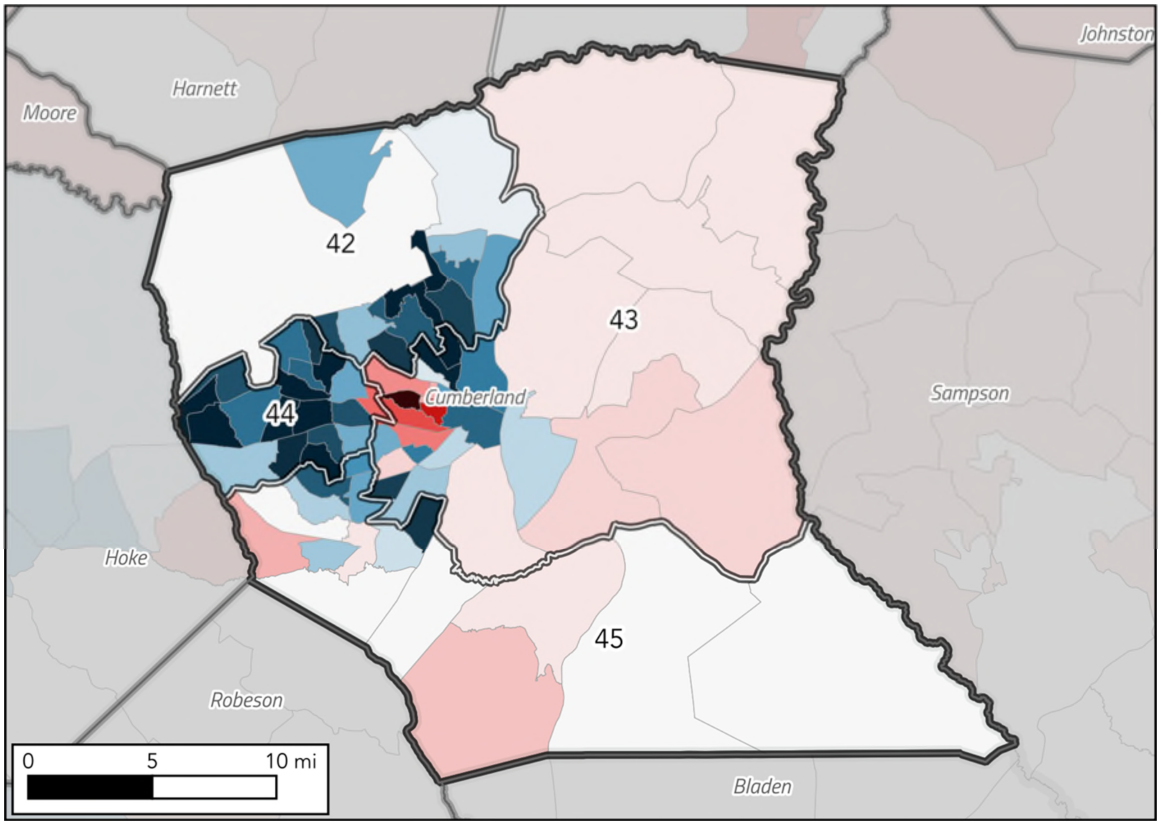


HDs 42, 43, 44, and 45: Cumberland County Cluster

Cumberland County is a heavily Democratic county, home to Fayetteville. Cumberland gave 58% of its two-party vote share to Joe Biden in 2020 and has not given the plurality of its votes for President to a Republican since 2004.

The enacted map creates two extremely competitive districts, HD-43 and HD-45 (with CCSC scores of D+1,334 and D+663, respectively) by splitting the Democratic-leaning City of Fayetteville into all four districts in the cluster. HD-43 picks up the most Republican VTDs in Fayetteville in a pattern that has partisan implications, making that district more competitive for first-term incumbent Republican Diane Wheatley. The district boundaries are also potentially confusing to voters. A citizen driving north on The All American Freeway would, in the span of about 3.5 miles, move from HD-43 to HD-44, then split the border between HD-43 and HD-44, then back into HD-44, form the border between HD-44 and HD-42, then move fully into HD-42. HD-45 includes the Republican and competitive VTDs on the south side of the county and moves into Fayetteville, but narrowly avoids the most Democratic-leaning VTDs in the city.

Map 43. VTD CCSC for the Cumberland County Cluster

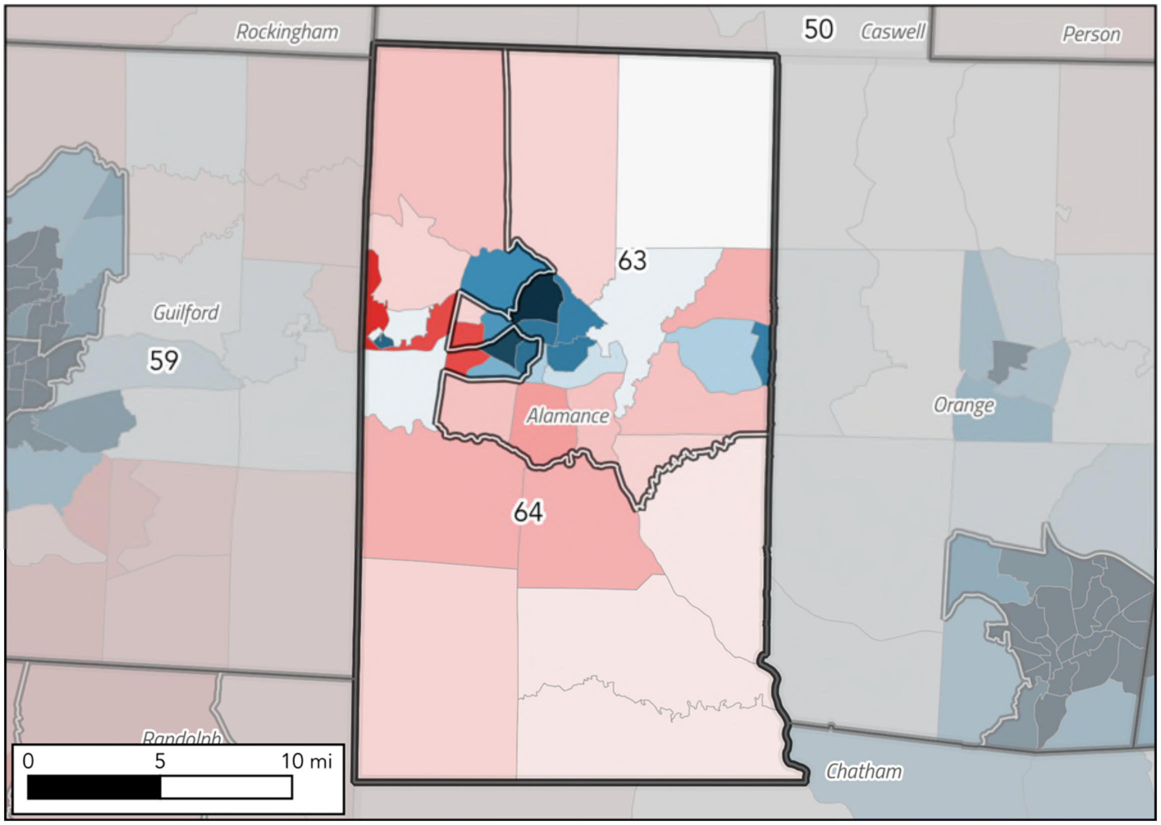


HDs 63 and 63: Alamance County Cluster

Alamance County is located between Guilford and Orange counties and includes the municipalities of Burlington, Graham, Mebane, Elon, Gibsonville, Green Level, Haw River, Ossipee, Swepsonville, and Alamance. The enacted map creates a heavily Republican HD-64 (R+8, R+13,572) and a competitive HD-63 (D+1, D+1,877) that could be challenging for the re-election of Democrat Ricky Hurtado, the only Latino legislator in North Carolina's General Assembly.

The enacted map takes a series of odd jogs around the City of Burlington in which three heavily Democratic VTDs are drawn into the heavily Republican HD-64, thus reducing the influence of those voters and leaving them walled off from HD-63 where they would be more likely to make a difference in the electoral outcome in a close district. This dovetail pattern does not follow municipal boundaries or other traditional communities of interest. At one point, the gap created between HD-63 and HD-64 is a mere three blocks wide.

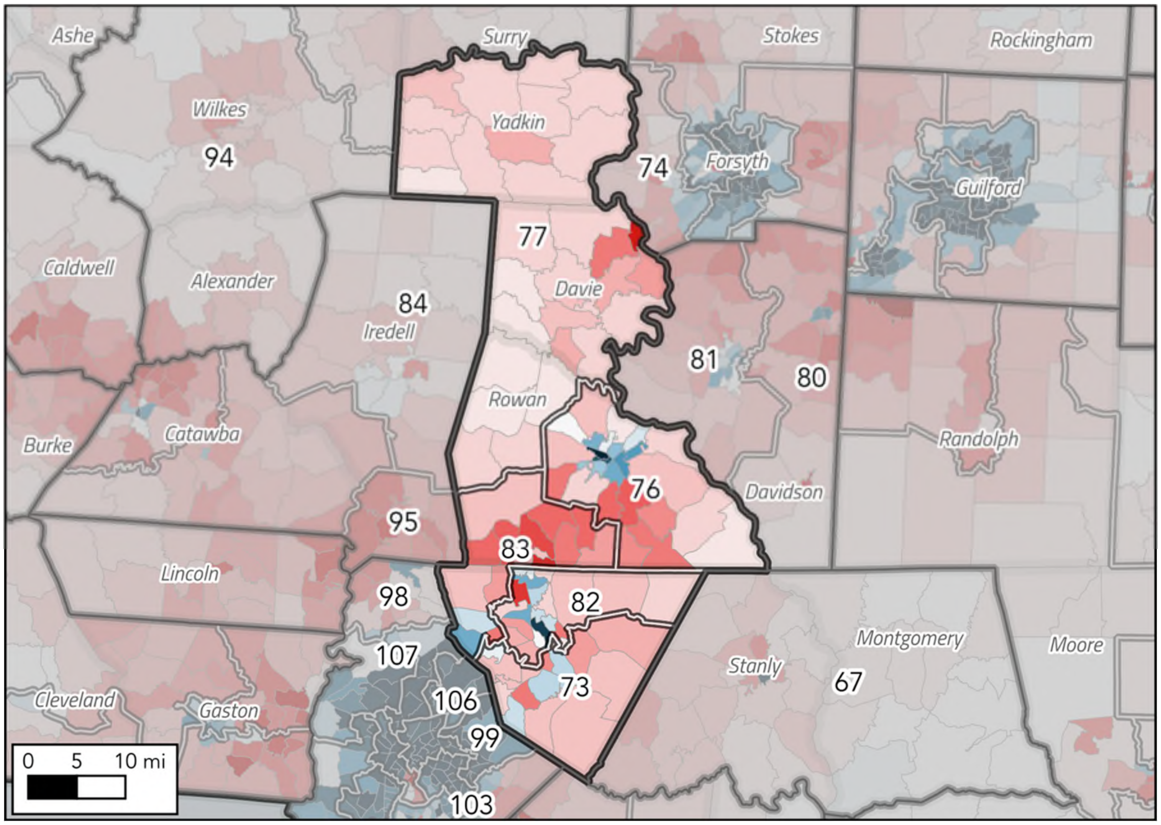
Map 44. VTD CCSC for the Alamance County Cluster



HDs 73, 76, 77, 82, and 83: Cabarrus, Davie, Rowan, and Yadkin County Cluster

This cluster is located northeast of Mecklenburg County. While the composition of these counties suggests that Republicans are likely to have an advantage in some of the potential districts in this cluster, the enacted map creates five Republican districts, ranging from a CPI of R+3 and CCSC score of R+5,578 to a CPI of R+25 and CCSC score of R+51,128. HD-82, which includes Concord and Kannapolis and is the most competitive district in the cluster as drawn, conspicuously excludes Democratic VTDs near the northeastern border of Mecklenburg County, which are placed in HDs 83 and 73.

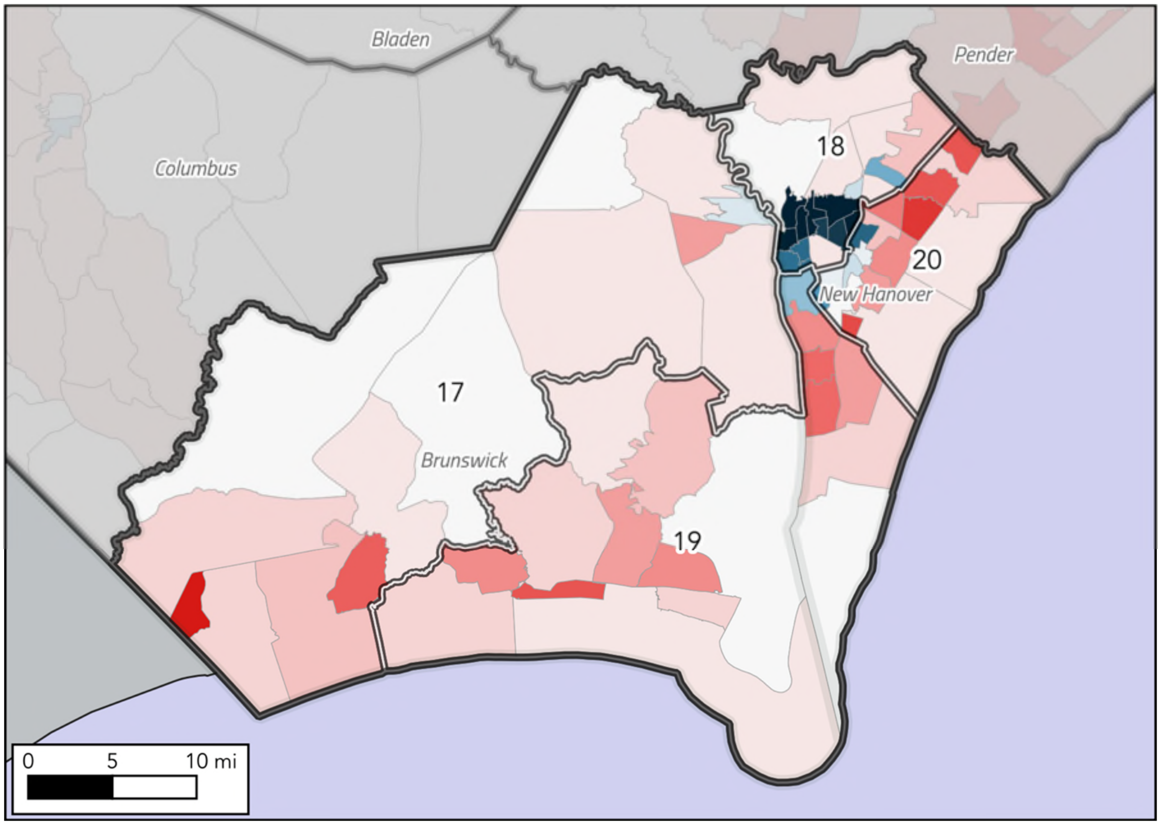
Map 45. VTD CCSC for the Cabarrus, Davie, Rowan, and Yadkin County Cluster



HDs 17, 18, 19, and 20: Brunswick and New Hanover County Cluster

The Brunswick-New Hanover cluster is located in eastern North Carolina and includes four House districts. Three of the four (HD-17, HD-19, and HD-20) lean towards the Republican Party, while HD-18 (D+11, D+20,338) packs Democratic voters in and around Wilmington, making the adjacent HD-20 (R+3, R+7,728) more competitive. The heavily Republican HD-19 also ensnares a Democratic-leaning VTD south of Wilmington, which keeps that VTD out of competitive HD-20.

Map 46. VTD CCSC for the Brunswick and New Hanover County Cluster



Conclusion

After analyzing the characteristics of all three maps as a whole, as well as the characteristics of each district in isolation, it is clear that the enacted maps will increase the number of Republicans in Congress and in the General Assembly, while decreasing the number of Democrats. Democratic voters in the vast majority of the congressional districts will have no chance at representation from a member of their own party and Republican voters in the congressional districts that pack Democrats will have no chance of representation from a member of their own party. Democratic voters are similarly disadvantaged in the Senate and House county clusters addressed above. This is not a result of natural packing or geographic clustering, but rather because the map-makers drew district lines in ways that, taken together, benefit the Republican Party. Not only do the enacted maps artificially create a substantial partisan advantage for which there is no apparent explanation other than gerrymandering, but the enacted maps also unnecessarily split communities of interest and will alter representational linkages in ways that, in some cases, have never been seen in North Carolina's history.

Christopher A. Cooper

Attachment A

December, 2021

Christopher A. Cooper

EDUCATION

Ph.D., University of Tennessee, Political Science (2002)

M.A., University of Tennessee, Political Science (1999)

B.A., Winthrop University, Political Science and Sociology (1997)

ACADEMIC POSITIONS

Madison Distinguished Professor (July 2019-Present)

Professor of Political Science and Public Affairs, Western Carolina University (2014-Present)

Associate Professor of Political Science and Public Affairs, Western Carolina University (2008-2014)

Associate Professor of Psychology (by Courtesy), Western Carolina University (2011-present)

Faculty Fellow, Institute for the Economy and the Future Western Carolina University (2002-2006)

Assistant Professor of Political Science and Public Affairs, Western Carolina University (2002-2008)

ADMINISTRATIVE POSITIONS

Director, Public Policy Institute, Western Carolina University (July 2008-July 2011; July 2021-present)

Department Head, Department of Political Science and Public Affairs, Western Carolina University (July 2012-July 2021; Interim from July 2011-June 2012)

Director, Master of Public Affairs (M.P.A.) Program, Western Carolina University (2005-2010)

INTERNATIONAL TEACHING

Guest Lecturer, Ludwigsburg University of Education, Ludwigsburg, Germany (May, 2018)

Guest Lecturer, Middelburg Center for Transatlantic Studies, Middelburg, the Netherlands (December, 2009; June 2012)

AWARDS

North Carolina Professor of the Year, Carnegie Foundation for the Advancement of Teaching (2013)

Board of Governors Teaching Award, WCU (2013)

University Scholar, WCU (2011)

Chancellor's Award for Engaged Teaching, WCU (2007)

Teaching-Research Award, WCU (2006)

Outstanding Achievement—Teaching, Service Learning Department (2005)

Oral Parks Award for the best faculty paper presented at the 2003 meeting of the North Carolina Political Science Association.

Artinian Professional Development Grant, Southern Political Science Association (2004; 2006)

Provost's Citation for Extraordinary Professional Promise, University of Tennessee (2002)

ADDITIONAL TRAINING

Social Network Analysis course through the Inter-university Consortium for Political and Social Research, Chapel Hill, NC (2010)

Spit Camp, Salimetrics, Inc, State College, PA (2010)

Deliberative Polling Institute, Stanford University (2008)

Hierarchical Linear Model course through the Inter-university Consortium for Political and Social Research, Amherst, MA (2005)

Summer Institute in Experimental Methods, Yale University (2003)

CATI and Ci3 training (2003)

Summer Institute in Political Psychology, Ohio State University (1999)

RESEARCH

BOOKS [2]

Cooper, Christopher A., and H. Gibbs Knotts. 2017. *The Resilience of Southern Identity*. Chapel Hill: University of North Carolina Press.
[Featured in the Durham Herald-Sun, Charleston City Paper, Statehouse Report (SC), Blue Ridge Public Radio (Asheville, NC), WFAE (Charlotte, NC), South Carolina Public Radio (Walter Edgar's Journal), WUNC (The State of Things), Georgia Public Radio (On Second Thought), Reviewed in the *Journal of Southern History*]

Cooper, Christopher A., and H. Gibbs Knotts, eds. 2008. *The New Politics of the Old North State*. Chapel Hill, NC: University of North Carolina Press.
[Featured in *Raleigh News and Observer*, Reviewed in *Journal of Southern History*, *North Carolina Historical Review*]

JOURNAL ARTICLES (PEER REVIEWED)
 # DENOTES STUDENT CO-AUTHOR

Cooper, Christopher A. "Innumeracy and State Legislative Salaries." *Public Opinion Quarterly*. 85(1): 147-160.

[Media coverage: Kate Elizabeth Queram; "Voters Have No Clue How Much State Lawmakers Make." Route Fifty. Sept 9, 2021. Jeremy Borden's Untold Story; Under the Dome Podcast (Sept 10); John Boye, "Paltry Pay for State Legislators is Embarrassing—and bad for Democracy." *Asheville Citizen Times*. December 4, 2021]

Cooper, Christopher A., Scott Huffmon, and H. Gibbs Knotts, and Seth McKee. 2021. "Heritage v. Hate: Assessing Opinions in the Debate Over Confederate Monuments and Memorials." *Social Science Quarterly*. 102(3): 1098-1110.

[Media coverage: This Week in South Carolina (SCETV)]

Cooper, Christopher A., M.V. Hood III, Scott Huffmon, Quintin Kidd, H. Gibbs Knotts, and Seth McKee. 2020. Switching Sides but Still Fighting the Civil War in Southern Politics. *Politics Groups, and Identities*.

Cooper, Christopher A. 2020. "Maybe They're Not So Different After All: The Uniform Effects of Personality on Job Satisfaction Among Government and Non-Government Workers." *Journal of Public and Nonprofit Affairs*. 6(1): 63-78.

Cooper, Christopher A. and John D. Gerlach. 2019. "Diversity Management in Action: Chief Diversity Officer Adoption in America's Cities." *State and Local Government Review*. 51(2): 113-121

Reinagel, Tyler, and Christopher A. Cooper. 2019. "Assessing the State of Mandatory Fees in America's Public Colleges and Universities: Causes and Consequences." *Social Science Quarterly* 101(2): 427-438.

Menickelli, Justin, Christopher A. Cooper, Chris Withnall, and Michael Wonnacott. 2019. "Analysis and Comparison of Lateral Head Impacts Using Various Golf Discs and a Hybrid III Head Form." *Sports Biomechanics*. 18(6).

Cooper, Christopher A., and H. Gibbs Knotts. 2019. "Do I Have to Take The GRE? Standardized Testing in MPA Admissions. *PS: Political Science & Politics*." 52(3): 470-475..

Cooper, Christopher A., H. Gibbs Knotts, and James Bourne#. 2018. "When the Personal Vote Isn't Enough: Voter Mobilization and the Failed Effort to Change the Form of Government in Columbia, SC." *Journal of Public and Nonprofit Affairs*. 4(3): 251-264.

Cooper, Christopher A. Cooper. "Not Just for Oprah Anymore: Incorporating Book Clubs Into Political Science Classes." *Journal of Political Science Education*.

Amira, Karyn, Christopher A. Cooper, H. Gibbs Knotts, and Claire Wofford. 2018. "The Southern Accent as a Heuristic in American Campaigns and Elections." *American Politics Research*. 46(6): 1065-1093.

[News Coverage: *Charleston Post and Courier* "Dang it! Politicians with Southern Accents seen as less honest, less intelligent."; PsyPost, "Candidates with a Southern accent are views more negatively—even in the South."; *US News and World Report; Charleston City Paper*]

Cooper, Christopher A. and Tyler Reinagel. 2017. "The Limits of Public Service Motivation: Confidence in Government Institutions Among Public Servants." *Administration and Society* 49(9): 1297-1317.

Chaffin, Latasha, Christopher A. Cooper, and H. Gibbs Knotts. 2017. "Furling the Flag: Explaining the 2015 Legislative Vote to Remove the Confederate Flag in South Carolina." *Politics and Policy*. 45: 944-963.

Cooper, Christopher A. Whittney Bridges#, and David M. McCord. 2017. "Personality and the Teaching of Public Administration: A Case for the Big Five." *Journal of Public Affairs Education*. 23: 677-690.

Cooper, Christopher A., H. Gibbs Knotts, and Jordan Ragusa. 2016. "The Constrained Governor? Gubernatorial Decision-Making on U.S. Senate Appointments." *Political Research Quarterly*. 69: 482-494.

Collins, Todd A., and Christopher A. Cooper. 2016. "The Case Salience Index, Public Opinion and Decision Making on the U.S. Supreme Court." *Justice System Journal*. 37: 232-245.

Menickelli, Justin, David Barney, Dan P. Grube, and Christopher A. Cooper. 2016. Disc Golf and Walking Benefits: A Pedometer Based Physical Activity Assessment. *International Journal of Physical Education, Fitness and Sports*. 5(1): 1-5.

Collins Todd A., and Christopher A. Cooper. 2015. "Making the Cases Real: Media Coverage of the U.S. Supreme Court." *Political Communication*. 32(1): 23-42

Cooper, Christopher A., and H. Gibbs Knotts. 2014. "Partisan Composition in Southern State Legislatures." *Southern Cultures*. 20: 75-89.
[Listed as One of *Southern Cultures*' "Top Ten Classroom Reads."]

Cooper, Christopher A., Dale Carpenter, Audrey Ranier,# and David M. McCord. 2014. "Personality and Job Satisfaction: Evidence from a Sample of Street Level Bureaucrats." *International Journal of Public Administration*. 37: 155-162.

Cooper, Christopher A., and H. Gibbs Knotts. 2013. "Overlapping Identities in the American South." *Social Science Journal*. 50: 6-12.

Cooper, Christopher A., Alan Socha# and Lauren Golden#. 2013. "The Big Five Personality Factors and Mass Politics." *Journal of Applied Social Psychology*. 43: 68-82.

Cooper, Christopher A., H. Gibbs Knotts, David M. McCord, and Andrew Johnson# 2012. "Taking Personality Seriously: The Five Factor Model and Public Administration." *American Review of Public Administration*. 43(3): 397-415.

Collins, Todd A., and Christopher A. Cooper. 2012. "Case Salience and Media Coverage of Supreme Court Decisions: Toward a New Measure." *Political Research Quarterly*. 65: 396-407.

Cooper, Christopher A., and H. Gibbs Knotts. 2012. "The Changing Relationship Between Race and Region: Opinions Towards Southerners from 1964 to 2008." *Social Science Quarterly*. 93: 58-75.

Cooper, Christopher A., H. Gibbs Knotts, and Katy Elders#.. 2011. "The Geography of Social Identity in Appalachia." *Southeastern Geographer*. 51: 457-472.

Bowen, Glenn, Carol Burton, Christopher A. Cooper, Laura Cruz, Anna McFadden, and Chesney Reich. 2011. "Listening to the Voices of Today's Undergraduates: Implications for Teaching and Learning. *Journal of the Scholarship of Teaching and Learning*.

Cooper, Christopher A., David McCord, and Alan Socha[#]. 2011. "Reassessing the College Sophomore Problem: The Case of Personality and Politics. *Journal of Psychology: Interdisciplinary and Applied.* 145:23-37.

Cooper, Christopher A., H. Gibbs Knotts and C. Don Livingston. 2010. "Regional Identity and Policy Choice." *Journal of Appalachian Studies.* 26-41.

Cooper, Christopher A., and H. Gibbs Knotts. 2010. "Rethinking the Boundaries of the South." *Southern Cultures.* Winter: 72-88.

[Featured in the *Washington Post.* Steve Hendrix, "D.C. Area and Dixie Drifting Farther and Farther Apart." Jan. 16 2011; Tracy Thompson, "Dixie is Dead." The Bitter Southerner Blog http://bittersoutherner.com/dixie-is-dead-tracy-thompson-defining-the-south/#.VW32TM7YmM4 ; Editorial, "Whistling Dixie, or Not." *Roanoke Times.* July 12, 2020]

Collins, Todd A., Christopher A. Cooper, and H. Gibbs Knotts. 2010. "Scholarly Productivity in Non-PhD Departments." *PS: Political Science and Politics.* 43: 509-514.

Socha, Alan[#], Christopher A. Cooper, and David M. McCord. 2010. "Confirmatory Factor Analysis of the M5-50: An Implementation of the IPIP Item Set." *Psychological Assessment.* 22: 43-49.

Cooper, Christopher A., and H. Gibbs Knotts. 2010. "Declining Dixie: Regional Identification in the Modern American South." *Social Forces.* 88(3) 1083-1102.

Cooper, Christopher A., and H. Gibbs Knotts. 2009. "Public Opinion on Land Use Planning." *Popular Government.* Fall: 24-28.

Cooper, Christopher A., and Martin Johnson. 2009. "Representative Reporters? Journalists and Media Bias in Context." *Social Science Quarterly.* 90: 387-406.

Cooper, Christopher A., H. Gibbs Knotts, and Moshe Haspel. 2009. "The Content of Political Participation: Letters to the Editor and the People Who Write Them." *PS: Political Science and Politics.* 42: 131-137.

[Featured in the *Raleigh News and Observer.* Ted Vaden. "Do Letters Reflect Regular Opinion?" October 8, 2006"]

Cooper, Christopher A., Moshe Haspel and H. Gibbs Knotts. 2009. "The Importance of Voterfiles for State Politics Research." *State Politics and Policy Quarterly.* 9: 102-121.

Cooper, Christopher A. 2008. "Reassessing Conference Goals and Outcomes: A Defense of Presenting Identical or Similar Papers at Multiple Conferences." *PS: Political Science and Politics.* 41: 293-296.

Brennan, Kathleen, and Christopher A. Cooper. 2008. "Rural Mountain Natives, In-Migrants, and the Cultural Divide." *Social Science Journal.* 45: 279-295.

Cooper, Christopher A., H. Gibbs Knotts, and Kathleen Brennan. 2008. "Trust in Government, Citizen Competence, and Public Opinion on Zoning." *Public Administration Review.* 68(3): 459-468.

Cooper, Christopher A., Anthony J. Nownes, and Martin Johnson. 2007. "Interest Groups and Journalists in the States." *State Politics and Policy Quarterly.* 7: 39-53.

[Reprinted in John R. Baker, ed. 2009. *The Lanahan Readings in State and Local Government: Diversity, Innovation, and Rejuvenation, 2nd Edition*. Lanahan Publishing.]

Cooper, Christopher A., and H. Gibbs Knotts. 2006. "Region, Race and Support for the South Carolina Confederate Flag." *Social Science Quarterly*. 87: 142-154.

Cooper, Christopher A., and Lilliard E. Richardson. 2006. "Institutions and Representational Roles in U.S. State Legislatures." *State Politics and Policy Quarterly*. 6: 174-194.

Richardson, Lilliard E., and Christopher A. Cooper. 2006. "E-mail Communication and Target Groups in US State Legislatures." *Policy Studies Journal*. 34: 113-129.

Cooper, Christopher A., Anthony J. Nownes, and Steven Roberts. 2005. "Perceptions of Power: Interest Groups in Local Politics." *State and Local Government Review*. 37: 206-216.
[Reprinted in John R. Baker, ed. 2009. *The Lanahan Readings in State and Local Government: Diversity, Innovation, and Rejuvenation, 2nd Edition*. Lanahan Publishing.]

Cooper, Christopher A., and Anthony J. Nownes. 2004. "Money Well Spent: An Experimental Investigation of the Effects of Advertorials on Citizen Opinion." *American Politics Research*. 32: 546-569.

Richardson, Lilliard E., Brian E. Russell, and Christopher A. Cooper. 2004. "Legislative Representation in a Single-Member Versus Multi-Member District System: The Arizona State Legislature." *Political Research Quarterly*. 57: 337-344.

Cooper, Christopher A., and H. Gibbs Knotts. 2004. "Defining Dixie: A State Level Measure of the Modern Political South." *American Review of Politics*. 25: 25-40 (Special Issues on Southern Politics).

Cooper, Christopher A. 2004. "Internet Use in the State Legislature." *Social Science Computer Review*. 22: 347-354.

Cooper, Christopher A., and Anthony J. Nownes. 2003. "Citizen Groups in Big City Politics." *State and Local Government Review*. 35: 102-111.

Cooper, Christopher A. 2002. "Media Tactics in the State Legislature." *State Politics and Policy Quarterly*. 2: 353-371.

Cooper, Christopher A. 2002. "E-Mail in the State Legislature: Evidence From Three States." *State and Local Government Review*. 34: 127-132.

Cooper, Christopher A. and Marc Schwerdt. 2001. "Depictions of Public Service in Children's Literature: Revisiting an Understudied Aspect of Political Socialization." *Social Science Quarterly*. 82: 614-630.

BOOK CHAPTERS

Cooper, Christopher A., and H. Gibbs Knotts. 2022. "Reliably Purple: The 2020 Presidential Election in North Carolina." In David Schultz and Rafael Jacob, eds. *Presidential Swing States, Third Edition*. Lexington Press.

Cooper, Christopher A., and H. Gibbs Knots. 2018. "North Carolina: Still Swingin' in the South." In David Schultz, and Rafael Jacob, eds. *Presidential Swing States, Second Edition*. Lexington Press.

Cooper, Christopher A., and H. Gibbs Knotts. 2015. "The Bluest Red State in America: North Carolina as a Swing State." In David Schultz and Stacy Hunter Hecht, eds. *Presidential Swing States: Why Only Ten States Matter*. Lexington Press.
[Recommended by *Choice*]

Cooper, Christopher A., and Mandi Bates. "Entertainment Media and Political Knowledge: Do People Get Any Truth out of Truthiness?" 2008. In Joseph Foy, ed., *Laughing Matters: Humor in American Politics*. University Press of Kentucky. Paperback edition published in 2010.
[Reviewed in the *New York Post* Aug. 2, 2008]

Cooper, Christopher A. "Multimember Districts and State Legislatures." 2008. In Bruce Cain, Todd Donovan, and Caroline Tolbert, eds. *Electoral Reform in the United States*. Brookings Institution Press.

Cooper, Christopher A. 2008. "The People's Branch: The North Carolina State Legislature." In *The New Politics of the Old North State*, ed. Christopher A. Cooper, and H. Gibbs Knotts. Chapel Hill: UNC Press.

Cooper, Christopher A, and H. Gibbs Knotts. 2008. "Traditionalism and Progressivism in the Old North State." In *The New Politics of the Old North State*, ed. Christopher A. Cooper, and H. Gibbs Knotts. Chapel Hill: UNC Press.

Cooper, Christopher A, and H. Gibbs Knotts. 2008. "Rethinking Progressivism and Governance in the Old North State." In *The New Politics of the Old North State*, ed. Christopher A. Cooper, and H. Gibbs Knotts. Chapel Hill: UNC Press.

Cooper, Christopher A., and H. Gibbs Knotts. 2004. "Packaging the Governor: Television Advertising in the 2000 Elections." In *Lights, Camera, Campaign! Media, Politics, and Political Advertising*, ed. David Schultz. Peter Lang Publishing: 101-120.

Cooper, Christopher A. and Anthony J. Nownes. 2002. "Textiles, Traditions and Scandal: Money in South Carolina Politics." In *Laboratories of Democracy: Money in State Politics*, ed. David Schultz. Durham: Carolina Academic Press: 133-154.

ENCYCLOPEDIA AND HANDBOOK ENTRIES

Cooper, Christopher A. 2008. "Multi-Member Districts." In the *Political Encyclopedia of the US States and Regions*, edited by Donald P. Haider-Markel, Michael Card, Keith Gaddie, Gary Moncrief, and Kenneth Palmer. Washington DC: CQ Press.

Cooper, Christopher A. 2008. "State Senator." In the *Encyclopedia of American Government and Civics*, edited by Michael A. Genovese and Lori Cox Han. New York; Facts on File.

Cooper, Christopher A., and Brian Noland. 2004. "Lobbying the Executive Branch." In *Research Guide to U.S. and International Interest Groups*, ed. Clive Thomas. Westport, CT: Praeger Press: 176-178.

EXPERT & TECHNICAL REPORTS

Cooper, Christopher, Blake Esselstyn, Gregory Herschlag, Jonathan Mattingly, and Rebecca Tippett. 2021. NC General Assembly County Clusterings from the 2020 Census. August 17. https://sites.duke.edu/quantifyinggerrymandering/files/2021/08/countyClusters2020.pdf

Cooper, Christopher, Blake Esselstyn, Gregory Herschlag, Jonathan Mattingly, and Rebecca Tippett. 2021. Legislative Clustering in North Carolina: Looking Towards the 2020 Census. July 16. https://sites.duke.edu/quantifyinggerrymandering/files/2021/07/Legislative-County-Clustering-in-North-Carolina.pdf

Cooper, Christopher A. 2020. Declaration of Christopher A. Coper, PhD. Submitted in National Urban League v. DeJoy. Case No. 1:20-cv-023971-GLR. September 2020.

Cooper, Christopher A. 2019. Rebuttal Report of Christopher A. Cooper, PhD. Submitted in Common Cause V. Lewis. 18: VCS 014001. June 7, 2019.

Cooper, Christopher A. 2019. Expert Report of Christopher A. Cooper, PhD. Submitted in Common Cause V. Lewis. 18: VCS 014001. April 8, 2019.

Cooper, Christopher A. *Survey Results for the Opt-In Process*. 2014. *Opt-In Survey Results*. Southwestern Regional Planning Commission/Appalachian Regional Commission.

Brennan, Kathleen, Christopher A. Cooper, and Inhyuck "Steve" Ha. 2014. *Regional Outlook Report, 2014*. Western Carolina University.

Cooper, Christopher A., H. Gibbs Knotts and Billy Hutchings#. 2010. *Public Opinion on the Town Square Property*. Report Prepared for the Town of Black Mountain based on original survey and focus group data.

Cooper, Christopher A., and Thomas Jones#. 2010. *Yancey County Schools Health Assessment Report*. Report Prepared for the Yancey County School District based on original survey data.

Cooper, Christopher A. 2008. *Citizen Satisfaction in Buncombe County, NC*. Report Prepared for Buncombe County based on original survey data.

Brennan, Kathleen, Christopher A. Cooper, and Inhyuck "Steve" Ha. 2008. *Regional Outlook Report, 2008*. Institute for the Economy and the Future, Western Carolina University.

Cooper, Christopher A., and Sarah Kehrberg#. 2007. *Yancey County Schools Health Assessment Report*. Report Prepared for the Yancey County School District based on original survey data.

Cooper, Christopher A., and Alison Melnikova#. 2006. *Citizen Satisfaction in Sylva, NC*. Report Prepared for the Town of Sylva based on original survey data.

Ha, Inhyuck, Kathleen Brennan, Christopher Cooper, Chester Pankowski, and Jay Denton. 2005. *The Impact of Western Carolina University on the Regional Economy*. Center for Regional Development, Western Carolina University.

Brennan, Kathleen, Christopher A. Cooper, and Inhyuck Ha. 2004. *Regional Outlook Report, 2004*. Center for Regional Development, Western Carolina University.

PUBLIC-FACING WRITING (OP-EDS, MAGAZINE PIECES, BLOGS, ETC.) [+ REGULAR CONTRIBUTIONS TO OLDNORTHSTATEPOLITICS.BLOGSPOT.COM)

Cooper, Christopher A. "Want Your vote to Count? Don't Ignore 2021. *Asheville Citizen Times.* October 17, 2021

Cooper, Christopher A. "State Legislators Make Big Decisions. So Why do They Get Tiny Paychecks? New Research Uncovered One Surprising Reason" The Monkey Cage/A Washington Post Blog. September 8, 2021. https://www.washingtonpost.com/politics/2021/09/08/state-legislators-make-big-decisions-so-why-do-they-get-tiny-paychecks/

Cooper, Christopher A. "Getting What We Pay For." *The Assembly* February 23, 2021. https://www.theassemblync.com/short-form/getting-what-we-pay-for/

Cooper, Christopher A. "Georgia, North Carolina, and the Illusion of 'Turning Blue.'" *Asheville Citizen Times.* February 7, 2021.

Cooper, Christopher A. "Eliminate Literacy Test from the NC Constitution." *Asheville Citizen Times.* January 10, 2021. Also appeared in the *Raleigh News and Observer.* January 17, 2021.

Cooper, Christopher A. "Election 2020 in Summary: Different Inputs, Similar Outputs." *Asheville Citizen Times.* November 23, 2020.

Cooper, Christopher A. "Absentee-by-Mail Balloting in North Carolina: The Mouse that Roars." Asheville Citizen Times. October 23, 2020.

Cooper, Christopher A. "NC Voters: Don't Ignore the Bottom of the Ballot." *Asheville Citizen Times.* September 29, 2020.

Cooper, Christopher A. "Keeping the Youth Vote Alive in a Time of Uncertainty." *Asheville Citizen Times.* September 7, 2020.

Cooper, Christopher. North Carolina's close Senate race is a puzzle in a purple state. *London School of Economics US Centre's American Politics and Policy Blog.* August 25, 2020. https://blogs.lse.ac.uk/usappblog/2020/10/21/north-carolinas-close-senate-race-is-a-puzzle-in-a-purple-state/

Cooper, Christopher. Why North Carolina may be one of the most important states in the 2020 election. *London School of Economics US Centre's American Politics and Policy Blog.* August 25, 2020. https://bit.ly/31qCgMi

Cooper, Christopher A. "In the World of North Carolina Politics and Beyond, What's Old is New Again." *Asheville Citizen Times.* August 8, 2020.

Cooper, Christopher A., M.V. Hood III, Scott Huffmon, Quintin Kidd, H. Gibbs Knotts, and Seth McKee. Southern Democrats' Split with Republicans over Confederate Symbols is More Recent Than You Might Think. *London School of Economics US Centre's American Politics and Policy Blog.* July 10, 2020.

Cooper, Christopher A. "The NC-11 Runoff: A Sneak Preview of Elections in the Time of Covid-19." *Asheville Citizen Times.* June 20, 2020.

Cooper, Christopher A. "Mark Meadows' Departure Has Opened the Door for a Unique Republican Runoff Election in North Carolina's 11th Congressional District." *London School of Economics US Centre's American Politics and Policy Blog*. June 19, 2020.

Cooper, Christopher A. "Let's end the Political Limbo in NCs 11th District." *Asheville Citizen Times*. May 15, 2020.

Cooper, Christopher A. "A Path to Victory in Bernie's Appalachia." *Asheville Citizen Times*. April 15, 2020.

Cooper, Christopher A. "Putting Runoff Elections in Context." *Asheville Citizen Times*. March 17, 2020.

Cooper, Christopher A. "Taking a Look at Those Online Candidate Quizzes Before Tuesday's Primary." *Asheville Citizen Times*. March 2, 2020.

Cooper, Christopher A. "The Rise of the Unaffiliated Voter." *Asheville Citizen Times*. February 15, 2020.

Cooper, Christopher A. "The Field is Clear: Learning From Candidate Filings in NC." *Asheville Citizen Times*. January 21, 2020.

Cooper, Christopher A. "Looking at 'Swing Counties' in the Swing State of North Carolina" *Asheville Citizen Times*. December, 15 2019.

Cooper, Christopher A. "Want Your Vote for President to Count? You Live in the Right Place" *Asheville Citizen Times*. November 9, 2019.

Cooper, Christopher A. "Don't Put the Cart Before the Horse; We've Got an Election Before 2020" *Asheville Citizen Times*. October 20, 2019.

Cooper, Christopher A. "Stacking the farm-team of future politicians with more female representation." Carolina Commentary. October 9, 2019.

Cooper, Christopher A. "Wisconsin Republicans' Corrupt Power Grab Echoes Republican Efforts in North Carolina in 2016." *NBC.com.* December 15, 2018.

Cooper, Christopher A. "Looking Down Ballot" *Asheville Citizen Times*. December 16, 2018.

Cooper, Christopher A. "And Now for Something Completely Different: Good News from the 2018 Elections." *Asheville Citizen Times*. November 19, 2018.

Cooper, Christopher A. "Taking Stock of What We Know Ahead of November 6." *Asheville Citizen Times*. October 20, 2018.

Cooper, Christopher A. "Contextualizing Constitutional Amendments." *Asheville Citizen Times.* September 17, 2018.

Cooper, Christopher A. "November is Coming: What We know—and don't—about How North Carolina Voters Will Lean. *Asheville Citizen Times*. August 26, 2018.

Cooper, Christopher A. "Lessons from Charlotte and the RNC." *Asheville Citizen Times*. July 23, 2018.

Christopher A. Cooper, H. Gibbs Knotts, and Jordan Ragusa. "When appointing a US Senator, governors act responsibly." *Raleigh News and Observer*. June 27, 2018.

Christopher A. Cooper. "The Voter ID Conversation I Wish We Were Having." *Asheville Citizen Times*. June 17, 2018.

Christopher A. Cooper. "North Carolina's Leaving the (Political) Party Behind." *Asheville Citizen Times*. May 20, 2018.

Christopher A. Cooper. "It's Hard to Have Democracy When Voters Have No Choice." *Asheville Citizen Times*. April 22, 2018.

Christopher A. Cooper. "He Zigs, He Zags." *Asheville Citizen Times*. August 2017.

Cooper, Christopher A. 2017. "Prospects for Democrats in the West." Carolina Commentary. August, 2017.

Cooper, Christopher A. "We Need More Women in Office." Asheville Citizen Times. February 22, 2017.

Cooper, Christopher A. "It's Time to Look Down Ballot for Real Impact." Asheville Citizen Times. December 12, 2017.

Cooper, Christopher A. "Provisional Ballots are new Hanging Chads." Asheville Citizen Times. November 18, 2016.

Cooper, Christopher A. "Trump Taught us That Rural Vote Matters." *Asheville Citizen Times*. November 14, 2016.

Cooper, Christopher A. "Election Almost Over; What Have We Learned?" *Asheville Citizen Times*. November 7, 2016.

Cooper, Christopher A, H. Gibbs Knotts, and Jordan Ragusa. "Governors Tend to Appoint Senators who most Resemble Voters in the State, Rather than Ideologues." London School of Economics United States Politics and Policy blog. October 8, 2016.

Cooper, Christopher A. "What to Take Away From Debate Season" Asheville Citizen Times. October 7, 2016.

Cooper, Christopher A. "What Can We Expect in the NC General Assembly Races? Asheville Citizen Times. September 9, 2016.

Cooper, Christopher A. "The Silver Lining May be Hard to See, But It's There." Asheville Citizen Times. August 12, 2016.

Cooper, Christopher A. "Still Swinging: NC Politics in National Context." Asheville Citizen Times. July 8, 2016.

Cooper, Christopher A. "The 2016 Primaries Are Over. What Did We Learn?" Asheville Citizen Times. June 10, 2016.

Cooper, Christopher A. "The Boogeymen in American Politics." Asheville Citizen Times. May 6, 2-16.

Cooper, Christopher A. "Examining the Relationship Between Political Partisanship and HB2." *Asheville Citizen Times*. April 17, 2016.

Cooper, Christopher A. "What Lessons from the 2016 Primary." *Smoky Mountain News*. March 30, 2016.

Cooper, Christopher A. "Thomas Jefferson, Donald Trump, and the Continued Relevance of Political Parties." *Asheville Citizen Times*. March 13, 2016.

Cooper, Christopher A. "Where's the Competition?" *Asheville Citizen Times*. February 14, 2016.

Cooper, Christopher A. "Winning Day is A-coming." *Asheville Citizen Times*. January 10, 2016.

Cooper, Christopher A. And H. Gibbs Knotts "The Confederate Flag Flap: Rapid Policy Change? Yes; Rapid Shift in Public Opinion? No. *Smoky Mountain News*. July 22, 2015.

Cooper, Christopher A. 2015. "Ignore the Polls; Don't Ignore the Endorsements." *Asheville Citizen Times*. June 26, 2015.

Cooper, Christopher A. "What Would the Students Do? Simulating the Work of the General Assembly in Cullowhee." *Asheville Citizen* Times. May 11, 2015.

Cooper, Christopher A. "North Carolina's Close Senate Race is a Puzzling Election in a Purple State." *London School of Economics and Politics Science American Politics Blog* October 21, 2014.

Cooper, Christopher A. "A Good News Story About the Election." *Asheville Citizen Times*. October 19, 2014.

Cooper, Christopher A. "Legislative Salary Increase Proposal Deserves a Chance to be Debated." *Asheville Citizen Times*. June 7, 2014.

Cooper, Christopher A. "Western North Carolina: Where Conservative Democrats Rule." *Asheville Citizen Times*. March 31, 2014.

Cooper, Christopher A., and H. Gibbs Knotts. "NC Counties Shift to Red." *Raleigh News and Observer*. March 14, 2013.

Cooper, Christopher A., and H. Gibbs Knotts. "Important Lessons from the 2012 Election." *Asheville Citizen Times*. November 25, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "Despite Late Night Comedy, Undecided Voters are No Joke." *Asheville Citizen Times*. October 28, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "From Poll Workers to Hanging Chads, Election Details Count." *Charleston Post and Courier*. October 6, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "The Rise of 'Moneyball' Politics." *Asheville Citizen Times*. September 30, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "In Elections, Details Matter." *Asheville Citizen Times*. August 27, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "It's the Economy, Stupid." *Asheville Citizen Times*. July 27, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "Let the Veepstakes Begin." *Asheville Citizen Times*. June 24, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "Thought the Election was Over? Get Ready for the Runoff." *Asheville Citizen Times*. May 27, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "A Closer Look at Same Sex Marriage Amendments." *Asheville Citizen Times*. April 27, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "Examining the Impact of Voter Mobilization in 2012." *Asheville Citizen Times*. March 25, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "With No Incumbent, What Makes a Favorite?" *Asheville Citizen-Times*. February 26, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "Primary Elections About More Than Winners and Losers." *Asheville Citizen-Times*. January 27, 2012.

Cooper, Christopher A., and H. Gibbs Knotts. "Occupy, Tea Party to Help Define 2012 Politics." *Asheville Citizen-Times*. December 21, 2011.

Cooper, Christopher A. and H. Gibbs Knotts. "What to Look for in the 2012 Election Season." *Asheville Citizen Times*. November 25, 2011.

Cooper, Christopher A. and H. Gibbs Knotts. "What do Business Names Say About Asheville?" *Asheville Citizen Times*. August 21, 2011.

Cooper, Christopher A. and H. Gibbs Knotts. "Redistricting 101: An FAQ on a Current Political Issue." *Asheville Citizen Times*. July 17, 2011.

Cooper, Christopher A., and H. Gibbs Knotts. "Be Skeptical of Both Sides in Debate Over N.C. Voter ID Law." *Charlotte Observer*. January 13, 2011.

Cooper, Christopher A., and H. Gibbs Knotts. "County Seats: the GOPs Rise to Parity." *Raleigh News and Observer*. December 8, 2010.

Cooper, Christopher A. "Reflections on the 'Far-Left Leanings' of the 11th Congressional District." *Asheville Citizen Times*. November 11, 2010.

Cooper, Christopher A., and H. Gibbs Knotts. "Conservative Democrats, Endangered Species and Rep. Heath Shuler." *Smoky Mountain News*. July 14, 2010.

Cooper, Christopher A., and H. Gibbs Knotts. "Local Government Doesn't Fare Well in Poll." *Smoky Mountain News*. July 14, 2010.

Cooper, Christopher A., and H. Gibbs Knotts. "Tea Party Catches on, but Impact on Election Still Hard to Gauge." *Smoky Mountain News*. July 14, 2010.

Cooper, Christopher A., and H. Gibbs Knotts. "Assessing the Tea Party Nationally and Locally." *Asheville Citizen Times*. March 2, 2010.

Cooper, Christopher A., and H. Gibbs Knotts. "Look for Opportunities in the Public Arena." *Asheville Citizen-Times*. June 5, 2009.

Cooper, Christopher A., and Thaddeus Huff[#]. "Future NC Leaders Get Valuable Learning Experience." *Asheville Citizen-Times*. May 15, 2009.

Cooper, Christopher A., and H. Gibbs Knotts. "Listening for the Voice of the People." *Durham Herald-Sun*. April 12, 2009.

Cooper, Christopher A., and H. Gibbs Knotts. "Here We Go Again—the 'Z' Word Returns." *Asheville Citizen Times*. April 8, 2009.

Cooper, Christopher A. "Shadowboxing Great for Sports TV, But Bad For Democracy." *Atlanta Journal-Constitution*. November 20, 2008.
[Reprinted in the *Asheville Citizen Times*. November 20, 2008.]

Cooper, Christopher A., and H. Gibbs Knotts. "Making Sense of This Historic 2008 Election." *Asheville Citizen Times*. November 9, 2008.

Cooper, Christopher A., and H. Gibbs Knotts "Survey Reveals Where WCU Students Stand in Presidential Race." *Asheville Citizen Times*. November 2, 2008.

Cooper, Christopher A., and H. Gibbs Knotts. "The State of Politics on NC Campuses." *Charlotteobserver.com*. November 1, 2008.

Cooper, Christopher A., and H. Gibbs Knotts. "Can Obama Pull off an Upset in the South?" *Asheville Citizen-Times*. October 19, 2008.

Cooper, Christopher, and Gibbs Knotts. "Candidates' Campaign Schedules Finely Tuned." *Asheville Citizen-Times*. September 7, 2008.

Cooper, Christopher, and Gibbs Knotts. "Voters: It's Healthy to Challenge Your Biases." *Asheville Citizen-Times*. July 27, 2008.

Cooper, Christopher A., and H. Gibbs Knotts. "Much Ado About Something: Vice-Presidential Selection in the 2008 Election." *Asheville Citizen-Times*. June 15, 2008.

Knotts, H. Gibbs, Christopher Cooper and Jewel Counts[#]. "Democratic Party's Process Undemocratic." *Charlotte Observer*. May 23, 2008.

Cooper, Christopher A., and H. Gibbs Knotts. "Political Mudslinging has a Long History in our Democracy." *Asheville Citizen-Times*. May 4, 2008.

Cooper, Christopher A., and H. Gibbs Knotts. "Race, Gender Intrude on Democratic Race." *Asheville Citizen-Times*. March 30, 2008.

Cooper, Christopher A., and H. Gibbs Knotts. "Turnout Tsunami." *Asheville Citizen-Times*. February 17, 2008.

Cooper, Christopher A., and H. Gibbs Knotts. "The Field Narrows." *Asheville Citizen-Times*. January 13, 2008.

Cooper, Christopher A., and H. Gibbs Knotts. "Tar Heels Need to Become More Aware of State Politics." *Asheville Citizen-Times*. April 8, 2007.

Cooper, Christopher A., and Niall Michelsen. "College Education Must Play a Role in Teaching Civic Responsibility." *Asheville Citizen-Times*. October 13, 2006.

Cooper, Christopher A., and H. Gibbs Knotts. "People Across the Nation Divided on Confederate Flag." *The Greenville (SC) News*. August 10, 2006.

Brennan, Kathleen, and Christopher A. Cooper. "WNC Natives and in-migrants Have More Common Values Than They know." *Asheville Citizen-Times*. December 12, 2004.

Cooper, Christopher A. "Opinion Polls, While not Perfect, Give Voice to the Public." *Asheville Citizen-Times*. October 20, 2004.

Cooper, Christopher A. "Kerry's choice of Edwards Unlikely to Have Large Impact on Election Outcomes." *Asheville Citizen-Times*. July 21, 2004.

Cooper, Christopher A. "A Money Spinner for the West." *Raleigh News and Observer*. July 1, 2004.

Cooper, Christopher A. "How to Increase Voter Turnout." *Charlotte Observer*. June 24, 2004. [Reprinted in the *Smoky Mountain News*.]

Cooper, Christopher A. "Trust in Government Declining, From City Hall to White House." *Asheville Citizen-Times*. June 13, 2004.

OTHER PUBLICATIONS [8]

Cooper, Christopher A., and H. Gibbs Knotts. 2020. "Tips for Talking with the Media." *The Department Chair*. 31(2): 10-11. (editor reviewed)

Cooper, Christopher A. 2020. "Back to Basics: Cultivating Community Among Faculty." *The Department Chair* 30(4): 3-4. (editor reviewed)

Cooper, Christopher A. 2020. Review of *The Rise and Fall of the Branchhead Boys: North Carolina's Scott Family and the Era of Progressive Politics*. *North Carolina Historical Review*.

Collins, Todd A., and Christopher A. Cooper. 2011. "The Case Salience Index: A Potential New Measure of Legal Salience." *Law and Courts Newsletter* 21: 5-7. (editor reviewed)

Cooper, Christopher A. 2010. Instructor's Manual for *State and Politics: Institutions and Reform*, *Second Edition*., by Todd Donovan, Christopher Mooney and Daniel Smith. Wadsworth Publishing.

Collins, Todd A., Christopher A. Cooper, and H. Gibbs Knotts. 2008. "Picturing Political Science." *PS: Political Science and Politics*, 42: 365. (editor reviewed)

Cooper, Christopher A. 2008. Instructor's Manual for *State and Politics: Institutions and Reform*, by Todd Donovan, Christopher Mooney and Daniel Smith. Wadsworth Publishing.

Cooper, Christopher A. 2006. Review of *Bringing Representation Home: State Legislators Among Their Constituencies*, by Michael A. Smith. *Perspectives on Politics* 2: 603-604.

CONFERENCE PRESENTATIONS

**Virtual*

"The Rise of the Unaffiliated Voter in North Carolina." Presented at the State of the Parties 2020 and Beyond Virtual Conference. Ray C. Bliss Institute for Applied Politics, University of Akron. November, 2021 (with J. Michael Bitzer, Whitney Ross Manzo, and Susan Roberts).*

"Redistricting in North Carolina." Panel Discussion at Redistricting and American Democracy Conference. Sanford School, Duke University. September, 2021.

"Is The Appalachian Voter Distinct?" Poster Presented at the Appalachian Studies Association. March, 2021.*

"Innumeracy and State Legislative Salaries." Presented at the North Carolina Political Science Association. February, 2021.*

"Roundtable: North Carolina and the 2020 Election." North Carolina Political Science Association. February 2021.*

"The Southern Voter." Presented at the Citadel Symposium on Southern Politics. March, 2020 (with Scott H. Huffmon, H. Gibbs Knotts, and Seth McKee).

"Cooper, Christopher A., Scott Huffmon, and, H. Gibbs Knotts. "The Politics of Southern Identity" Presented at the Biennial Meeting of the Southern Studies Forum. Odense, Denmark. April, 2019

"Heritage v. Hate: Assessing Opinions in Debate Over Confederate Monuments and Memorials." Presented at the Annual Meeting of the South Carolina Political Science Association. February, 2019 (with Scott H. Huffmon, H. Gibbs Knotts, and Seth McKee).

"Still Fighting the Civil War? Southern Opinions on the Confederate Legacy?" Presented at the Biennial Meeting of the Citadel Symposium on Southern Politics. March, 2018 (with M.V. Hood III, Scott H. Huffmon, Quentin Kidd, H. Gibbs Knotts, and Seth C. McKee).

"Leaving the (Political) Party in the South: Unaffiliated Voters and the Future of the Southern Electorate." Presented at the Auburn University Montgomery Southern Studies Conference. February, 2018.

"The Resilience of Southern Identity." Presented at the Biennial Meeting of the Southern American Studies Association. March, 2017 (with H. Gibbs Knotts).

"The Five Factor Model, Public Service Motivation, and Person-Organization Fit." Presented at the Northeastern Conference for Public Administration. Harrisburg, PA. November, 2016.

"Furling the Flag: Examining the Legislative Vote to Remove the Confederate Flag from the Statehouse Grounds in South Carolina." Presented at the Citadel Symposium on Southern Politics. March, 2016 (with Latasha Chaffin and H. Gibbs Knotts).

"Tuition vs. Fees: Breaking Down the Ballooning Costs of Attendance in America's Public Colleges." Presented at the Northeastern Conference for Public Administration. Arlington, VA. November, 2015 (with Tyler Reinagel).

"Charter Reform in City Government: The Case of Columbia, SC." Presented at the Annual Meeting of the Southeastern Conference for Public Administration. Charleston, SC. October, 2015 (with James Bourne and H. Gibbs Knotts).

"The Bluest Red State in America: North Carolina as a Swing State." Presented at the Annual Meeting of the Midwest Political Science Association. Chicago, IL. April, 2015 (with H. Gibbs Knotts)

"Personality Predictors of Job Satisfaction in Public Administrators." Presented at the Annual Meeting of the Southeastern Psychological Association. Hilton Head, SC. March, 2015 (with John Luke McCord).

Kaysing, Nicole, Erin Leonard, Adam Keath, Justin Menickelli and Christopher A. Cooper. "Perceived Sexual Orientation of Women in Sports and Non-Sport Contexts. 2015 SHAPE America National Convention and Expo. Seattle, WA March, 2015.

Menickelli, Justin, Maridy Trom, Tom Watterson, Christopher A. Cooper and Dan Grube. "Activity Monitor Accuracy in Assessing Caloric Expenditures in Obese Adults." 2015 SHAPE America National Convention and Expo. Seattle, WA March, 2015.

"The Resilience of Southern Identity." Presented at the AUM Southern Studies Conference 2015. February 2015 (with Gibbs Knotts).

"Personality and Nonprofit Management." Presented at the Northeastern Conference on Public Administration. October, 2014.

"What Do Wilbur Zelinsky and the Beatles Have in Common?" Presented at the Annual Meeting of the Association of American Geographers. Tampa, FL. April 2014 (with Gibbs Knotts)

"Blue Beacon in the South, or the New South Carolina? North Carolina Politics in the 21st Century" Presented at the Citadel Symposium on Southern Politics. Charleston, SC. February, 2014 (with Gibbs Knotts)

"A 'Court' of Public Opinion Influence on Judicial Decision-Making in the U.S. Supreme Court." Presented at the Public Choice Society Conference. March, 2014 (with Todd Collins).

"Appointed Senators: Treadmill to Oblivion or Stairway to Success?" Presented at the Southern Political Science Association. Orlando, FL. January, 2014 (with Gibbs Knotts)

"Unpacking Southern Identity." Presented at the Southern American Studies Association Meeting. Charleston, SC. February, 2013 (with Gibbs Knotts)

"Southern Identity Revisited." Presented at the Southern Political Science Association. Orlando, FL. January, 2013 (with Gibbs Knotts)

"Reassessing Case Salience." To be presented at the American Political Science Association. New Orleans, LA. August, 2012 (with Todd Collins). [Conference was cancelled due to Hurricane]

"The Southern Focus Poll Revisited." Presented at the Citadel Symposium on Southern Politics. Charleston, SC. February, 2012 (with Gibbs Knotts).

Menickelli, J., Smith, J., Claxton, D, Troy, M., Cooper, C., & Grube, D. (2012, March). Validity of the Walk4Life MVP Pedometer for Measuring Steps and Moderate-to-Vigorous Physical Activity. Presented at the AAHPERD Convention, Boston.

Menickelli, J., Tuten, C., Cooper, C., Grube, D., Claxton, D., Barney, D. & Lyksett, J. (2012, March). Disc Golf and Walking Benefits: A Pedometer-Based Exercise Assessment. Presented at the AAHPERD Convention, Boston.

"In Search of Meaning in Southern And Dixie Business Names." Presented at the Annual Meeting of the North Carolina Political Science Association. Charlotte, NC. February, 2011 (with Gibbs Knotts and Hope Alwine#).

"Media Coverage of the Burger Court." Presented at Southern Political Science Association. New Orleans, LA. January, 2011 (with Todd A. Collins).

"Measuring Legal Salience." Presented at the Annual Meeting of the Midwest Political Science Association. Chicago, IL. April, 2010 (with Todd A. Collins).

"Love 'Em or Hate 'Em: Opinions of Southerners between 1964 and 2008." Presented at the Citadel Symposium on Southern Politics, March, 2010 (with Gibbs Knotts).

"The Geography of Social Identity in Appalachia." Presented at the Annual Meeting of the North Carolina Political Science Association. Durham, NC. February, 2010 (with Gibbs Knotts and Katy Elders).

"Methodological Tools in SoTL" Presented at the International Society for the Scholarship of Teaching and Learning. Bloomington, IN. October, 2009 (with John Habel, Mary Jean Herzog, and Kathleen Brennan).

"Guided by Voices: Understanding Student Learning." Presented at the International Society for the Scholarship of Teaching and Learning. Edmonton, AL. October, 2008 (with Anna McPhadden, Chesney Reich, Glenn Bowen, Laura Cruz, and Carol Burton).

"Two Approaches to Place and Civic Engagement." Presented at the American Democracy Project. Snowbird, UT. June, 2008 (with Sean O'Connell).

"Overlapping Identifies: Investigating the Causes and Consequences of Social Identify in the South." Presented at the Citadel Symposium on Southern Politics, March, 2008 (with Gibbs Knotts, presenter).

"The Importance of Voter Files for State Politics Research." Presented at the Annual Meeting of the Southern Political Science Association. New Orleans, LA. January, 2008 (with Gibbs Knotts and Moshe Haspel).

"Beyond Racial Threat." Presented at the Annual Meeting of the American Political Science Association. Chicago, IL. September, 2007 (with Gibbs Knotts and Moshe Haspel).

"News Media and the State Policy Process: Perspectives from Legislators and Political Professionals." Presented at the 7th Annual Conference on State Politics and Policy. Austin, TX. February, 2007 (with Martin Johnson).

"Politics and the Press Corps: Reporters, State Legislative Institutions and Context." Presented at the Annual Meeting of the American Political Science Association. Philadelphia, PA. August, 2006 (with Martin Johnson).

"Politics and the Press Corps: Reporters, State Legislative Institutions and Context." Presented at the 6th Annual Conference on State Politics and Policy. Lubbock, TX. May, 2006 (with Lilliard Richardson).

"The Impact of Multi-Member Districts on Descriptive Representation in U.S. State Legislatures, 1975-2002." Presented at the 6th Annual Conference on State Politics and Policy. Lubbock, TX. May, 2006 (with Lilliard Richardson).

"Trust in Government, Citizen Competence and Public Opinion on Zoning." Paper presented at the Annual Meeting of the North Carolina Political Science Association. High Point, NC. March, 2006 (with Gibbs Knotts and Kathleen Brennan).

"Casework in U.S. State Legislatures." Presented at the Annual Meeting of the Southern Political Science Association. Atlanta, GA. January, 2006 (with Lilliard Richardson).

"Voice of the People: Letters to the Editor in America's Newspapers." Presented at the Annual Meeting of the American Political Science Association. Washington, DC. August, 2005 (with H. Gibbs Knotts).

"Newsgathering in America's Statehouses." Presented at the 5th Annual Conference on State Politics and Policy. East Lansing, MI. May, 2005 (with Martin Johnson).

"Media Coverage of Scandal and Declining Trust in Government: An Experimental Analysis of 9/11 Commission Testimony." Presented at the Annual Meeting of the Midwest Political Science Association. Chicago, IL. April, 2005 (with Anthony Nownes).

"Beyond Dixie: Race, Region, and Support for the South Carolina Confederate Flag." Presented at the Annual Meeting of the North Carolina Political Science Association. Pembroke, NC. March, 2005 (with H. Gibbs Knotts).

"Media Bias and American Statehouse Reporting." Presented at the Annual Meeting of the Southern Political Science Association. New Orleans, LA. January, 2005 (with Martin Johnson).

"The Impact of Institutional Design on State Legislative Representation." Presented at the 4th Annual Conference on State Politics and Policy. Kent, OH. April, 2004 (with Lilliard Richardson).

"Defining Dixie: Searching for a Better Measure of the Modern Political South." Presented at the 2004 Citadel Symposium on Southern Politics. March, 2004 (with H. Gibbs Knotts).
[Also presented at the Annual Meeting of the North Carolina Political Science Association. Elon University. March, 2004.]

"Negotiating Newsworthiness: Organized Interests and Journalists in the States." Presented at the Annual Meeting of the Southern Political Science Association. New Orleans, LA. January, 2004 (with Anthony J. Nownes).

"State Legislators in the Internet Age." Presented at the Annual Meeting of the American Political Science Association. Philadelphia, PA. August, 2003. (with Lilliard Richardson).

"Descriptive Representation in Multi-Member Districts, 1975-2002." Presented at the Annual Meeting of the Midwest Political Science Association. Chicago, IL. April, 2003 (with Lilliard Richardson).

"The Consequences of Multi-Member Districts in the State Legislature." Presented at the 3rd Annual Meeting of the Conference on State Politics and Policy. Tucson, AZ. March, 2003 (with Lilliard Richardson).

"I Learned it From Jay Leno: Entertainment Media in the 2000 Election." Presented at the Annual Meeting of the South Carolina Political Science Association. Rock Hill, SC. February 2003 (with Mandi Bates). Also presented at the Annual Meeting of the North Carolina Political Science Association. Elon, NC.

"Do Advertorials Work?" Presented at the Annual Meeting of the Southern Political Science Association. Savannah, GA. November 2002 (with Anthony Nownes).

"Legislative Representation in the Face of Direct Democracy." Presented at the 2nd Annual Conference on State Politics and Policy. Milwaukee, WI. May, 2002 (with Lilliard E. Richardson).

"Local Citizen Groups." Presented at the Annual Meeting of the Western Political Science Association. Long Beach, CA. March 2002 (with Anthony J. Nownes).

"Internet Use in the State Legislature." Presented at the Annual Meeting of the Western Political Science Association. Las Vegas, NV. March, 2001.

"Media Consumption in the State Legislature." Presented at the Annual Meeting of the Western Political Science Association. Las Vegas, NV. March 2001.

"Media and the State Legislature." Presented at the Annual Meeting of the American Political Science Association. Washington, DC. September, 2000.

"Depictions of Public Service in Children's Literature." Presented at the Annual Meeting of the International Society for Political Psychology. Seattle, WA (with Marc Schwerdt). July, 2000.

"Former State Legislators in the U.S. Congress During the 1990's." Presented at the Annual Meeting of the Southern Political Science Association. Atlanta, GA. (with Lilliard E. Richardson). August, 1999.

INVITED TALKS AND COMMUNITY SPEAKING ENGAGEMENTS

**Virtual*

"State and Local Government in NC," Leadership Asheville. December, 2021.

"The Resilience of Southern Identity." West Forum, Winthrop University. November, 2021 (with Gibbs Knotts).

"Running Elections in NC—an Insider's Perspective." Panel for Carolina Public Press. November, 2021.*

"North Carolina Politics Primer." Presented to Leadership Asheville Seniors. November, 2021.*

Co-host and Co-Moderator for Sylva Town Commission Debate. October, 2021*

"Redistricting." Presented to Politica. October, 2021*

"The Swain County Electorate." Presented to Indivisible, Swain County.*

"The Jackson County Electorate." Presented to the Jackson County NC Democratic Women.

"Introduction to North Carolina Government." Presented at the Science Policy Bootcamp and NC STEM Policy Fellowship Orientation. Sigma Chi.* June, 2021.

"The Landscape of North Carolina Politics." Presented to the NC League of Municipalities Conference, April, 2021.*

"Politics 2021" Presented to the Hendersonville Rotary. February, 2021.*

"Election Recap." Presented to NC Association of City and County Managers." February, 2021.*

"State and Local Government in North Carolina." Presented to Leadership Asheville, January 2021.*

"Election 2020: In the Rear View Mirror." Presented to Leadership Asheville Foundation. November, 2020.*

"Election 2020: In the Rear View Mirror." Presented to Sylva Rotary. November, 2020.*

"Election 2020." Presented to Leadership Asheville Seniors. October, 2020.*

"North Carolina Politics." Presented to University of Chicago Harris School Alumni Association. October 2020. *

"Election Data." Guest Lecture for Gerry Cohen's Election Law Class at the Duke University Sanford School of Public Policy. October, 2020. *

"Election 2020." City of Burlington, NC. October 2020. *

"Election 2020" Haywood Sunrise Rotary Club. October, 2020. *

Election 2020 from the Bottom Up." Asheville Chamber of Commerce Executive Committee. September 2020. *

"Election 2020." Policy on Tap. Asheville Chamber of Commerce. September 2020.

"North Carolina Elections 2020." Folkmoot. Waynesville, NC. September, 2020. *

"Measuring, Mapping and Interpreting Southern Identity." Guest Lecture for Derek Alderman's Geography of the South class. University of Tennessee, Knoxville. *

"Thoughts on Election 2020." Leadership Asheville Buzz Breakfast. August, 2020). *

"Local, Regional, and State Political Climate." Asheville Rotary Club. July, *

"Political Polarization: Causes and Consequences." Givens Estate. May, 2020; *

"Gerrymandering." Hinton Rural Life Center. February, 2020.

"Elections 2020." Hendersonville Rotary Club.

Moderator, 11th Congressional District Democratic Forum. Jackson County Library. February, 2020.

"State and Local Elections 2020." Presented at the Leadership Asheville Foundation. January, 2020.

"North Carolina Redistricting." Presented at the Asheville Chamber of Commerce. December, 2019.

"State and Local Government." Presented at Leadership Asheville. December, 2019.

"Politics 2020." Roundable on NC Spin (UNC-TV)

"A User's Guide to the 2020 Election." Presented at Life@WCU (two presentations). November, 2019.

"The Resilience of Southern Identity." Presented at Clemson University's Osher Lifelong Learning Institute. (with Gibbs Knotts). November 8, 2019.

"The Resilience of Southern Identity." Presented at the West Forum, Winthrop University. November, 2018.

"2018 Elections." Presented to the Foundation Board of Blue Ridge Public Radio. November, 2018.

"2018 Elections." Roundtable on NC Spin (UNC-TV).

"The Future of the Two-Party System." Presented at Leadership Asheville Foundation. October, 2018

"The 2018 Election" Presented at the Beth HaTePhelia Congregation Brotherhood Luncheon. October, 2018

"The 2018 Constitutional Amendments." Presented at the Cathedral of All Souls. Asheville, NC. October, 2018.

"Elections and North Carolina Politics in 2018." Presented at the NC Local Government Budget Officers Association Annual Summer Meeting. Atlantic Beach, NC. July 2018.

"State and Local Government in North Carolina." Leadership Asheville. December, 2018.

"Politics 2017." Presented at Life@WCU (two presentations). November, 2018.

Moderated 11th Congressional District Democratic Primary Debate. Canton, NC. April, 2018.

"The Resilience of Southern Identity." Madstone Café and Books. September, 2017.

Moderated Asheville City Council Debate. Givens Estate. August, 2017.

"Politics in Western North Carolina." Presented at the Hinton Rural Life Center. June, 2017.

"Redistricting." Presented at the FairVote Forum, Haywood Community College. June, 2017.

"Redistricting." Presented to the Asheville Chamber of Commerce. May, 2017.

"Man is, by Nature, a Political Animal." Presented at the Science Café. Sylva, NC. March, 2017.

"State of State Politics." Presented to Leadership Asheville Foundation Luncheon. March, 2017.

"Raising Your Voice: Contacting Your Representatives in a Polarized Age." Presented at the Haywood County Library. March, 2017.

"Politics 2017." Presented to the NC City/County Manager's Association in Durham, NC. February 2017.

"Election 2016." Presented at the WCU Alumni Association Meeting in Charlotte, NC. October, 2016.

Speaker and Moderator for Buncombe County Commissioner Debate. October, 2016.

"Election 2016." Presented at the WCU Alumni Association Meeting in Atlanta, NC. October, 2016.

"Election 2016." Presented at the South Asheville Rotary Club. October, 2016.

"Election 2016." Presented at the Buncombe County Rotary Club. October, 2016.

"Election 2016." Presented at the Sylva Rotary Club. October, 2016.

"Election 2016." Presented at Beth Hatephelia Brotherhood Lunch. October, 2016.

"Politics 2016." Presented at Life@WCU. Cullowhee and Asheville. October 2016.

"Political Polarization." Presented to the Buncombe County League of Women Voters. June 2016.

"Congress Today." Presented at Life@WCU. Cullowhee, and Asheville. November, 2015.

"Politics 2015." Presented at the Highlands Leadership Series. Highlands, NC. July, 2015.

"Politics in North Carolina." Presentation to the Nonprofit Pathways Policy Conference. January, 2015.

"Polarization in Politics." Presented at the Givens Estate, Asheville, NC. June 2015.

"Politics Today in North Carolina." Presented at Leadership Asheville. Asheville, NC. February, 2015.

"North Carolina For Nonprofits." Presented at the Nonprofit Pathways Public Policy Briefing. January 2015.

"Regional Outlook Report." Presented at Lead WNC, Cullowhee, NC. November, 2014.

"North Carolina Politics." Presented at Leadership Asheville, Asheville, NC. November, 2014.

"Election 2014." Presented at Beth Hatephelia Synagogue. Asheville, NC. October 2014.

"Electoral Politics in the United States." Presented to the Finance Directors for America's Motor Speedways. October, 2013.

"The Current State of American Civics." 2nd Annual Social Work Conference: Citizenship and Civility: Working Together for Practical Advocacy in a Polarized Era. May, 2013.

"Election 2012." Presented at Sylva Rotary Club. Sylva, NC, October, 2012.

"Election 2012." Presented at Leadership Asheville. Asheville, NC, October, 2012.

"Election 2012." Keynote address to the Motor Speedway Finance Officers. September, 2012.

"Election 2012 in North Carolina." Keynote address to the North Carolina Association of Electrical Cooperatives. September, 2012.

"Election 2012." Keynote address to the North Carolina City/County Manager's Association Summer Meeting. June, 2012.

"What Do The Data Tell Us About Hunger?" Presented at Leadership Asheville. Asheville NC, April, 2012.

"Public Opinion on Second Home Development." Presented at the Symposium on Second Home Development. Asheville, NC April, 2011.

"North Carolina Politics" (with Gibbs Knotts). Presented to the Association of North Carolina Budget Officers. Grove Park Inn, Asheville, NC. 2010.

"Engaged Scholarship and the Public Policy Institute." Presented to the Morehead State Leadership Institute, 2009.

"Progressivism in North Carolina Politics" (with Gibbs Knotts). Presented at the John Locke Foundation. Raleigh, NC, June, 2008.

"Political Change in Western North Carolina." Presented at the Economic Forecast Forum, sponsored by the NC Association of Bankers and the NC Chamber of Commerce. Raleigh, NC, January, 2008.

"Multi-Member Districts." Electoral Reform: 2006 and Beyond Conference. Columbus, OH, January, 2007.

"Rhetoric on Representation." University of California, Riverside, November, 2006.

"The Importance of Undergraduate Research." Presentation to the Winthrop University Undergraduate Research Expo. February, 2006.

"Perspectives on Economic Development Research." Presentation to Business Librarians in North Carolina. August, 2005.

"The Importance of a Political Science Education." Presentation to Winthrop University Pi Sigma Alpha Chapter Keynote speaker, Pi Sigma Alpha initiation, Winthrop University, February 2003.

CONTRACTS AND GRANTS

"Policymaking in the Shadows: Collaborative Governance, University Governing Boards and the New Politics of Higher Education." Graduate School and Research. $5000.

"Opt-In Survey." 2013. $8,896.

"Public Opinion on the Town Square Property in Black Mountain, NC." 2010. $6,000.

"French Broad River Congestion Management Plan." 2010. Subcontract from The Louis Berger Group. $5000.

"Evaluating Health Risk in Yancey County Schools." 2010. $500.

"Know Your Region." A Contract with the US Economic Development Administration. 2009. Co-PI with John Hensley. $50,000.

"American Youth Congress." 2009. NC Civic Education Consortium/Z Smith Reynolds. $6000.

"Voter Education Initiative." 2008. NC Campus Compact. $500.

"Citizen Satisfaction in Buncombe County." 2007. $16,577.

"Evaluating Health Risk in Yancey County Schools." 2007. $500.

"Regional Outlook Report." 2007. Internal Contract with the Institute for the Economy and the Future. $6,500.

WCU Summer Research Fellowship. 2007. $1500.

Co-Principal Investigator (with H. Gibbs Knotts). Sponsored contract with the city of Asheville, NC to consult about the design of a citizen satisfaction survey. $3,000.

WCU Summer Research Grant, 2001. $5000.

Yates Dissertation Fellowship, UTK, 2001. $5000.

Undergraduate Education Improvement Grant, UTK Department of Political Science, 2001. $1000.

Dissertation Fellowship, UTK Department of Political Science, 2001. $700.

TEACHING

COURSES TAUGHT

Election Administration (Graduate)
State and Local Governance (Graduate)
Political Analysis (Undergraduate)
State and Local Government (Undergraduate, Traditional and Distance Education)
Political Parties, Campaigns and Elections (Undergraduate)
Research Methods for Public Affairs (Graduate)
Southern Politics (Undergraduate)
Public Policy Analysis (Graduate)
Public Affairs Capstone Experience (Graduate)
Public Affairs Administration (Graduate)
Simulation in American Politics (Undergraduate)
Election 2012 (Undergraduate)
Interdisciplinary Approaches to the Study of Politics (Undergraduate, Freshman Seminar)
Introduction to American Government (Undergraduate)
Mass Media and American Politics (Undergraduate)
Civic Engagement (Undergraduate)
The University Experience (Undergraduate)
Advanced Writing in Political Science (Undergraduate)
Public Administration (Undergraduate)
Internship in Political Science (Undergraduate)
Co-op in Political Science (Undergraduate)
MPA Internship Experience (Graduate)
Metropolitan Government (Graduate)
Capstone in Public Affairs (Graduate)
A variety of independent studies on state politics and elections

THESIS & DISSERTATION COMMITTEES

Christopher Franklin (EdD, 2016)
John Luke McCord (MA, Psychology, 2016, Chair)
Amy Jones (EdD, 2014)
Whitney Bridges-Campbell (MA, Psychology, 2013)
Kimberlee Cooper (MA, Psychology, 2013)
David Solomon (MA, Psychology 2012)
Christopher Holden (MA, Psychology, 2012)
Jenny Smith (MA, HHP, 2011)
Benjamin Locklair (MA, Psychology, 2011)

Brandon Rice (MA, English, 2010)
Andrew Johnson (MA, Psychology , 2010)
Heidi Turlington (MA HHP, 2009)
Joe Hurley (MA, History 2006)

SERVICE

SERVICE TO THE PROFESSION

External Reviewer for Tenure and/or Promotion Cases at:
Furman University
University of Minnesota, Duluth

External Program Reviewer for:
Missouri State University Political Science, MPA, and International Studies
Tennessee Tech University Political Science
University of West Florida Political Science
Western Carolina University Higher Education Student Affairs MA Program
Western Carolina University International Programs and Services
Western Carolina University Mountain Heritage Center

Editorial Boards, Disciplinary Committees, and Section Chair Duties at Conferences

Editorial Board, Journal of Election Administration Research and Practice (2021-)
Editorial Board, Social Science Journal (2021-)
Executive Committee Member, North Carolina Political Science Association (2021-)
Chair, State Politics and Policy Quarterly Best Paper Award Committee (2021-2022)
Chair, Student Paper Committee, North Carolina Political Science Association (2021-)
Consultant, Greensboro History Museum Project Democracy 20/20 Exhibit (2021)
Section Chair for State and Local Politics Section of the Southern Political Science Association (2008)

Reviewer for [since 2010]:
American Journal of Political Science
American Political Science Review
American Politics Research
American Review of Politics
American Review of Public Administration
American Sociological Review
Association of American Geographers
Congress and the Presidency
European Journal of Personality
Geography Compass
Group Processes and Intergroup Relations
International Journal of Health Policy and Management
International Journal for the Scholarship of Teaching and Learning
International Public Management Journal
International Review of Public Administration
Journal of Appalachian Studies
Journal of Food Science Education
Journal of Hate Studies
Journal of Information Technology and Politics

Journal of Political Science
Journal of Political Science Education
Journal of Politics
Journal of Public and Nonprofit Affairs
Journal of Public Administration Research and Theory
Journal of Public Affairs Education
Justice System Journal
Landscape Research
Legislative Studies Quarterly
Personality and Individual Differences
PLOS ONE
Political Behavior
Political Communication
Political Research Quarterly
Politics and Policy
PS: Political Science and Politics
Public Administration Review
Public Opinion Quarterly
Public Budgeting and Finance
Public Management Review
Public Personnel Management
Public Performance and Management Review
Review of Public Personnel Administration
Social Science Journal
Social Science Quarterly
Social Forces
Southeastern Geographer
State and Local Government Review
State Politics and Policy Quarterly
Social Problems
Social Science and Medicine
Social Science Journal
Southeastern Geographer
Southern Cultures
Urban Affairs Review
Oxford University Press
University of South Carolina Press
Routledge
Rowman and Littlefield
Palgrave McMillan
CQ Press
Carnegie Foundation for the Advancement of Teaching
National Science Foundation

Discussant and Panel Chair Duties at Conferences

Discussant for panel on "Congressional Politics." Citadel Symposium on Southern Politics. March, 2020.

Discussant for panel on "Electoral Reform in North Carolina." North Carolina Political Science Association. February, 2011.

Chair for panel on "Economic Development Policies." North Carolina Political Science Association. Durham, NC. February, 2010.

Chair for panel on "The Future of State Politics." Southern Political Science Association. New Orleans, LA. January, 2008.

Discussant for panel on "Electoral Reform." American Political Science Association. Chicago, IL. September, 2007.

Discussant for panel on "Disaster: Politics and Policy." Policy History Conference. Charlottesville, VA. June, 2006.

Chair and Discussant for panel on "Issues in Electoral Politics." North Carolina Political Science Association. High Point, NC. March, 2006.

Discussant for panel on "Issues in American Politics." North Carolina Political Science Association. High Point, NC. March, 2006.

Discussant for panel on "North Carolina Politics." Citadel Symposium on Southern Politics. Charleston, SC. February, 2006.

Chair and discussant for panel on "State Policy. American Political Science Association. Washington, DC. September, 2005.

Discussant for panel on state politics. Annual Meeting of the Midwest Political Science Association. Chicago, IL. April, 2005.

Chair and Discussant for panel on "Electoral Politics." Annual Meeting of the North Carolina Political Science Association. Cullowhee, NC. March, 2004.

Discussant, "State Legislative Elections." Annual Meeting of the Southern Political Science Association. New Orleans, LA. January, 2004.

Discussant and Chair, "Highlighting Student Research." Annual Meeting of the South Carolina Political Science Association. Rock Hill, SC. February 2003.

Discussant and Chair, "Media Coverage of Elections and Representation." Annual Meeting of the Southern Political Science Association. November, 2002.

UNIVERSITY, COLLEGE & DEPARTMENT SERVICE

Current and Continuing

- Dept. of Political Science, Tenure, Promotion and Reappointment Committee (2008-present)
- MPA Committee (2002-present)
- Coulter Faculty Commons Advisory Board (2016-)
- University Collegial Review Committee (2020-)
- Congressional Internship Selection Committee (2018-)
- Committee on National and International Scholarships and Awards (2020-)
- Chair, Search Committee to hire Government Affairs Liaison/Deputy Chief of Staff

Previous Service

- Pathfinders Task Force to Select New Learning Management System (2020)
- Provost Search Committee (2020)
- Bookstore Director Search Committee (2020)
- Student Assessment of Instruction Task Force (2018-2019)
- Task Force to Select New Assessment Software (2018-2019)
- Regional Conference Planning Committee (2012-2016)
- Editor, Faculty Forum (2016-2019)
- COACHE survey task force (2015-2016)
- Facilitator, Leadership Summit (2015)
- Faculty Senate (2009-2015)
- SAI Standardization Task Force (2015)
- Academic Policy Review Council (2013-2015)
- Arts and Sciences Tenure, Promotion and Reappointment Committee (2008-2014)
- Chair, Search Committee for Public Administration Faculty (2015)
- Book Store Task Force (2014)
- Search Committee for Public Administration Faculty (2014)
- Search Committee to hire an Assistant Professor in Public Administration (2012-2013)
- Chair, search committee to hire a visiting assistant professor in International Relations
- Chair, search committee to hire a lecturer in American Politics and Global Issues
- Search Committee for Research Development Specialist (2014)
- Search Committee for Human Geography (2014)
- Chair, Search Committee to hire Comparative Politics Faculty (2013)
- Chair, Faculty Affairs Caucus (2010-2011; 2012-2013)
- Dean of Arts and Sciences Search Committee (2012-2013
- Faculty Affairs Caucus (2009-2014)
- Faculty Senate Planning Team (2010-2011; 2012-2013)
- Chair, 2020 Commission Subcommittee on Community Partnerships (2012)
- Chair, Search Committee to hire an Administrative Support Associate in the Department of Political Science and Public Affairs (2012)
- Chair, Search Committee to hire a Research Support Associate in the Coulter Faculty Center (2011)
- Search Committee to hire an Assistant Professor in Parks and Recreation Management (2012)
- Search Committee to hire an Assistant Professor in Public Administration (2012)
- Search Committee to hire a Visiting Assistant Professor in Public Administration (2012)
- College of Business Research Award Committee (2012)
- Institutional Review Board (2005-2011)
- Mountain Heritage Center Program Assessment Team (201!)
- Chair, American Democracy Project (2010-2011)
- Arts and Sciences Program Prioritization Task Force (2011)
- Cullowhee Revitalization Task Force (2010)
- Chair, Department Graduate Recruitment Committee
- Chair, Department Graduate Comps Committee
- Chair, Department Graduate Internship Committee
- International Relations Search Committee (2010)

- WCU/Dillsboro Partnership Task Force (2009-2010)
- QEP Assessment Committee (2007-2010)
- Arts and Sciences Teaching Award Committee (2009-2010)
- Co-Chair Social Science Research Forum (2007-2010)
- Chair, MPA Director Search Committee (2009-2010)
- Public Administration Search Committee (2009-2010)
- Chair, MPA Director Search Committee (2008-2009)
- Public Administration Search Committee (2008-2009)
- International Relations Search Committee (2008-2009)
- Chair, Graduate Research Grant subcommittee of the Research Council (2008)
- College Restructuring Task Force (2008-2009)
- Athletics Committee (2006-2009)
- Graduate Council (2006-2009)
- Research Council (2005-2008)
- Chair, Graduate Research Grant subcommittee of the Research Council (2008)
- Co-chair, Integration of Learning Award subcommittee of the Student Learning Committee (2008)
- Outreach and Engagement Committee for UNC-Tomorrow (2008)
- Humphrey Fellows Steering Committee (2007-2008)
- Chair, Public Administration Search Committee (2007-2008)
- Chair, Institutional Review Board (2005-2007)
- Chair, Public Administration Visiting Search Committee (2007)
- Public Law visiting assistant professor search committee (2006)
- International Relations visiting instructor search committee (2006)
- Congress to Campus Coordinator (2006)
- President, University Club (2006-2007)
- Arts and Sciences Strategic Planning Committee (2005-2007)
- Arts and Sciences Dean's Advisory Board (2006-2007)
- Committee Chair, National Youth Congress (April, 2005)
- Scholarship of Teaching and Learning Committee (2005-2006)
- Committee on Student Learning (2005-2008)
- ICPSR Representative for WCU (2004-2007)
- Created and Directed WCU faculty Quantitative Research Forum (2004-2005)
- Congress to Campus Coordinator (2004)
- Center for Regional Development Director Search Committee (2003)
- Public Administration Search Committee (2003)
- Co-op and Internship Coordinator, Dept. of Political Science, WCU (2002-2006)
- Webmaster, WCU Department of Political Science (2002-2007)

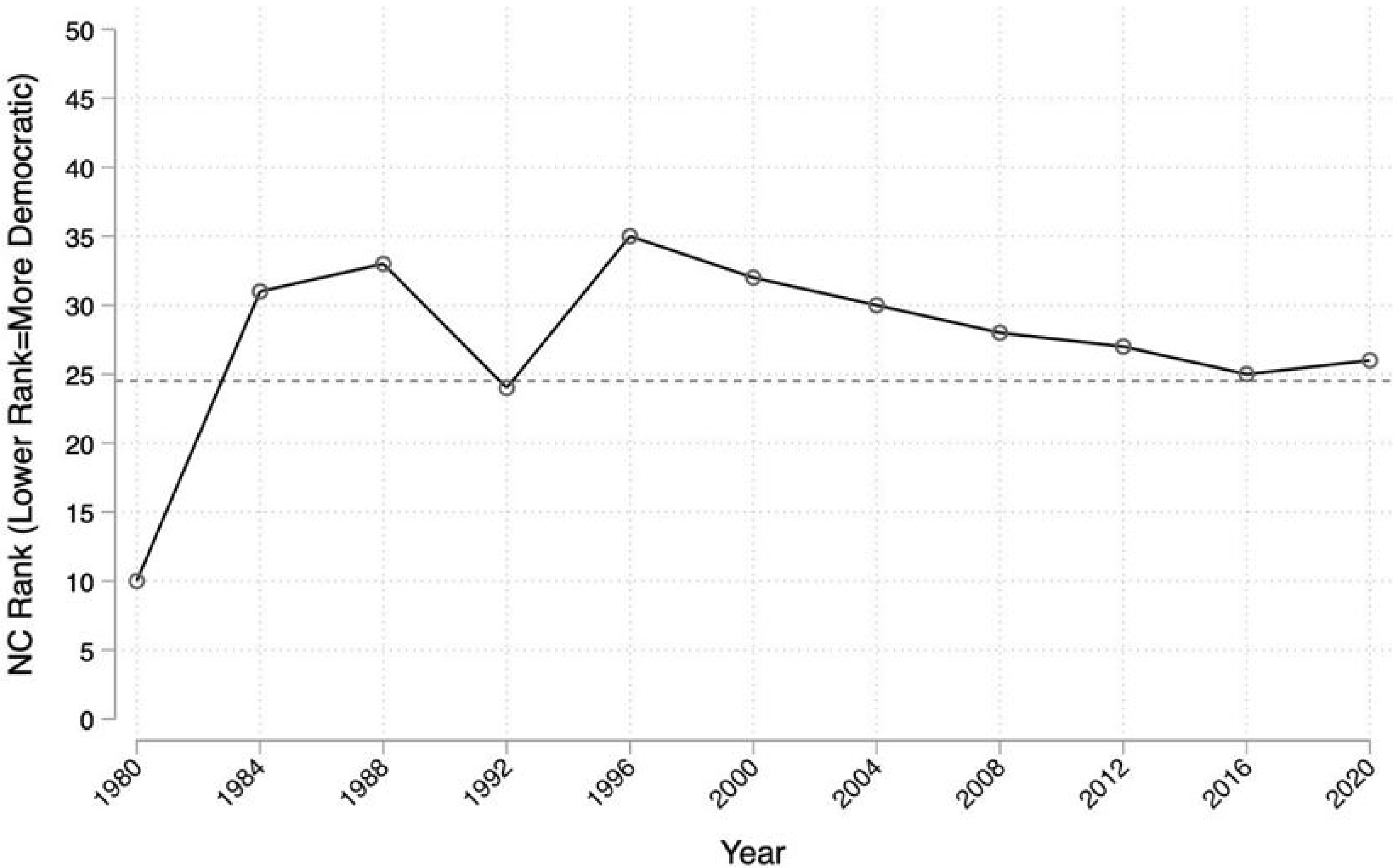
MEDIA APPEARANCES, ON-CAMPUS AND COMMUNITY SPEAKING

**Virtual*

- Quoted thousands of times in such media outlets including BBC (TV and Radio), CNN, Fox News, *New York Times*, *National Public Radio (All Things Considered, Weekend All Things Considered, Morning Edition), Christian Science Monitor*, Vox, *Washington Post, Wall Street Journal, Financial Times, ESPN.com, USA Today, Detroit Free Press, Raleigh News and Observer, Boston Herald*, *Business Insider, Asheville-Citizen*

Times, Charlotte Observer, Winston Salem Journal, National Journal, Rock Hill Herald, Smoky Mountain News, Hendersonville Times, Sylva Herald, Mountain Express, Yahoo Singapore News, Carolina Journal, Blue Ridge Public Radio, WUNC, WFAE, Roll Call, Waynesville Mountaineer, Voice of America, Zoomer Radio (Toronto, Canada), WLOS TV (Asheville, NC), WATV, WRAL (Raleigh, NC), WCNC (Charlotte, NC), WFSC, WJLA (Washington DC) and KISS FM, Spectrum News and many more.

Figure 1. North Carolina Rank in Democratic Vote Share for President Among the 50 States



Data Source: David Liep's Atlas of U.S. Presidential Elections

Figure 2. Two-Party Vote Share in the 2020 Presidential Election

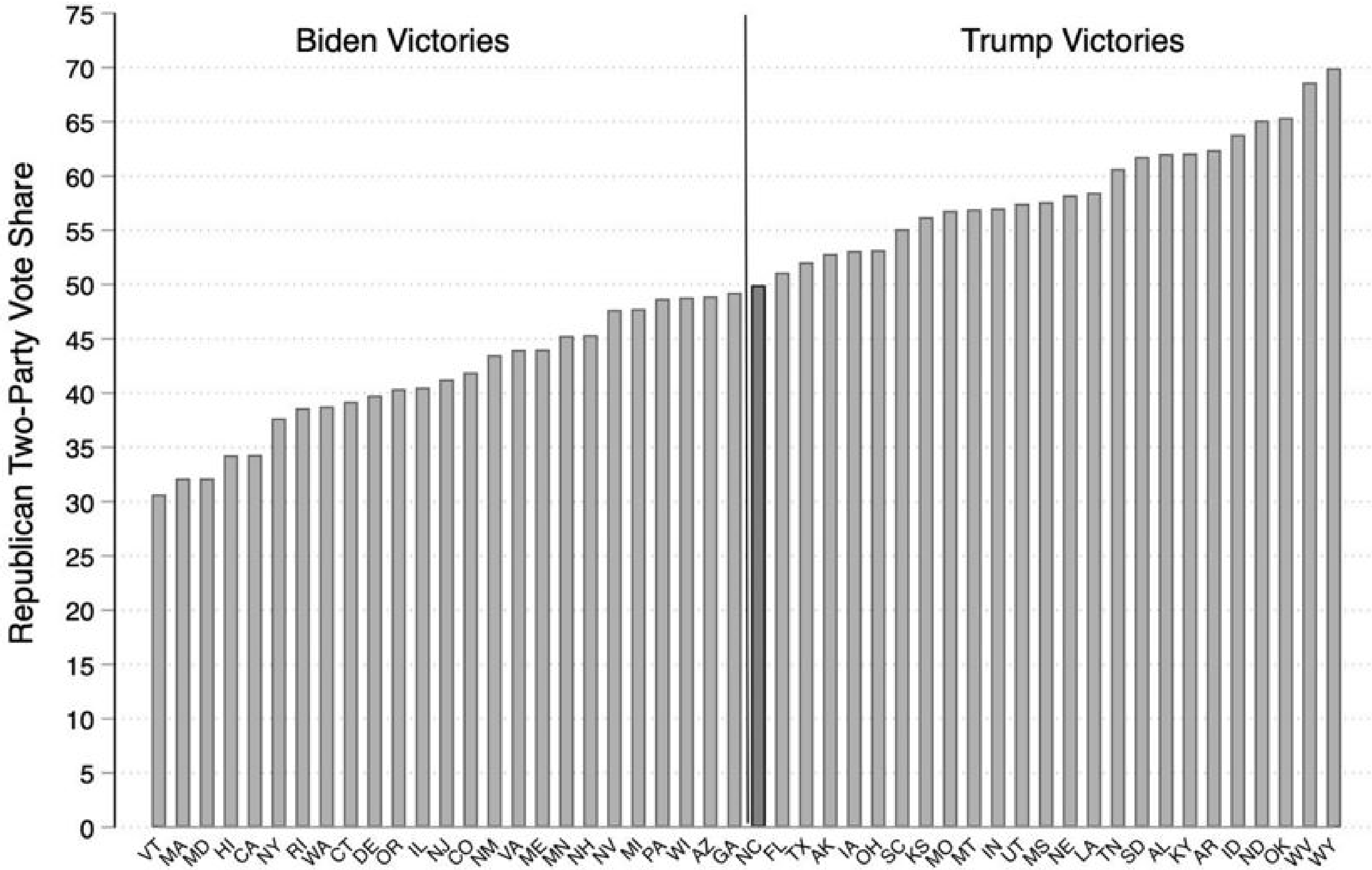
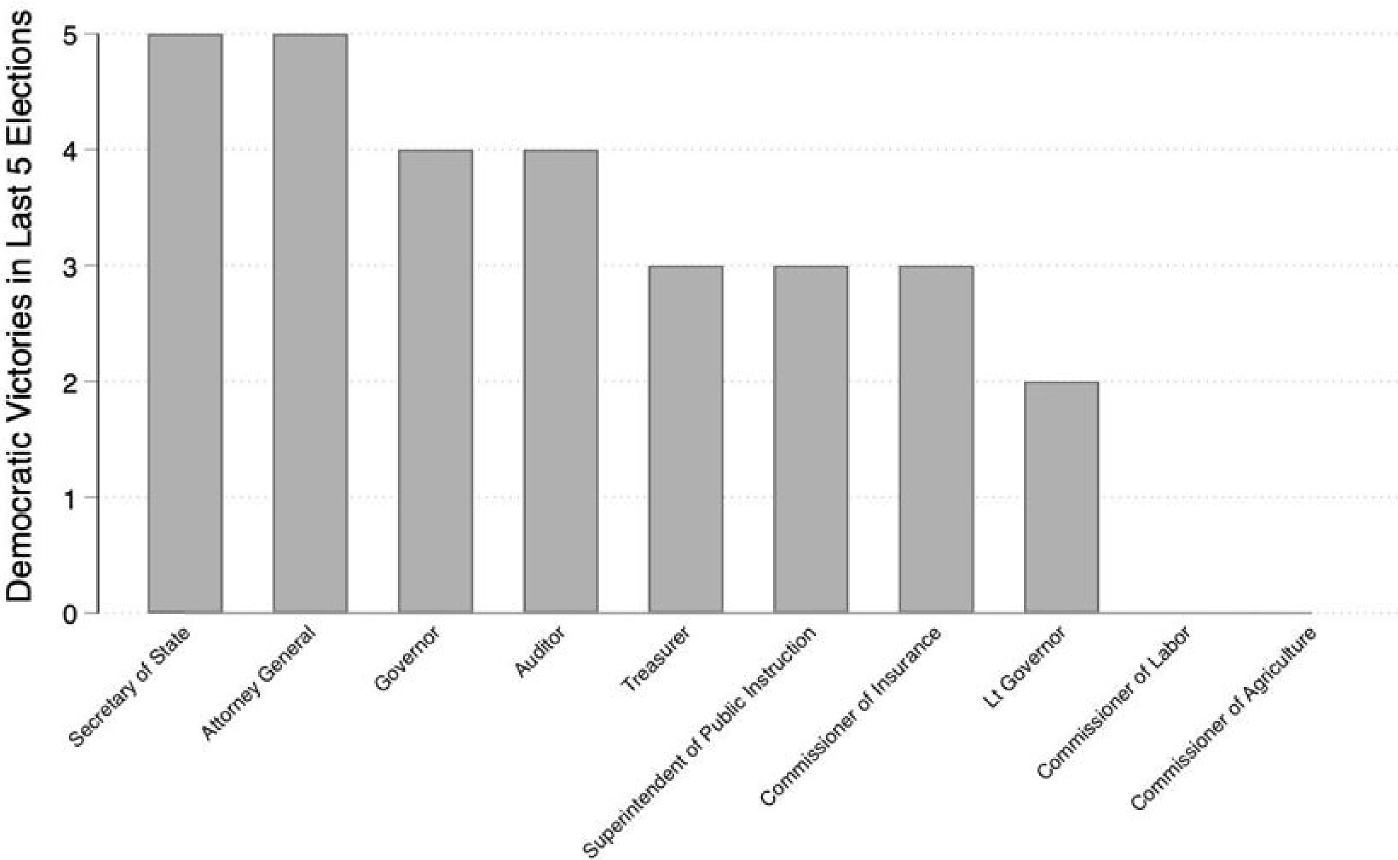


Figure 3. Results of The Last Five Council of State Elections



Note: Calculated from NC State Board of Eletions data. Council of State elections take place every four years.

Figure 4. Voter Registration in North Carolina

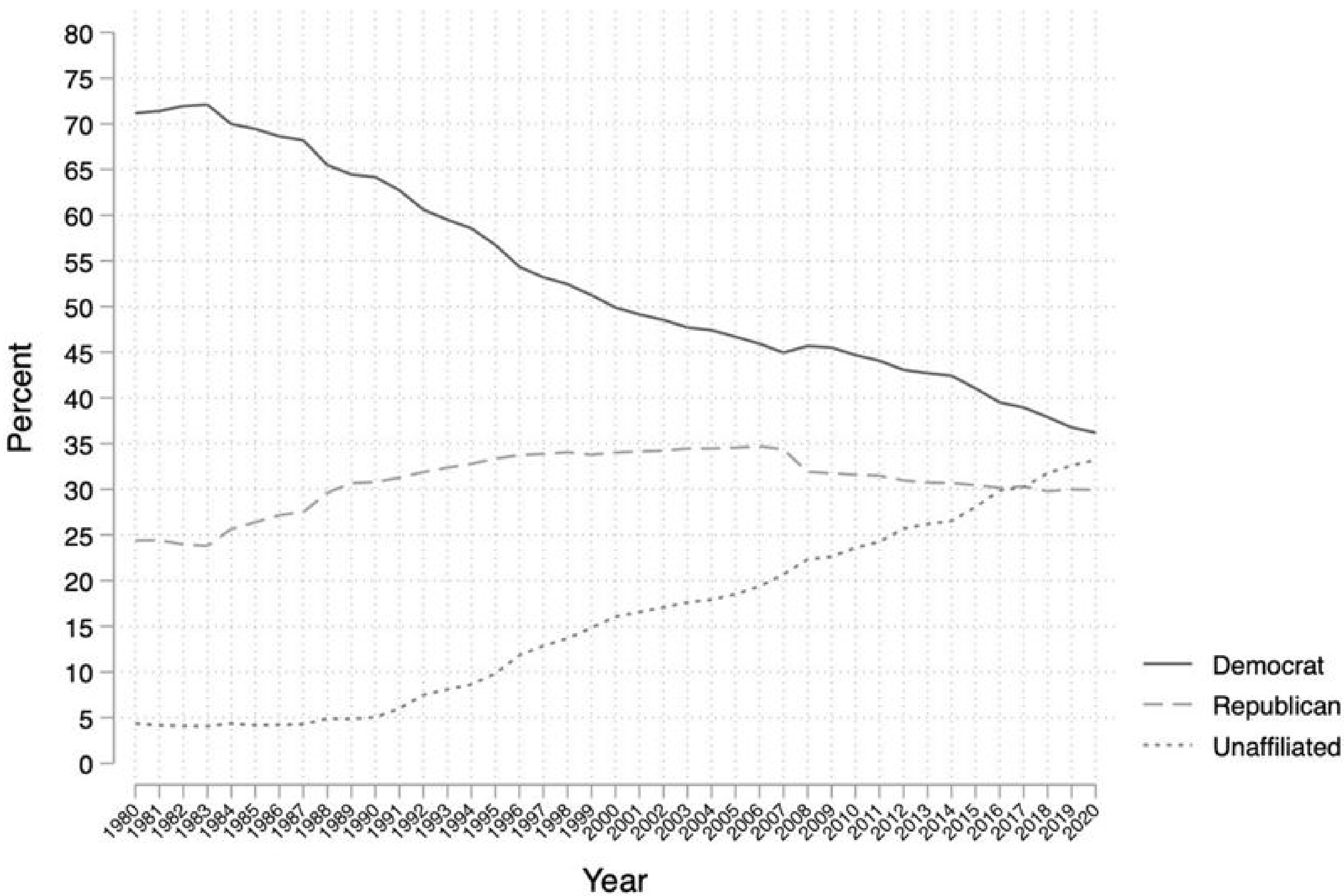


Figure 5. Comparing Votes and Seats in North Carolina's Congressional Delegation, 2012-2020

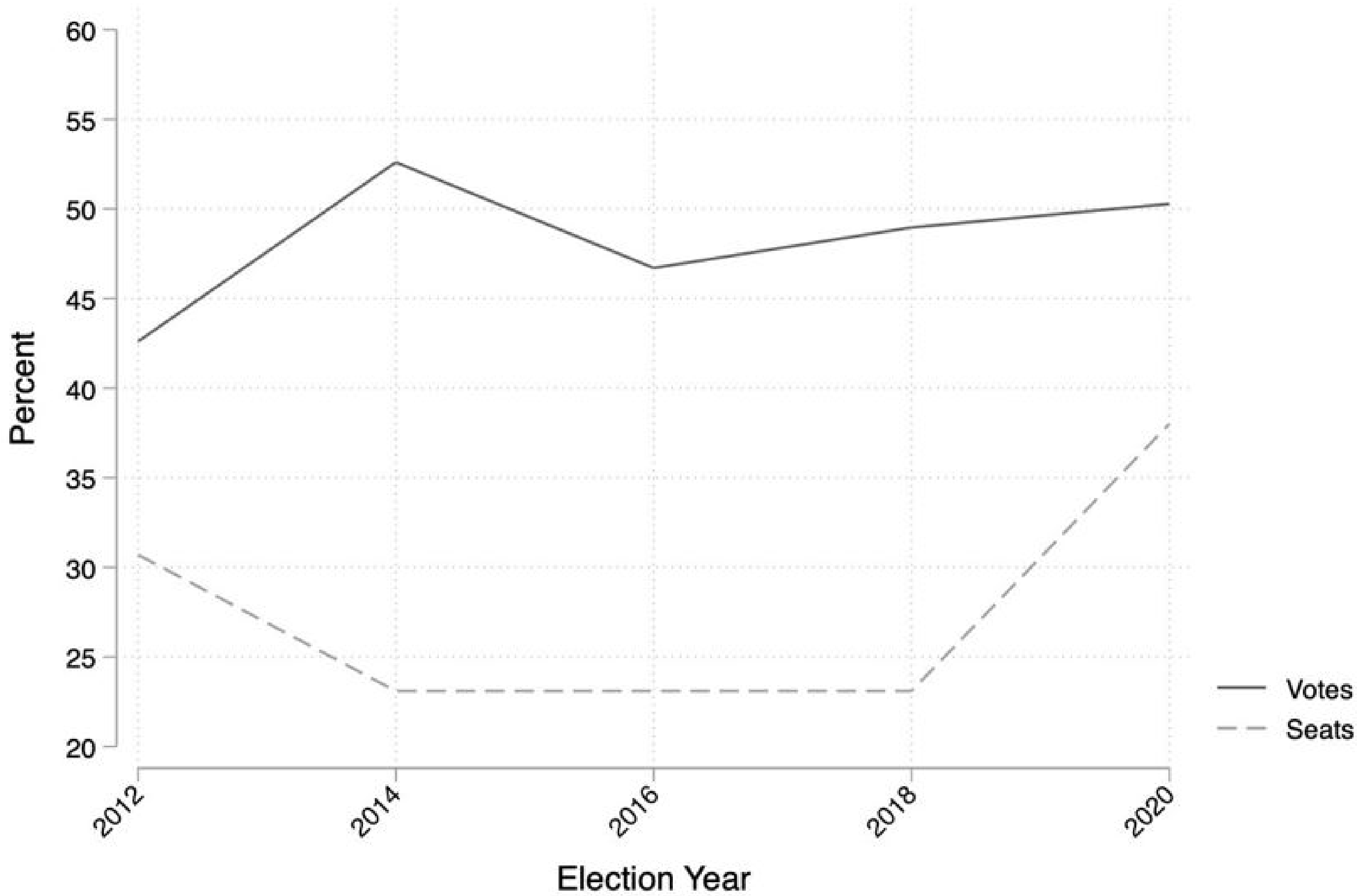


Figure 6. Comparing Votes and Seats in the North Carolina Senate, 2012-2020

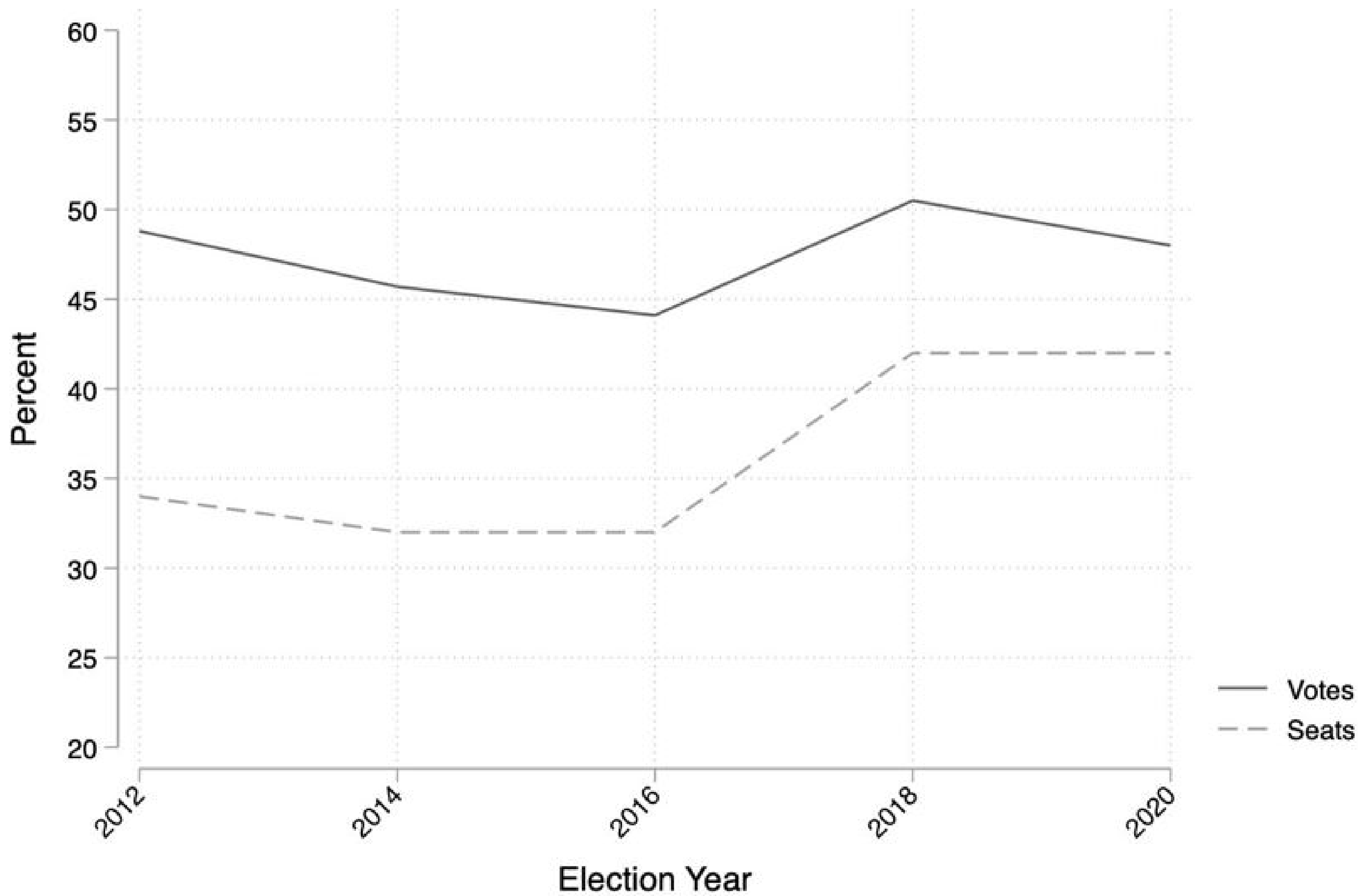


Figure 7. Comparing Votes and Seats in the North Carolina House, 2012-2020

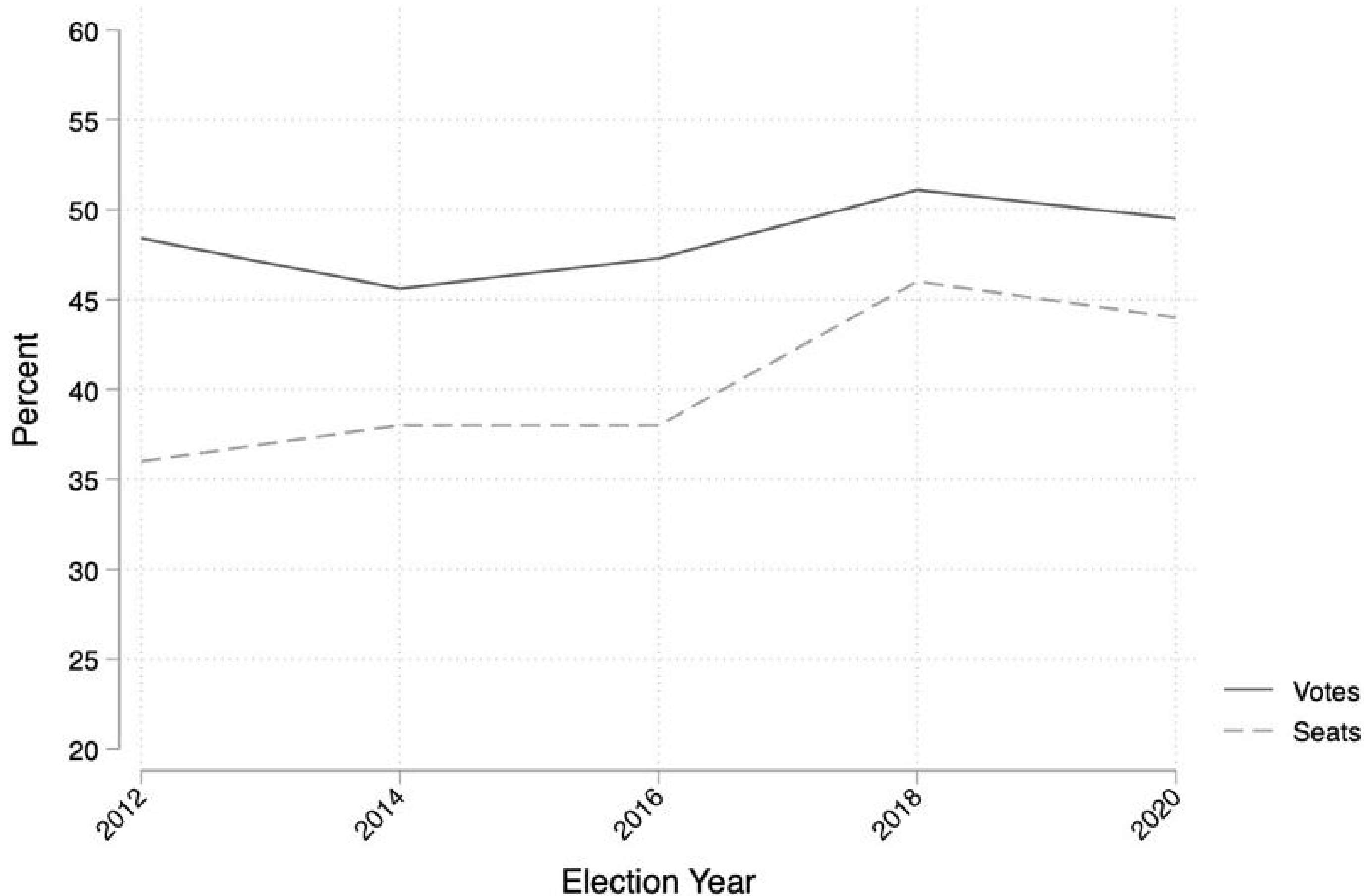
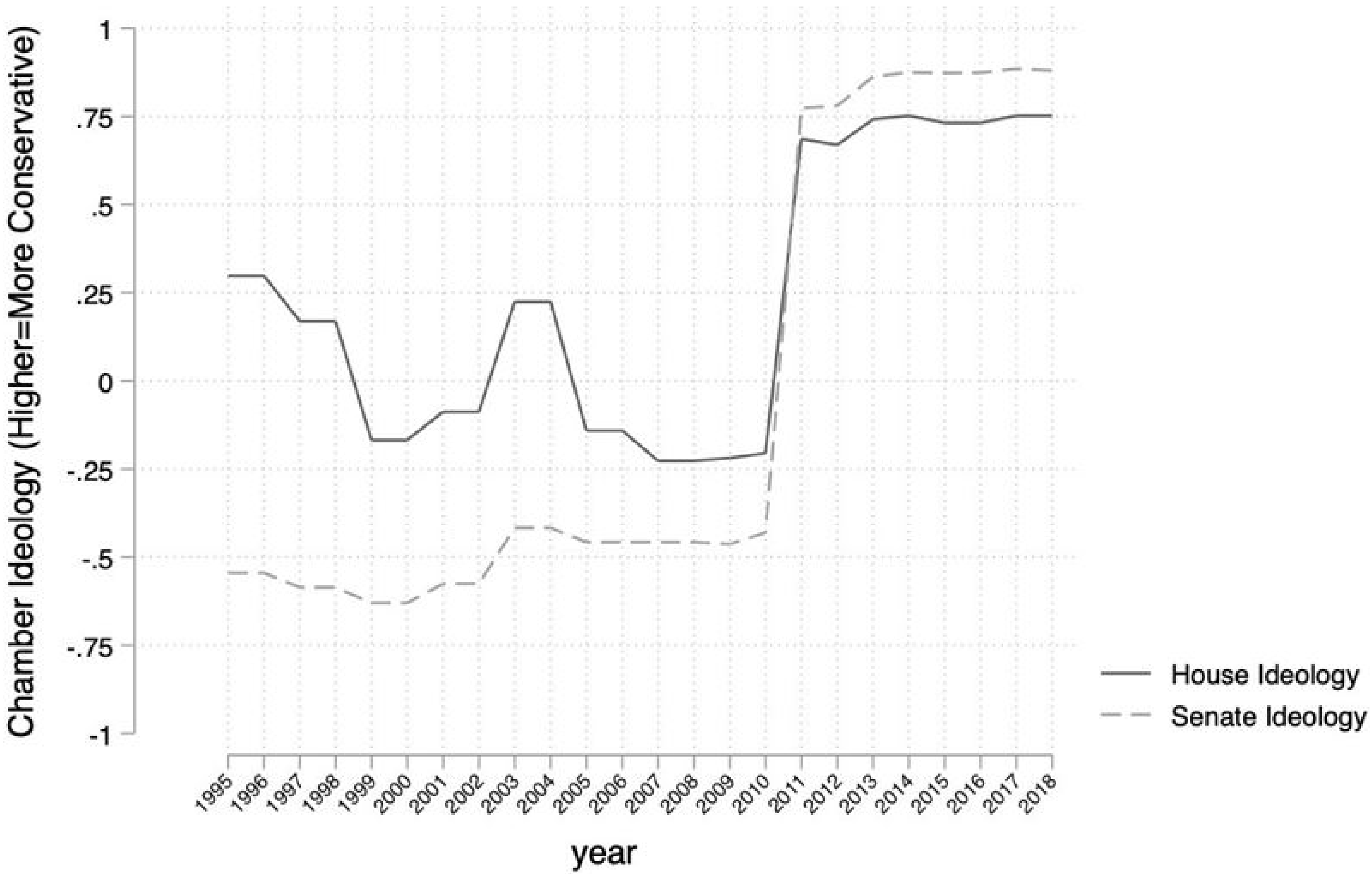
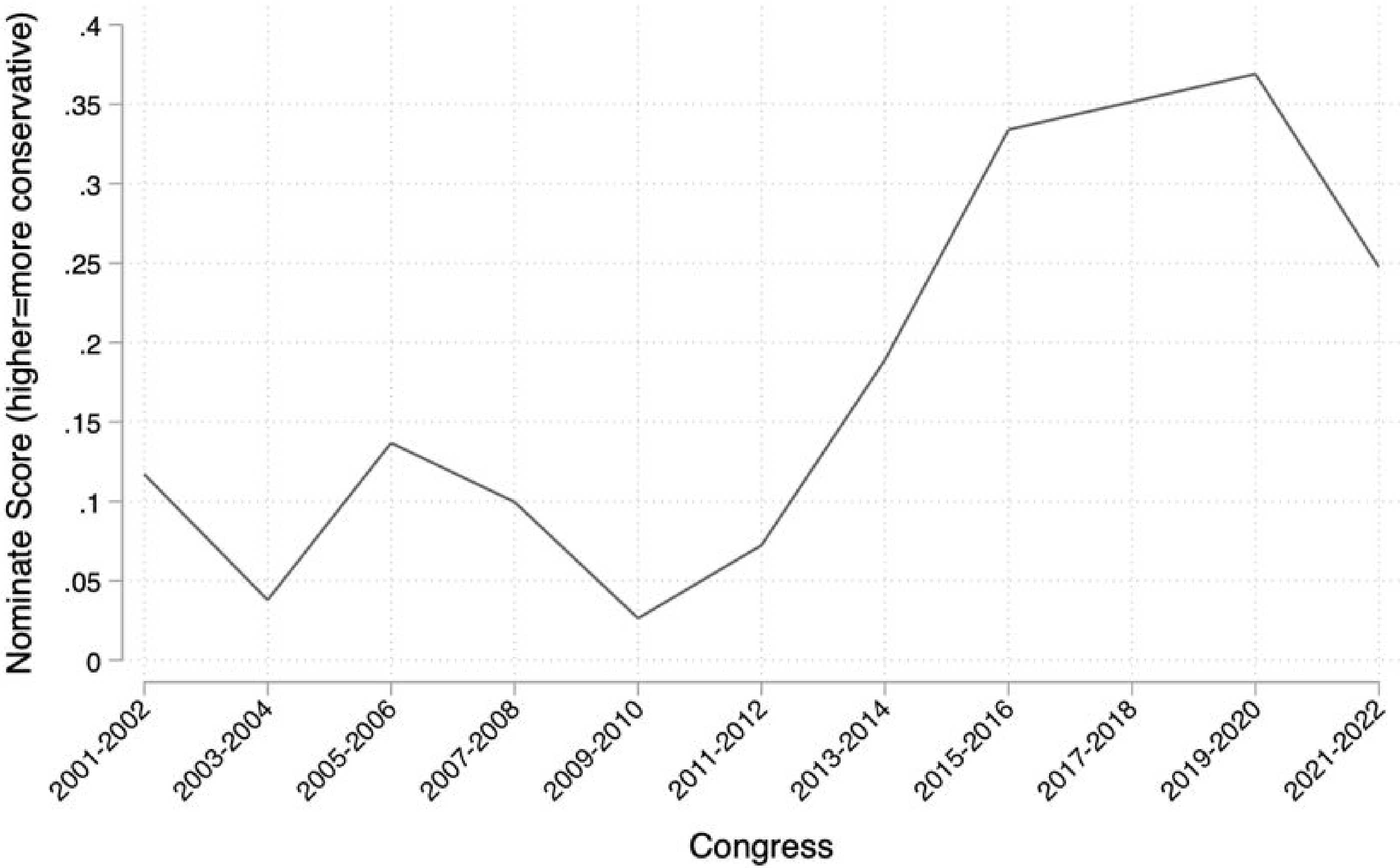


Figure 8. Chamber Estimates of North Carolina General Assembly Ideology, 1995-2018



Source: American Legislatures Project (Schor and McCarty 2020)

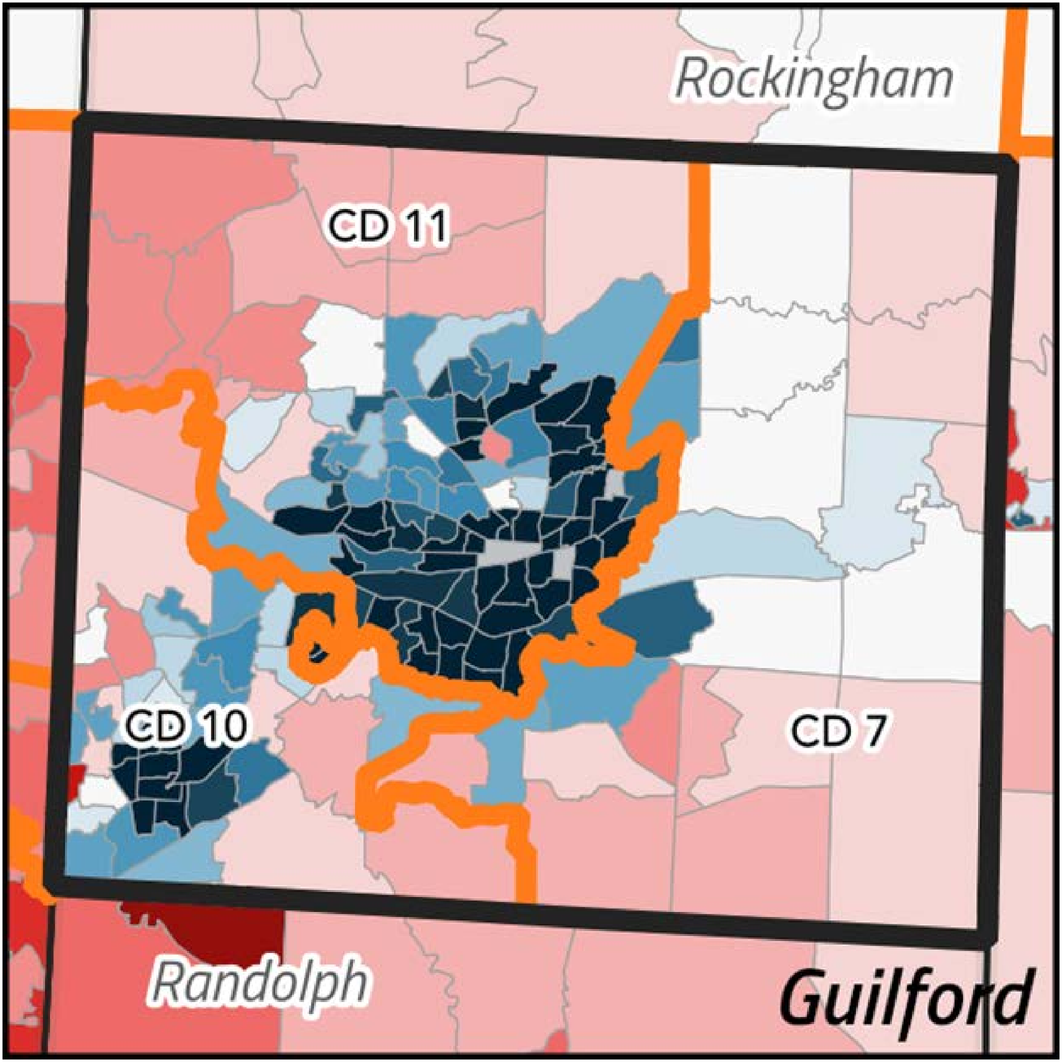
Figure 9. Nominate scores of North Carolina's congressional delegation, 2001-2002 Congress through 2021-2022 Congress

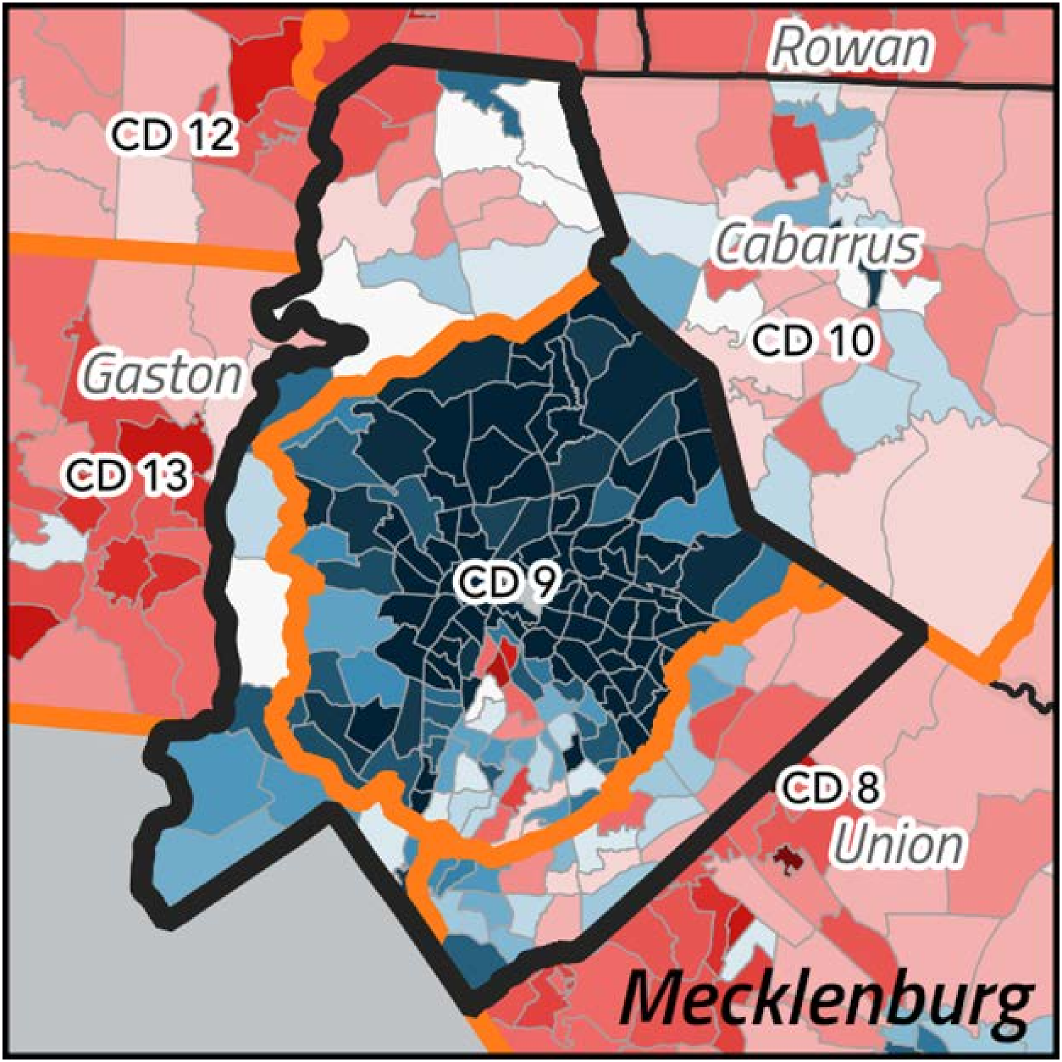


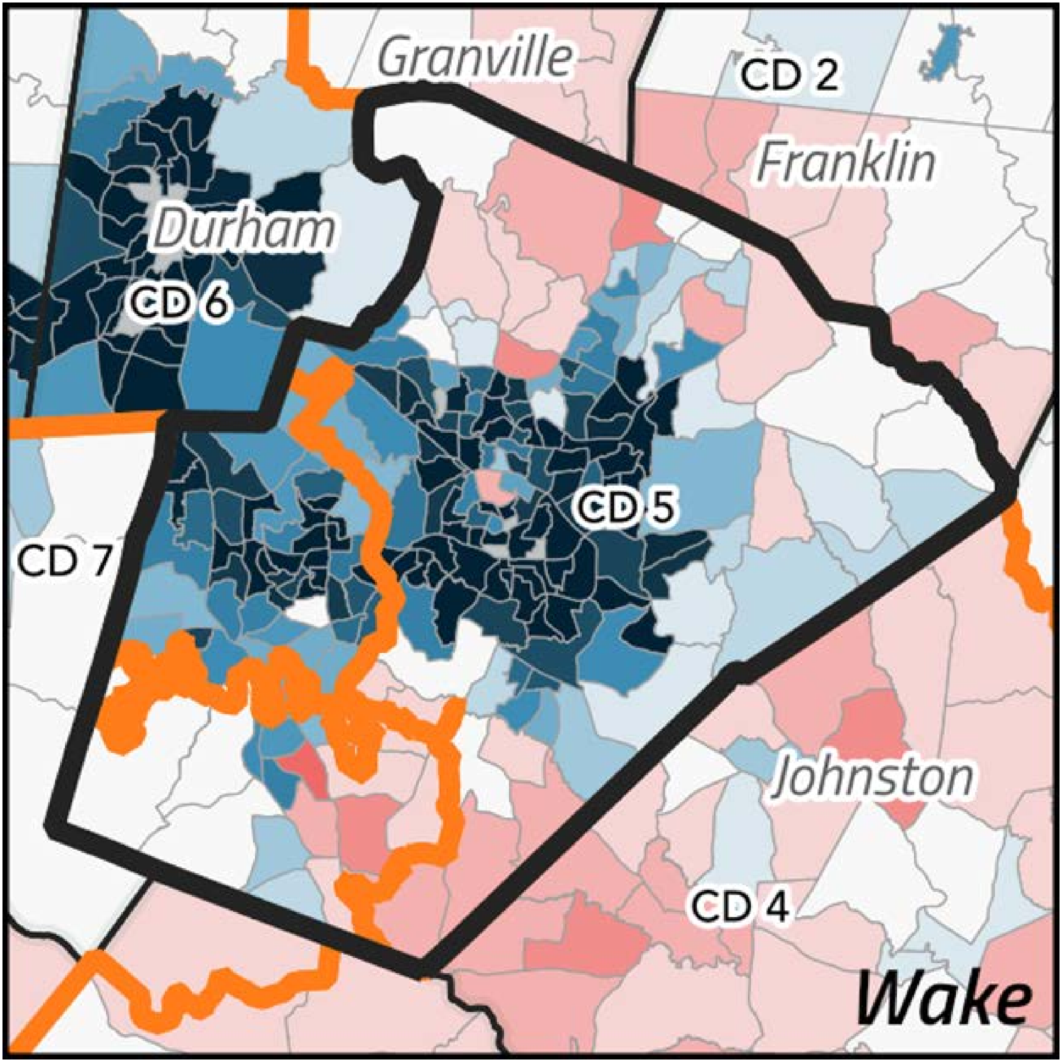
Source: Lewis et al. (2021)

Table 1. Summary Data for Each Enacted Congressional District

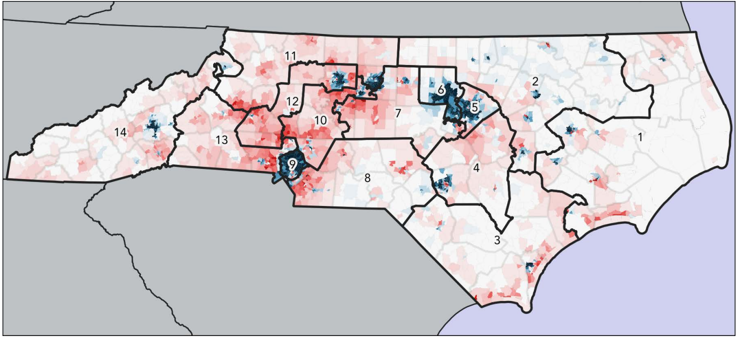
District	PVI	CCSC	Trump Perc
1	R+10	R +98,969	57%
2	Even	D +40,396	48%
3	R+10	R +111,451	58%
4	R+5	R +28,045	53%
5	D+12	D +227,327	34%
6	D+22	D +374,786	25%
7	R+11	R +115,682	57%
8	R+11	R +125,842	57%
9	D+23	D +325,717	25%
10	R+14	R +156,833	60%
11	R+9	R +94,407	57%
12	R+9	R +102,404	56%
13	R+13	R +150,187	60%
14	R+7	R +58,387	53%


Rockingham
CD 11
CD 10
CD 7
Randolph
Guilford

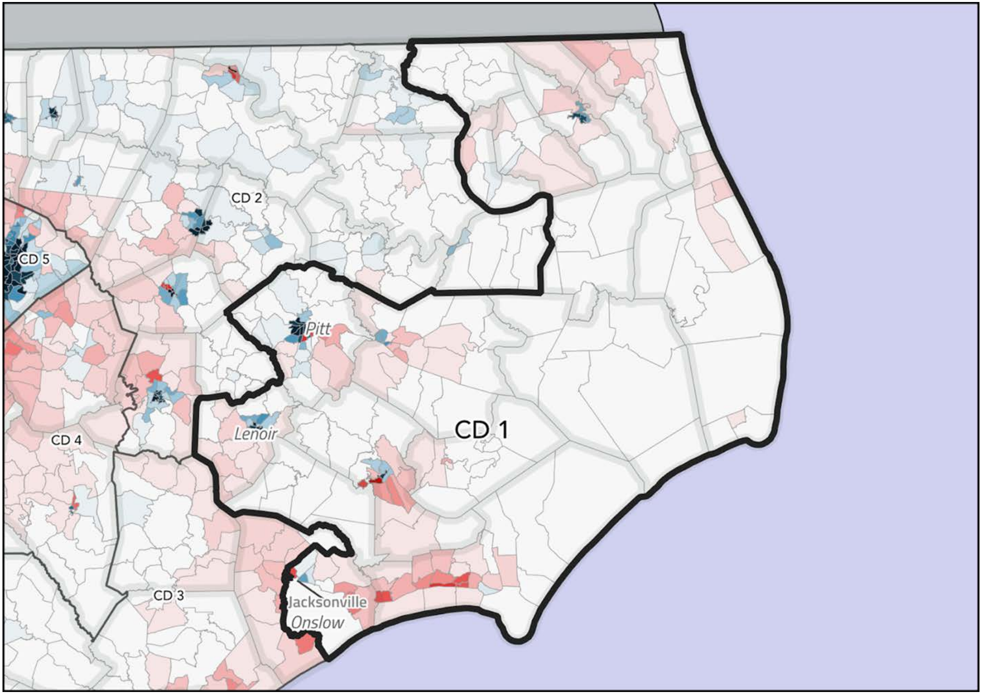

Rowan
CD 12
Cabarrus
CD 10
Gaston
CD 13
CD 9
CD 8
Union
Mecklenburg


Granville
CD 2
Franklin
Durham
CD 6
CD 5
CD 7
Johnston
CD 4
Wake

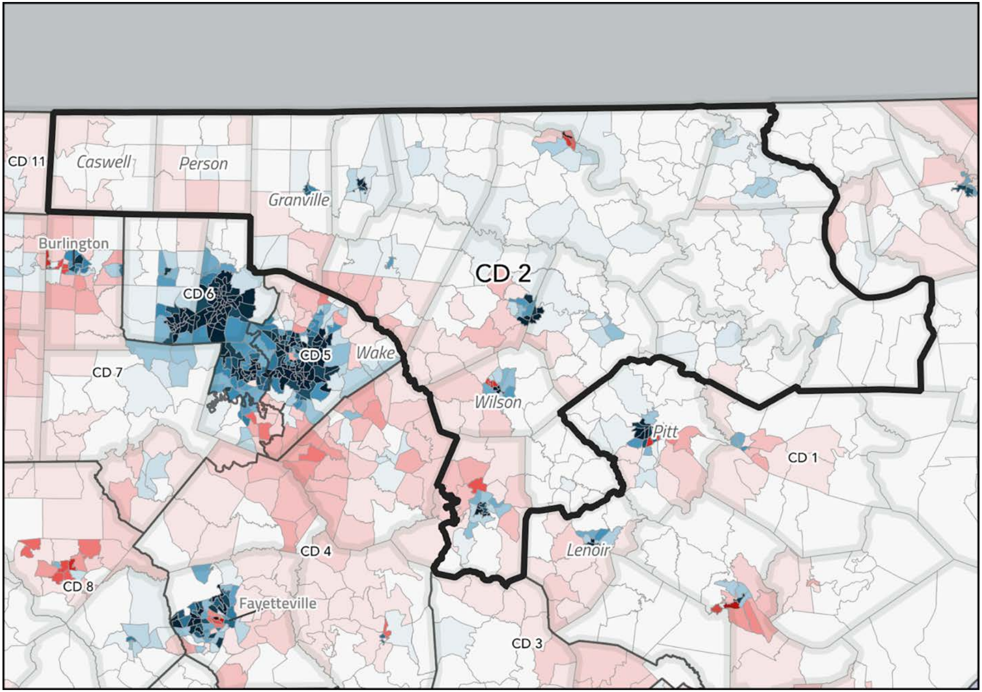
– Ex. 4348 –



– Ex. 4349 –

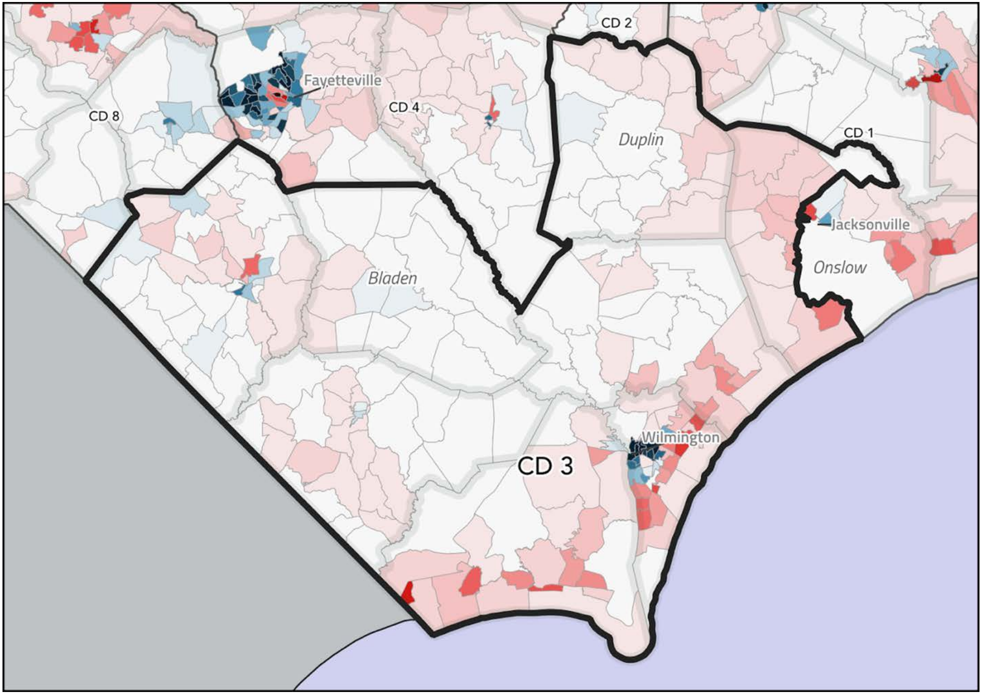


– Ex. 4350 –

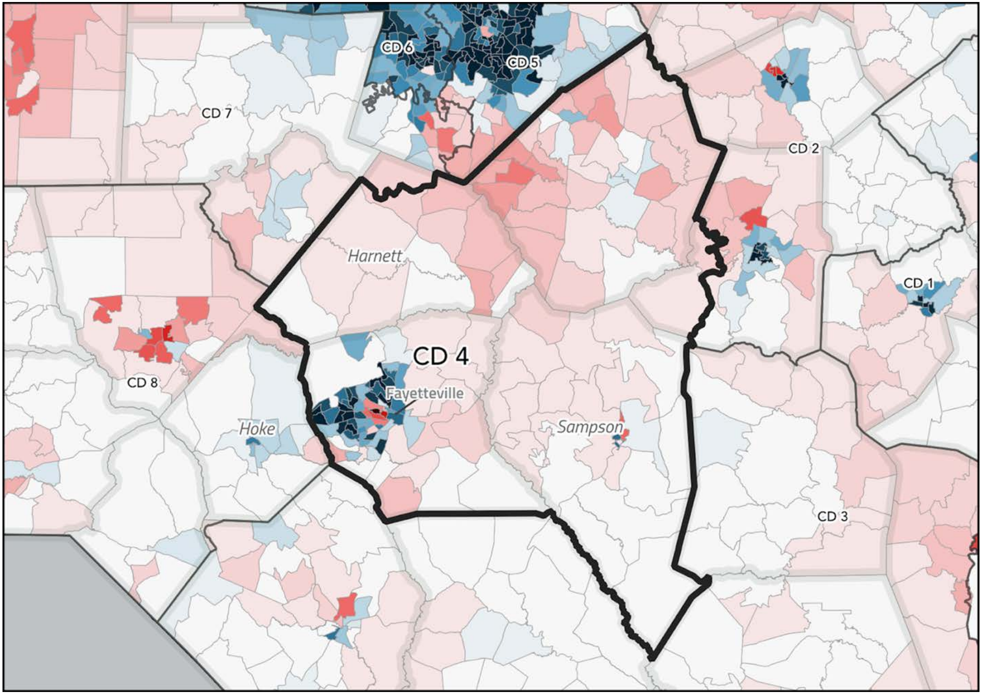


CD 11
Caswell
Person
Granville
Burlington
CD 2
CD 6
CD 5
Wake
CD 7
Wilson
Pitt
CD 1
Lenoir
CD 4
CD 8
Fayetteville
CD 3

– Ex. 4351 –

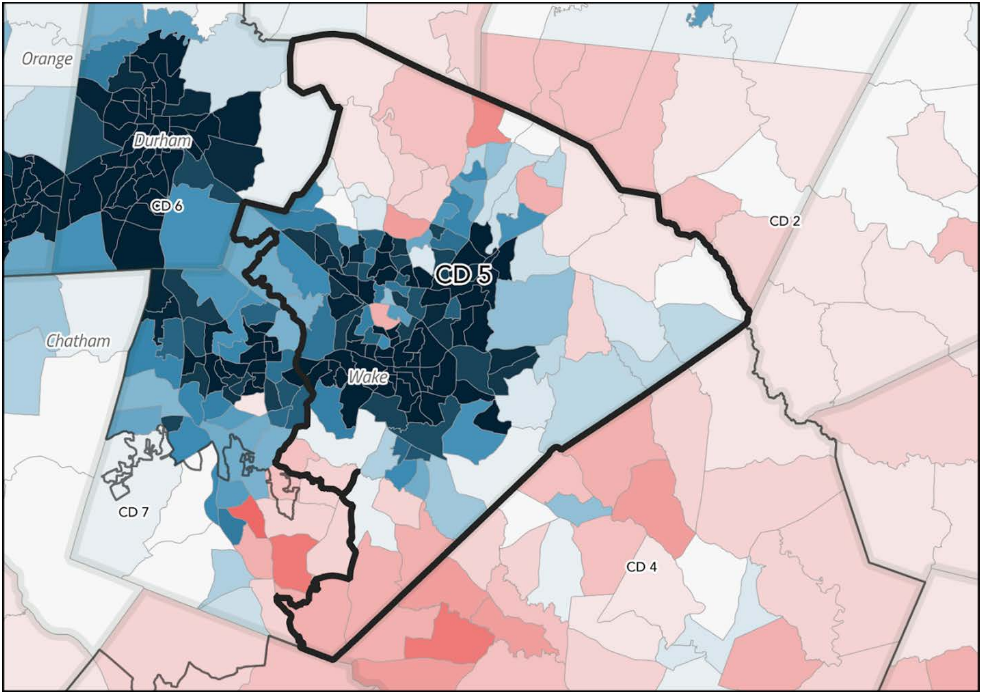

CD 2
Fayetteville
CD 8
CD 4
Duplin
CD 1
Jacksonville
Bladen
Onslow
Wilmington
CD 3

– Ex. 4352 –

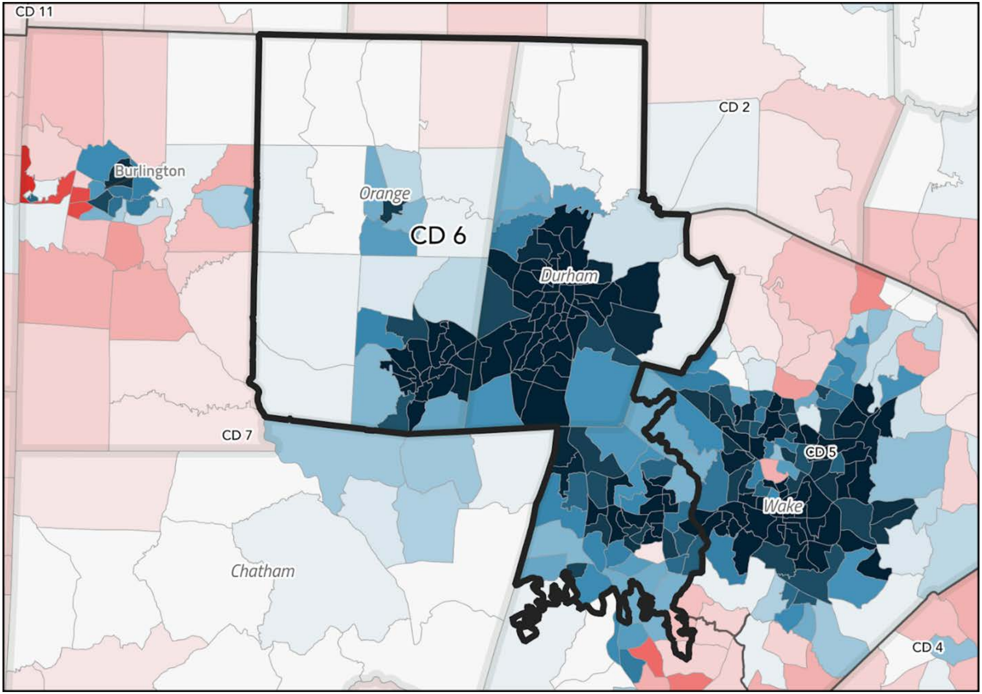


CD 6
CD 5
CD 7
CD 2
Harnett
CD 1
CD 4
Fayetteville
CD 8
Hoke
Sampson
CD 3

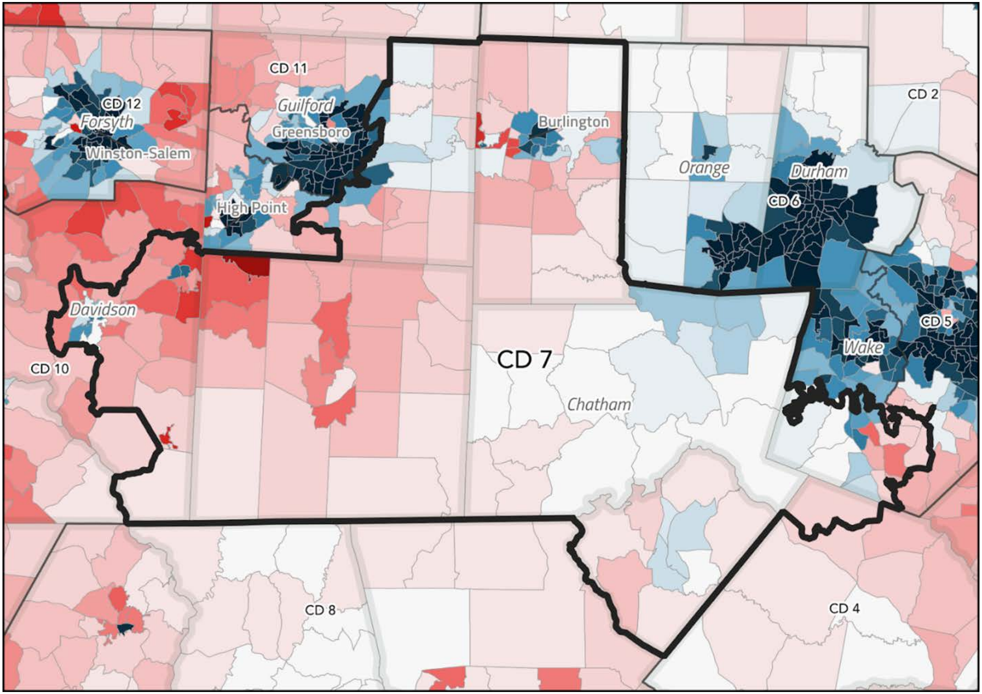
– Ex. 4353 –



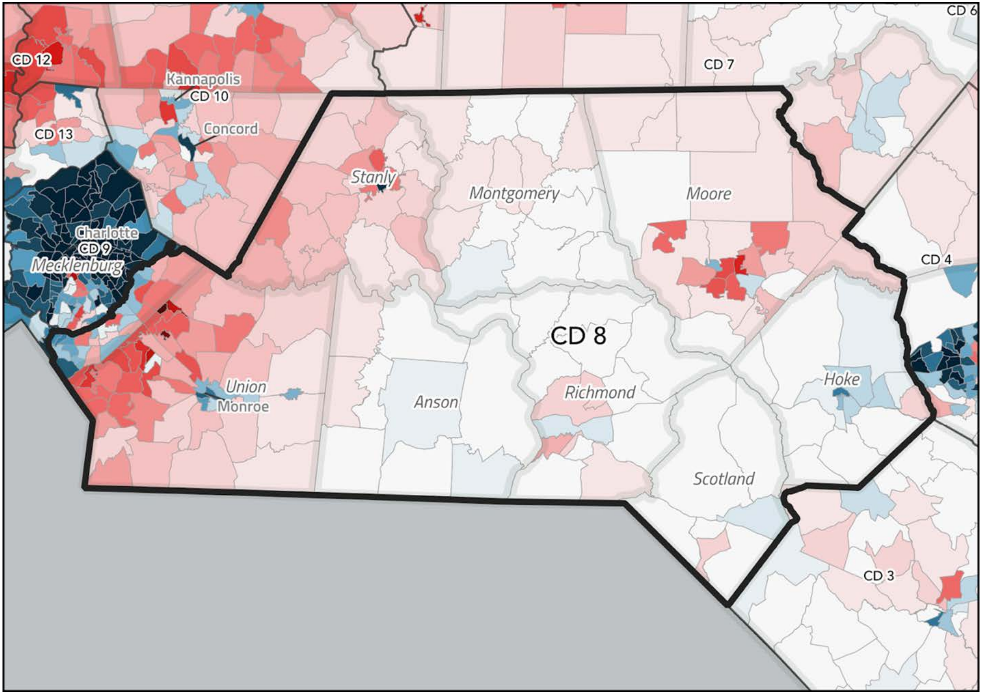
– Ex. 4354 –


CD 11
CD 2
Burlington
Orange
CD 6
Durham
CD 7
CD 5
Wake
Chatham
CD 4

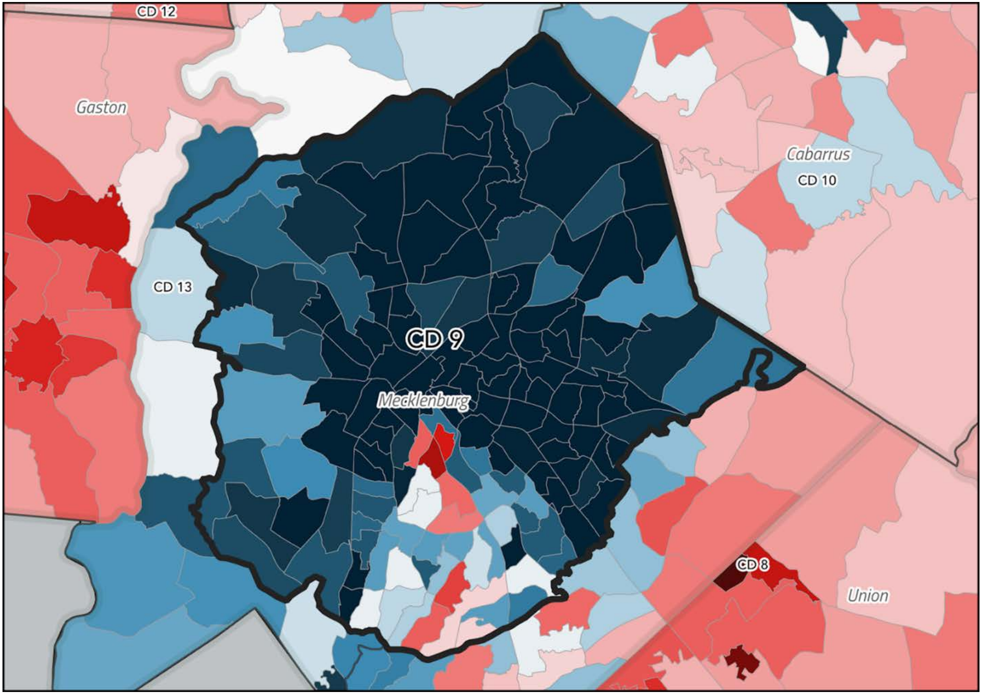
– Ex. 4355 –


CD 11
CD 12
Forsyth
Winston-Salem
Guilford
Greensboro
High Point
Burlington
Orange
Durham
CD 6
CD 2
Davidson
CD 10
CD 7
Chatham
CD 5
Wake
CD 8
CD 4

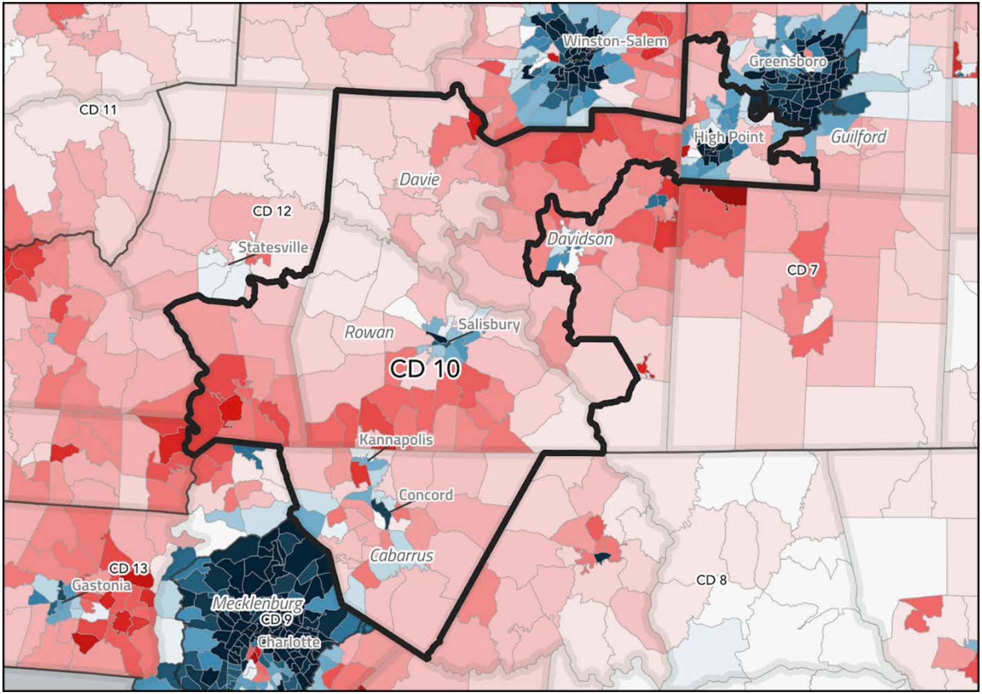
– Ex. 4356 –


CD 6
CD 12
CD 7
Kannapolis
CD 10
Concord
CD 13
Stanly
Montgomery
Moore
Charlotte
CD 9
Mecklenburg
CD 4
CD 8
Union
Monroe
Anson
Richmond
Hoke
Scotland
CD 3

– Ex. 4357 –

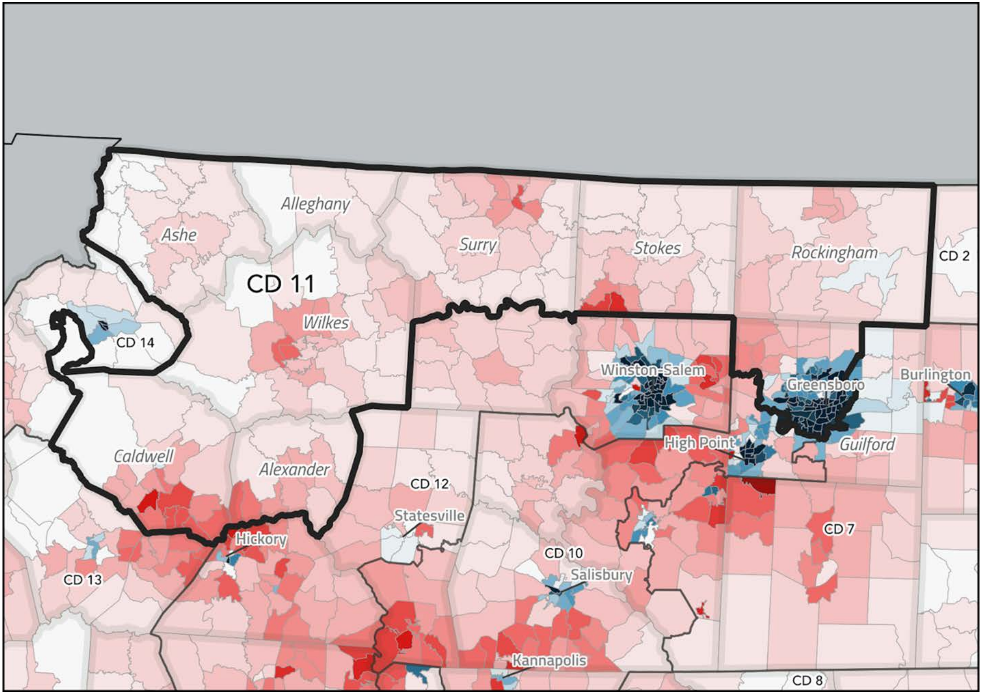


– Ex. 4358 –



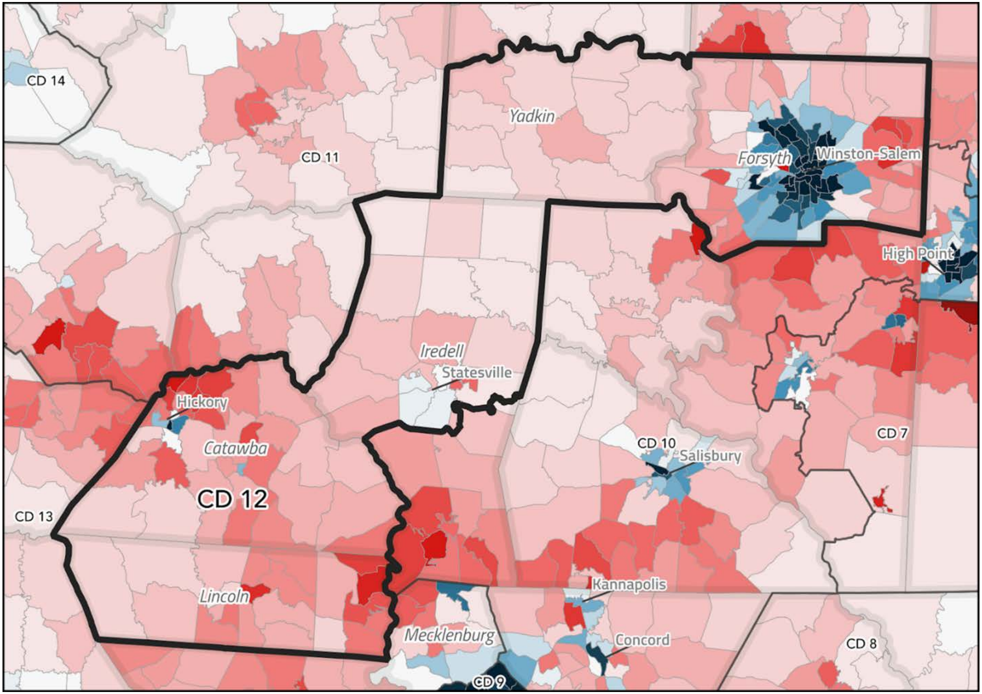
Winston-Salem
Greensboro
CD 11
High Point
Guilford
Davie
CD 12
Statesville
Davidson
CD 7
Salisbury
Rowan
CD 10
Kannapolis
Concord
Cabarrus
CD 13
Gastonia
CD 8
Mecklenburg
CD 9
Charlotte

– Ex. 4359 –

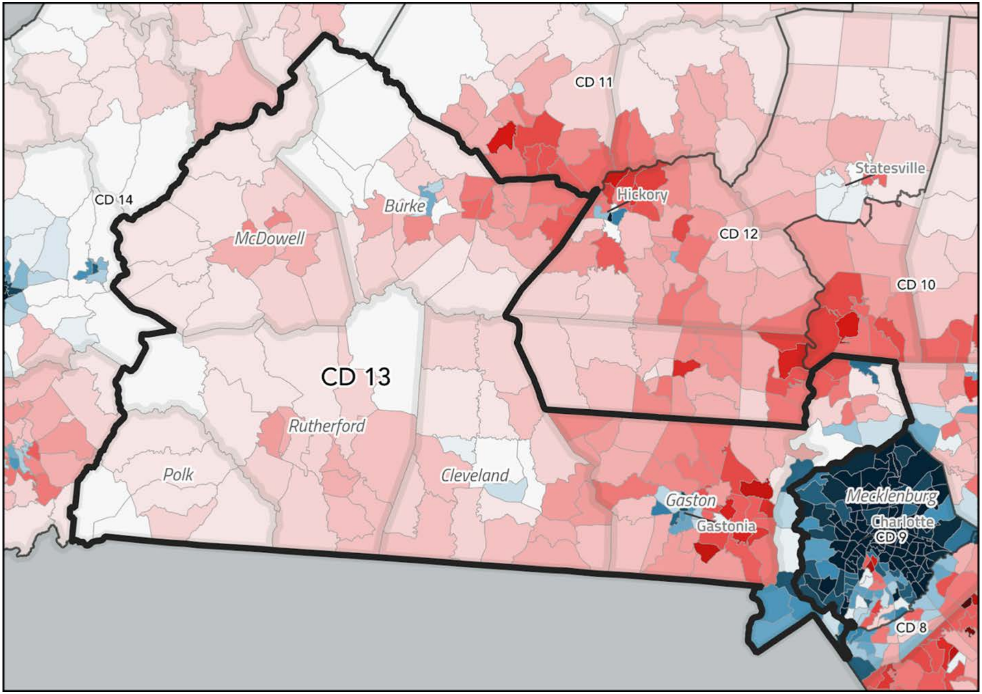


Alleghany
Ashe
Surry
Stokes
Rockingham
CD 2
CD 11
Wilkes
CD 14
Winston-Salem
Greensboro
Burlington
High Point
Guilford
Caldwell
Alexander
CD 12
Statesville
CD 7
Hickory
CD 10
Salisbury
CD 13
Kannapolis
CD 8

– Ex. 4360 –

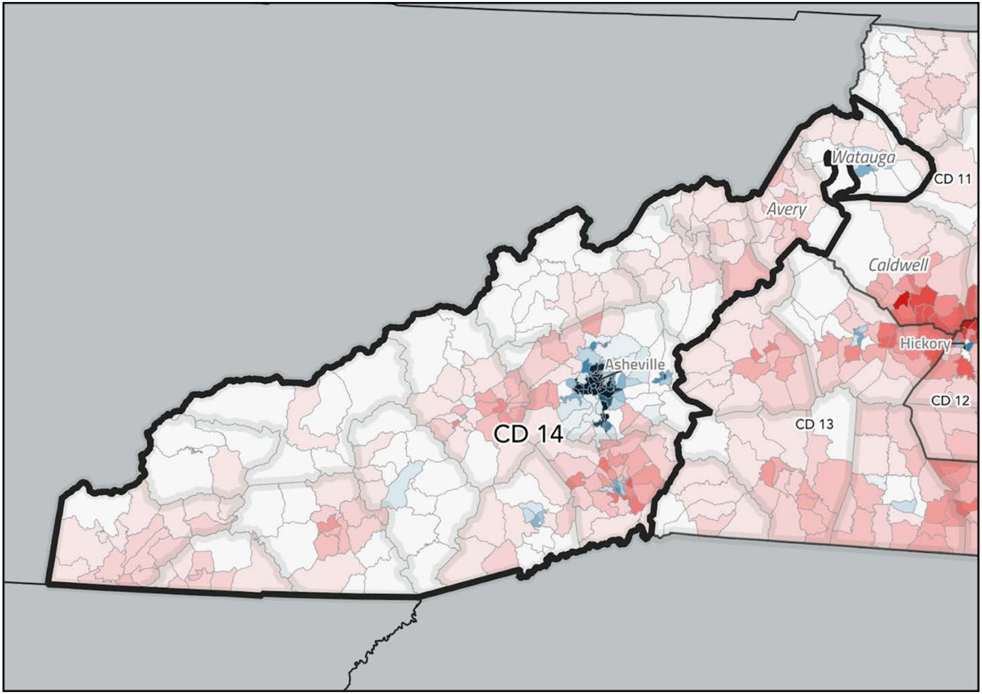

CD 14
CD 11
Yadkin
Forsyth
Winston-Salem
High Point
Iredell
Statesville
Hickory
Catawba
CD 12
CD 13
CD 10
Salisbury
CD 7
Lincoln
Kannapolis
Mecklenburg
Concord
CD 8
CD 9

– Ex. 4361 –

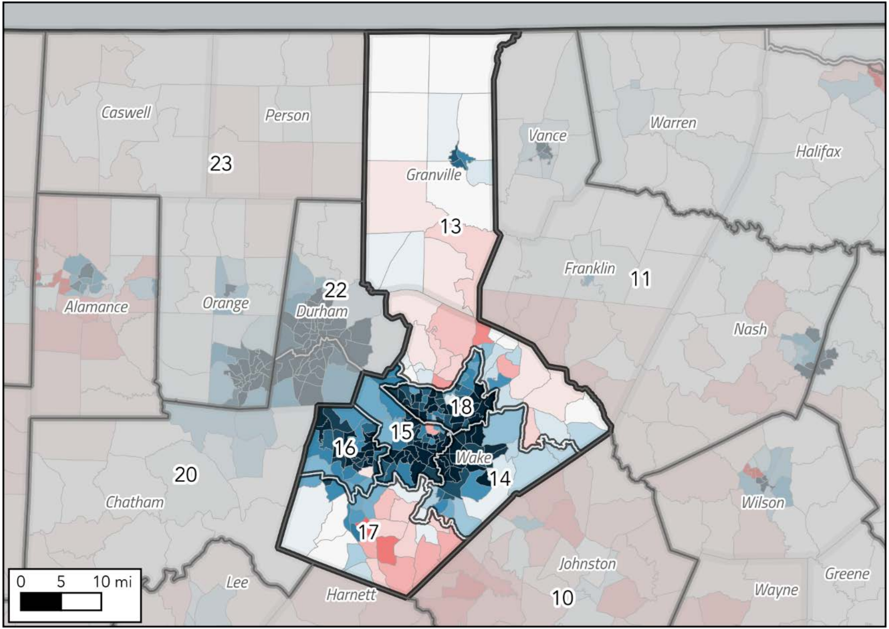


CD 11
CD 14
Statesville
Hickory
Burke
McDowell
CD 12
CD 10
CD 13
Rutherford
Polk
Cleveland
Gaston
Gastonia
Mecklenburg
Charlotte
CD 9
CD 8

– Ex. 4362 –

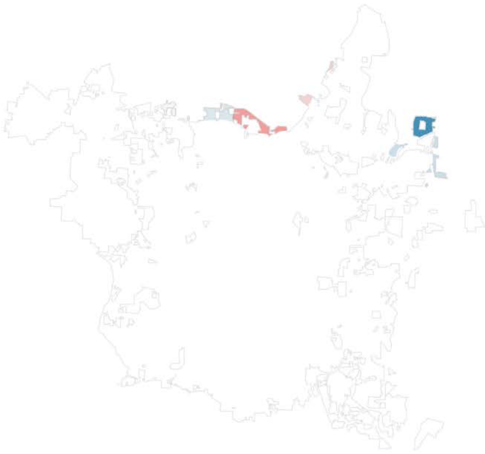

Watauga
CD 11
Avery
Caldwell
Hickory
Asheville
CD 12
CD 13
CD 14

– Ex. 4363 –

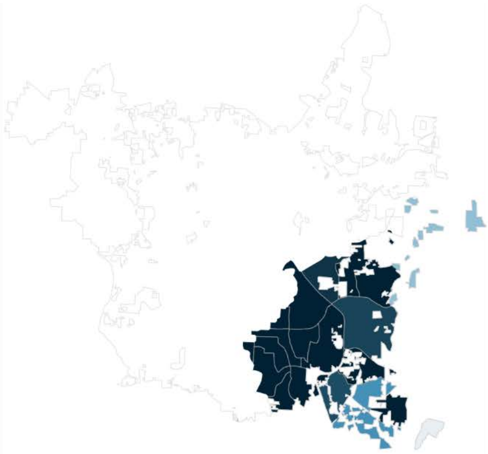


Caswell
Person
23
Granville
Vance
Warren
Halifax
13
Alamance
Orange
22
Durham
Franklin
11
Nash
18
15
16
Wake
14
20
Chatham
17
Wilson
Johnston
Lee
Harnett
10
Wayne
Greene
0 5 10 mi

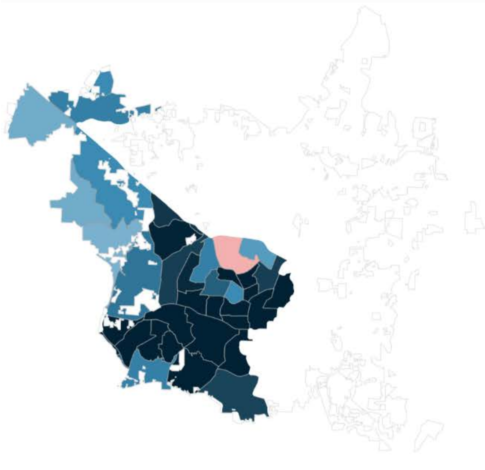
– Ex. 4364 –



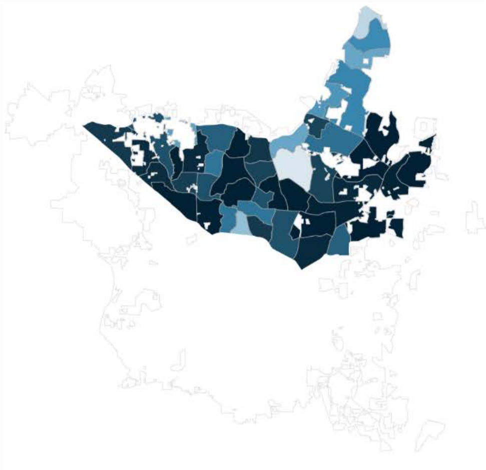
Portions of Raleigh City Limits (Shaded)
in Senate District 13



Portions of Raleigh City Limits (Shaded)
in Senate District 14



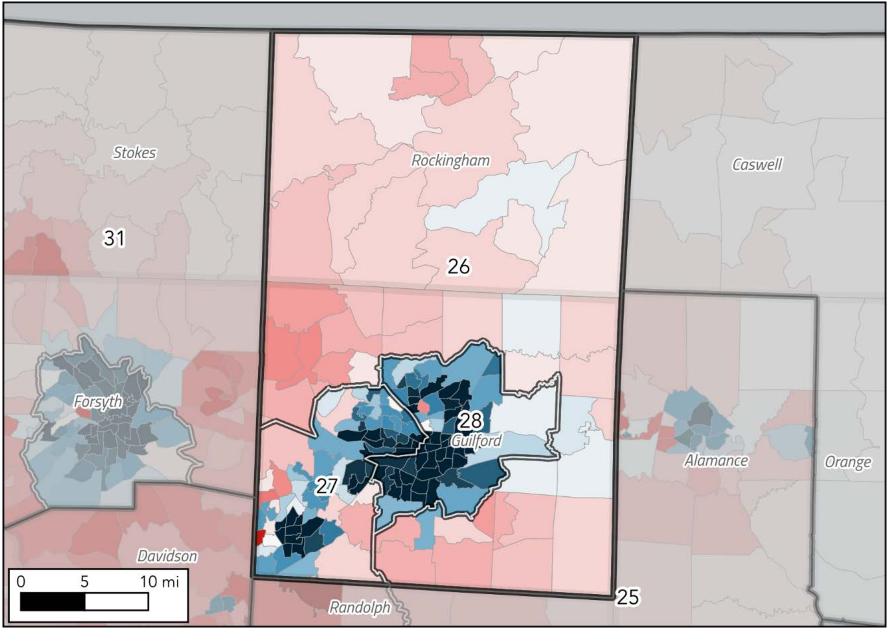
Portions of Raleigh City Limits (Shaded)
in Senate District 15



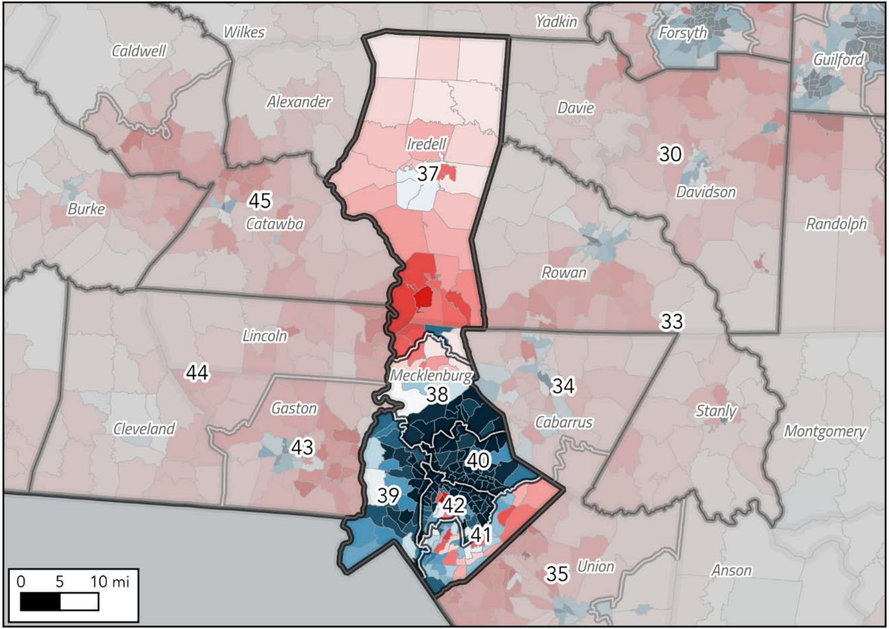
Portions of Raleigh City Limits (Shaded)
in Senate District 18

Very small portions of west and northwest Raleigh have been assigned to SD 16, 17, 20 and 22 (not shown)

– Ex. 4365 –

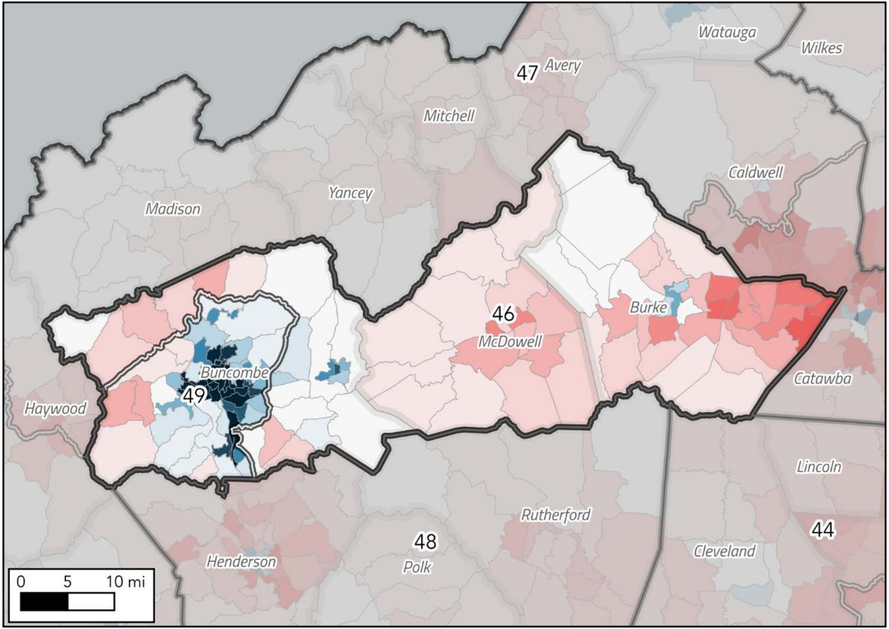

Stokes
Rockingham
Caswell
31
26
Forsyth
28
Guilford
Alamance
Orange
27
Davidson
0
5
10 mi
25
Randolph

– Ex. 4366 –

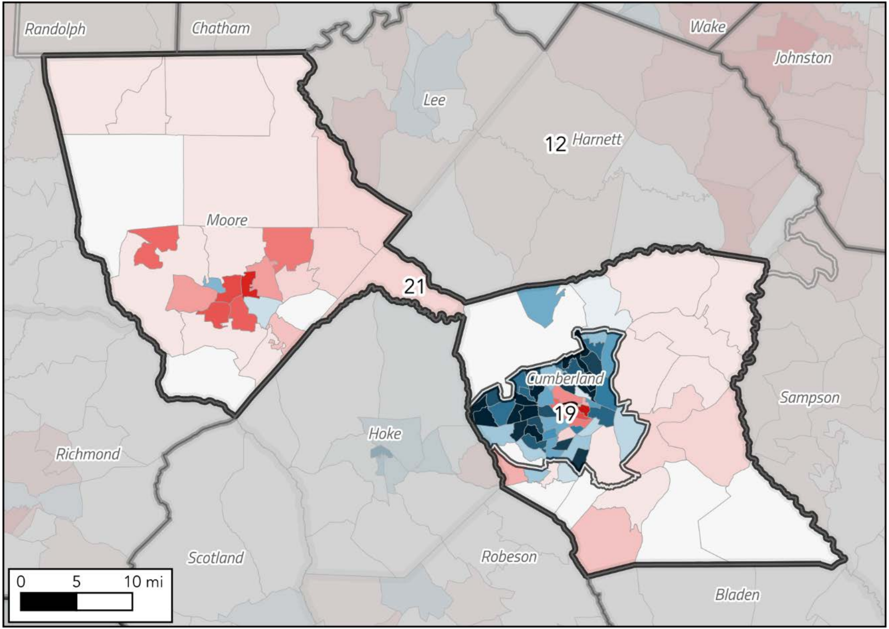


Wilkes
Caldwell
Yadkin
Forsyth
Guilford
Alexander
Davie
Iredell
37
30
Davidson
45
Burke
Catawba
Randolph
Rowan
33
Lincoln
44
Mecklenburg
38
34
Gaston
Cabarrus
Stanly
Cleveland
Montgomery
43
40
39
42
41
35
Union
Anson
0
5
10 mi

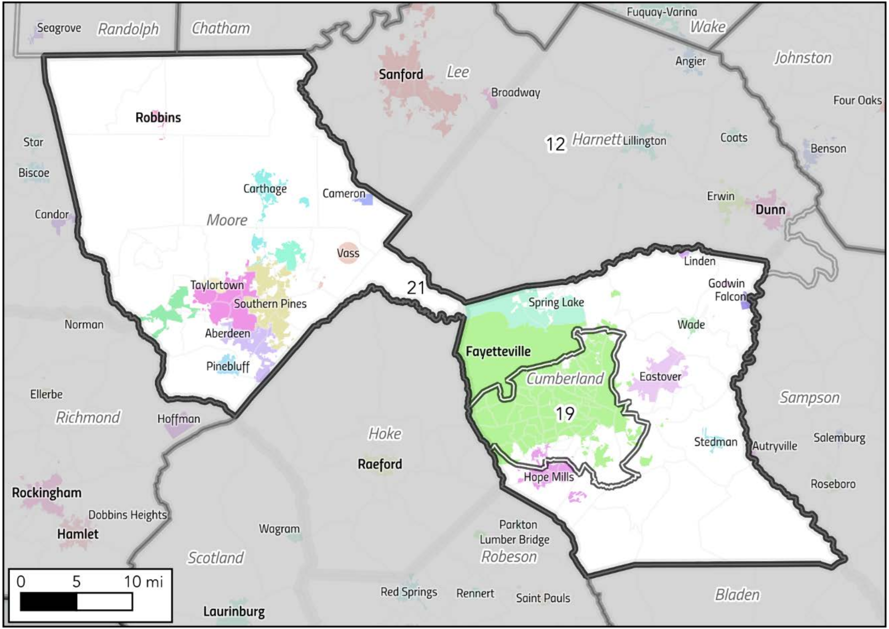
– Ex. 4367 –


Watauga
Wilkes
47
Avery
Mitchell
Caldwell
Yancey
Madison
46
Burke
McDowell
Buncombe
49
Catawba
Haywood
Lincoln
Rutherford
44
48
Cleveland
Henderson
Polk
0
5
10 mi

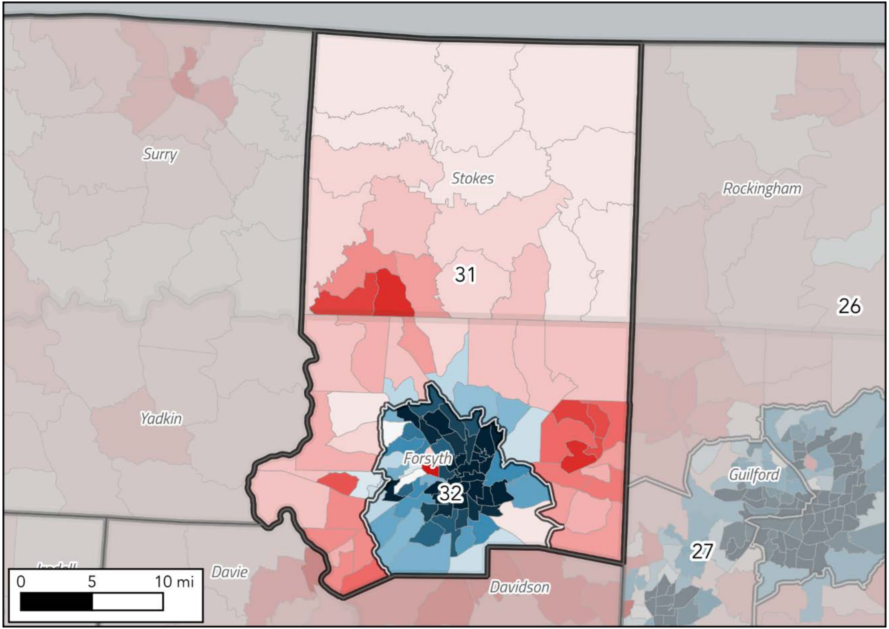
– Ex. 4368 –


Randolph
Chatham
Wake
Johnston
Lee
12 Harnett
Moore
21
Cumberland
19
Sampson
Hoke
Richmond
Scotland
Robeson
Bladen
0
5
10 mi

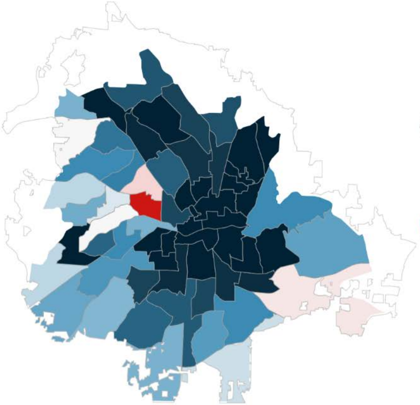
– Ex. 4369 –


Seagrove
Randolph
Chatham
Fuquay-Varina
Wake
Sanford
Lee
Angier
Johnston
Broadway
Four Oaks
Robbins
12
Harnett
Lillington
Coats
Benson
Star
Biscoe
Carthage
Cameron
Erwin
Dunn
Candor
Moore
Vass
Linden
Taylortown
21
Godwin
Falcon
Southern Pines
Spring Lake
Norman
Aberdeen
Wade
Fayetteville
Pinebluff
Cumberland
Eastover
Ellerbe
Sampson
Richmond
Hoffman
19
Hoke
Stedman
Salemburg
Autryville
Raeford
Hope Mills
Roseboro
Rockingham
Dobbins Heights
Parkton
Hamlet
Wagram
Lumber Bridge
Scotland
Robeson
0
5
10 mi
Red Springs
Rennert
Saint Pauls
Bladen
Laurinburg

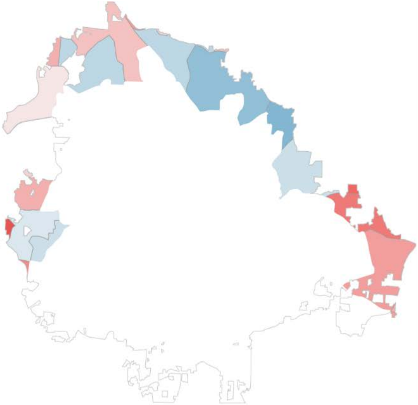
– Ex. 4370 –


Surry
Stokes
Rockingham
31
26
Yadkin
Forsyth
32
Guilford
27
Davie
Davidson
0
5
10 mi

– Ex. 4371 –

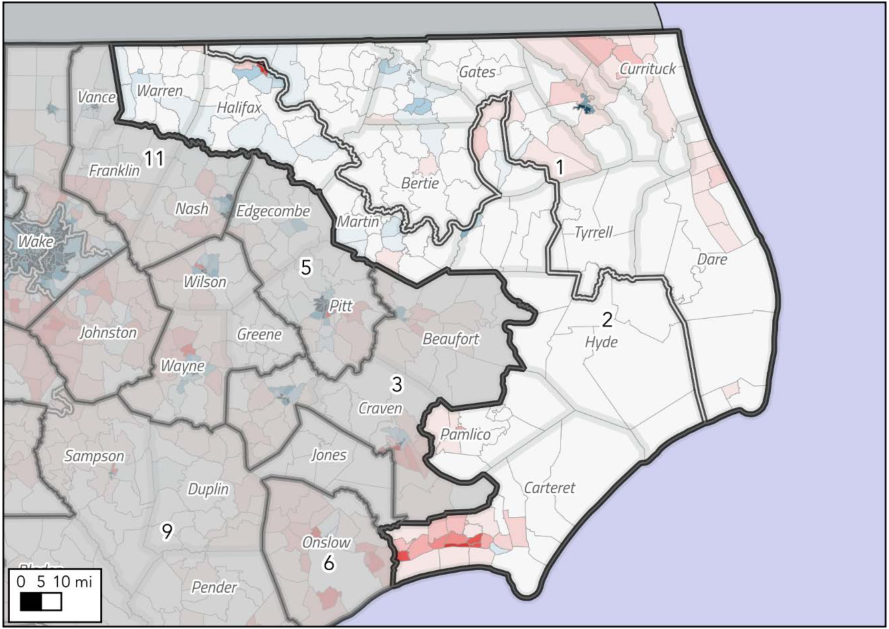


Portions of Winston-Salem City Limits (Shaded)
in Senate District 32



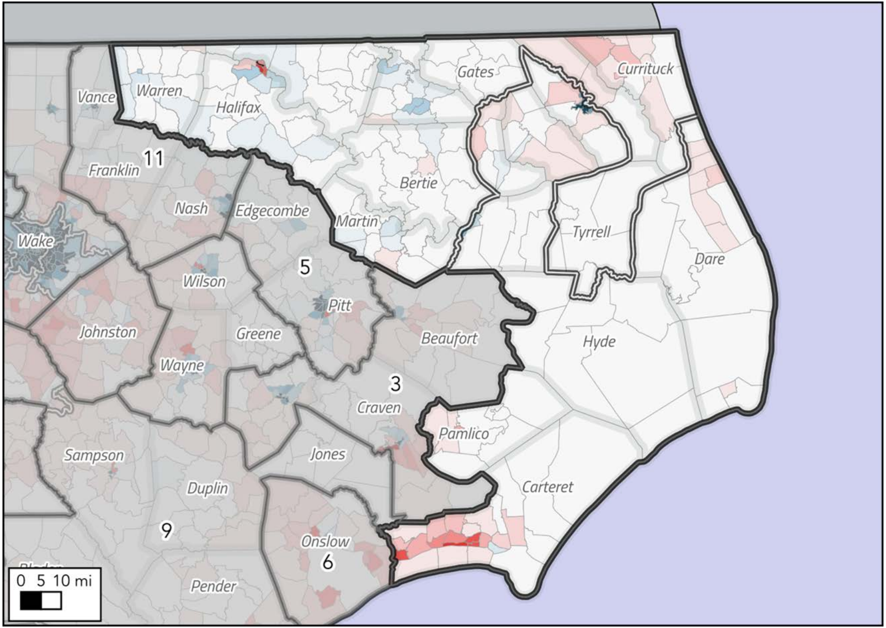
Portions of Winston-Salem City Limits (Shaded)
in Senate District 31

– Ex. 4372 –

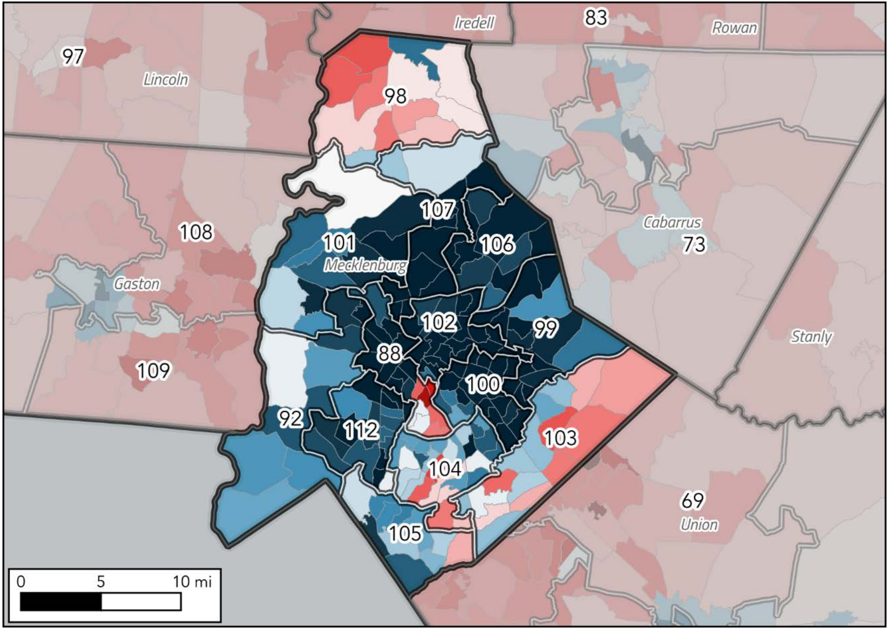


Gates
Currituck
Warren
Vance
Halifax
11
Franklin
1
Bertie
Nash
Edgecombe
Martin
Wake
Tyrrell
Dare
5
Wilson
Pitt
2
Hyde
Johnston
Greene
Beaufort
Wayne
3
Craven
Pamlico
Sampson
Jones
Duplin
Carteret
9
Onslow
6
Pender
0 5 10 mi

– Ex. 4373 –

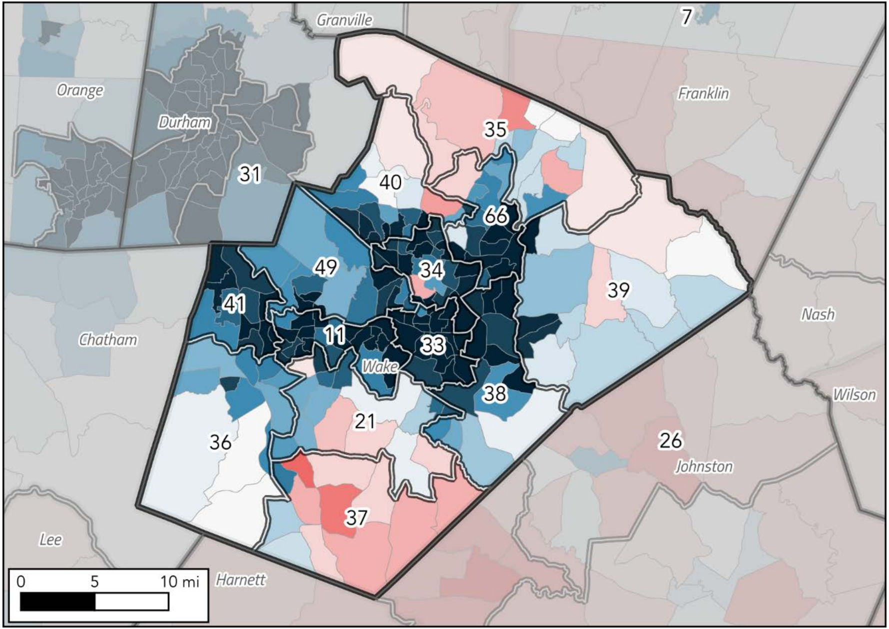

Gates
Currituck
Vance
Warren
Halifax
11
Franklin
Bertie
Nash
Edgecombe
Martin
Tyrrell
Wake
5
Dare
Wilson
Pitt
Johnston
Greene
Beaufort
Hyde
Wayne
3
Craven
Pamlico
Sampson
Jones
Carteret
Duplin
9
Onslow
6
0 5 10 mi
Pender

– Ex. 4374 –



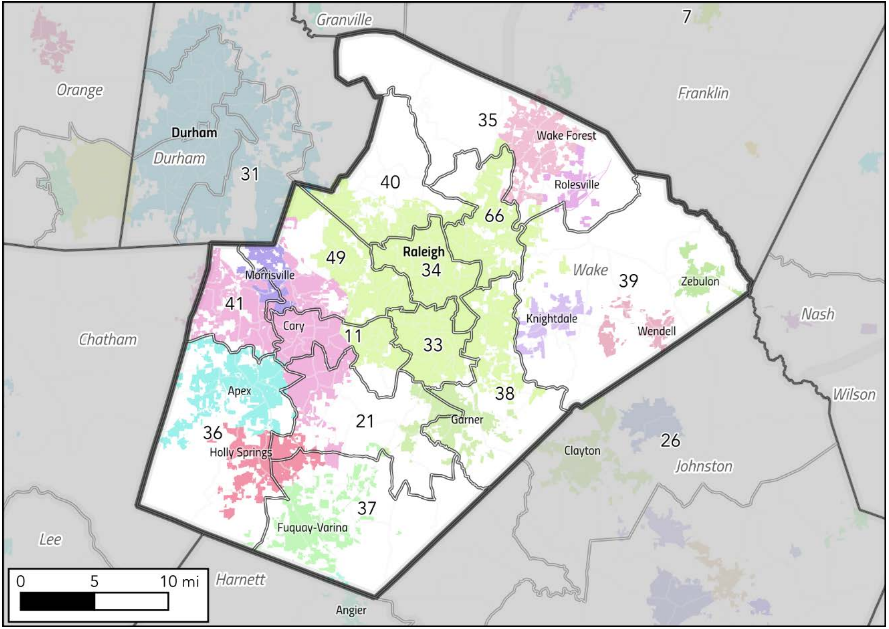
Iredell
83
Rowan
97
Lincoln
98
107
108
101
106
Cabarrus
73
Mecklenburg
Gaston
102
99
Stanly
88
109
100
92
112
103
104
69
Union
105
0
5
10 mi

– Ex. 4375 –



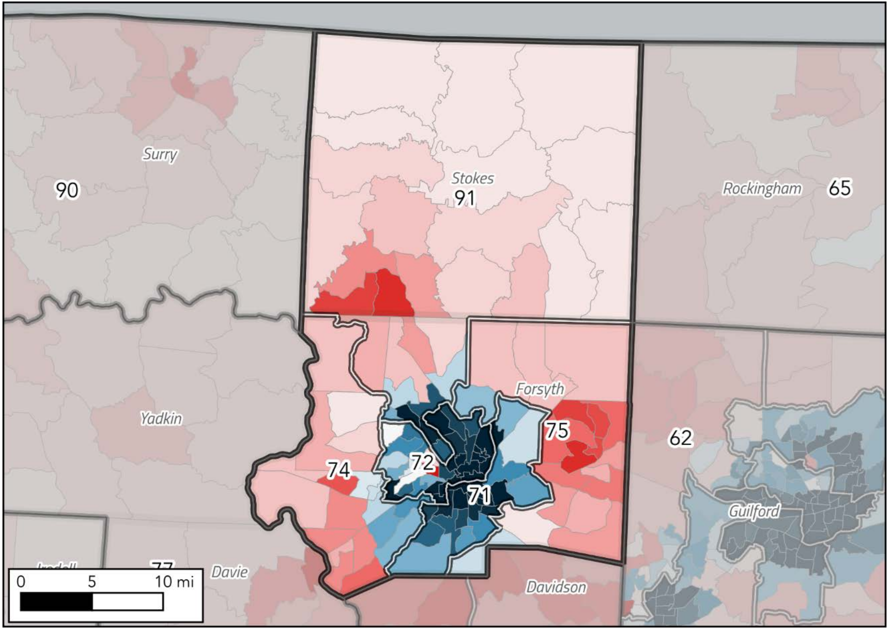
Granville
7
Orange
Durham
Franklin
35
31
40
66
49
34
39
41
Nash
11
Chatham
33
Wake
38
Wilson
21
36
26
Johnston
37
Lee
0
5
10 mi
Harnett

– Ex. 4376 –

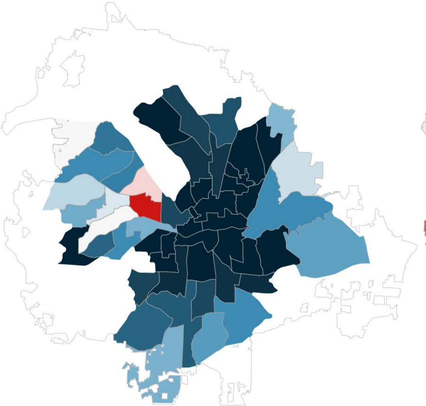


Granville
7
Orange
Franklin
Durham
Durham
31
35
Wake Forest
Rolesville
40
66
Raleigh
34
49
Morrisville
Wake
39
Zebulon
41
Cary
11
Knightdale
Wendell
Nash
Chatham
33
Apex
38
Wilson
21
Garner
36
Holly Springs
26
Clayton
Johnston
37
Fuquay-Varina
Lee
0
5
10 mi
Harnett
Angier

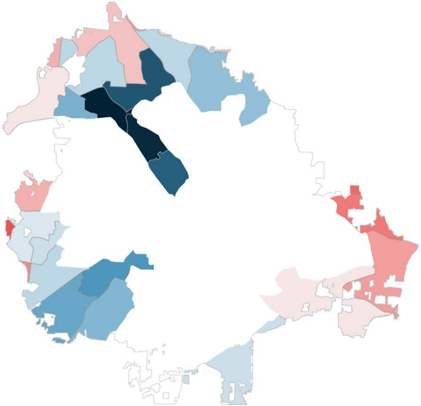
– Ex. 4377 –


Surry
90
Stokes
91
Rockingham
65
Forsyth
75
62
Yadkin
72
74
71
Guilford
Davie
Davidson
0
5
10 mi

– Ex. 4378 –

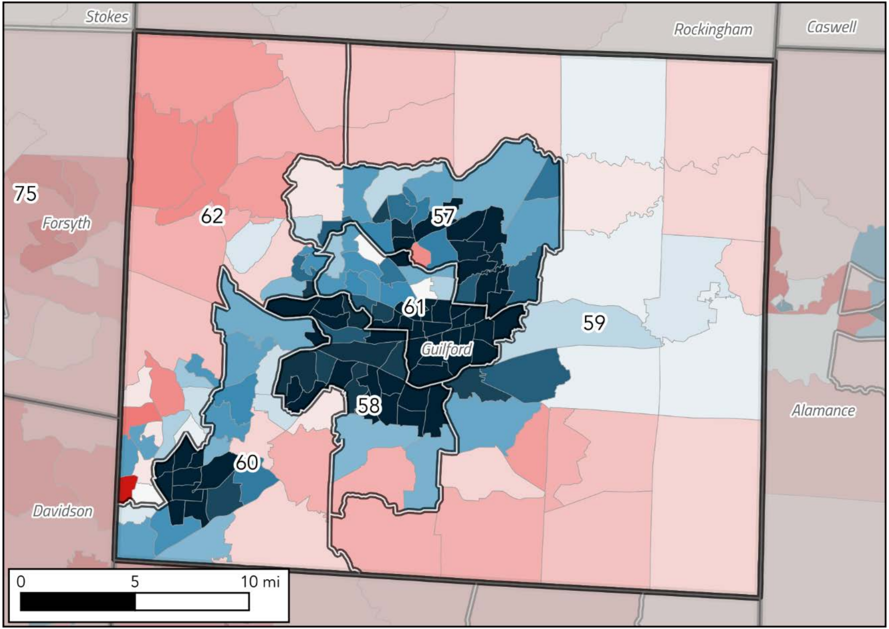


Portions of Winston-Salem City Limits (Shaded)
in House District 71, 72



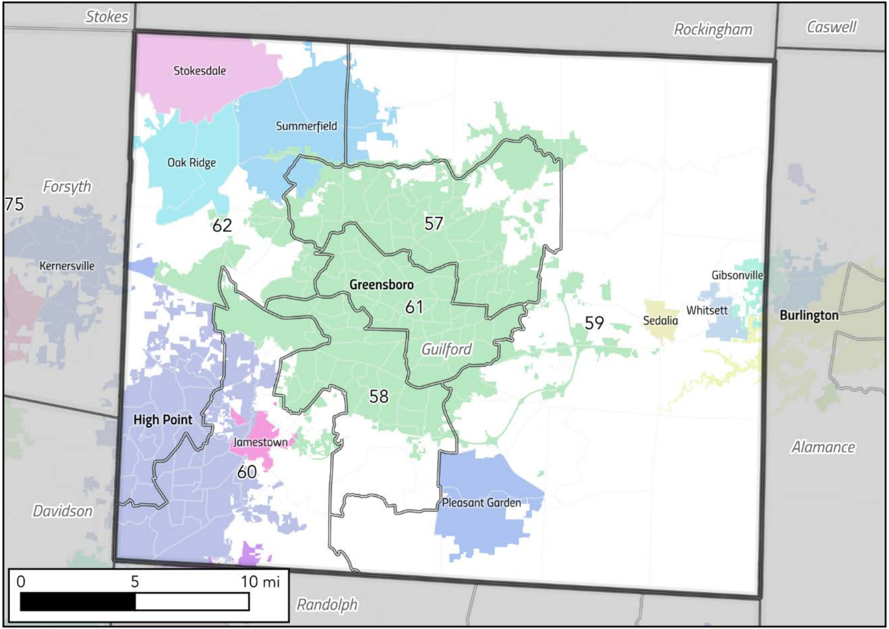
Portions of Winston-Salem City Limits (Shaded)
in House District 74, 75, 91

– Ex. 4379 –



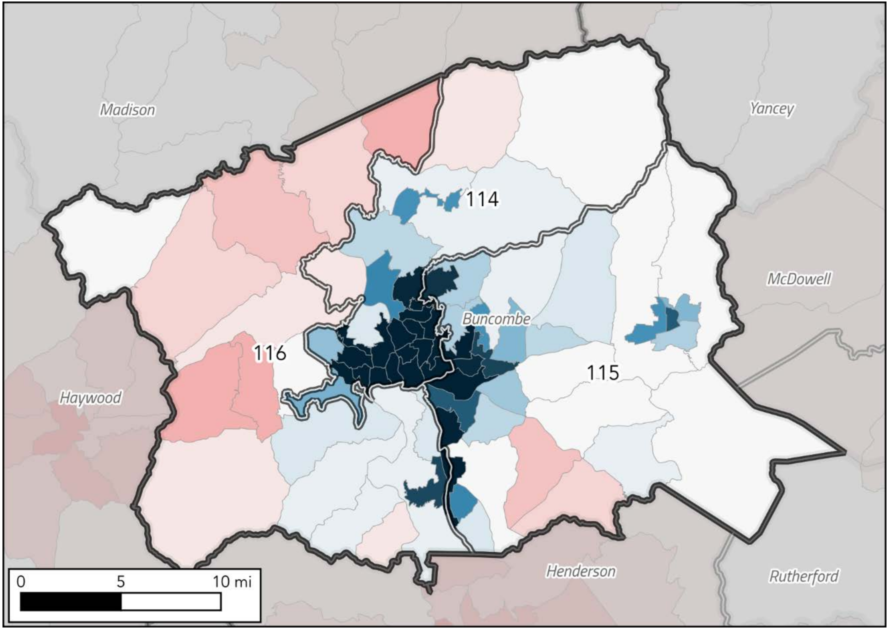
Stokes
Rockingham
Caswell
75
Forsyth
62
57
61
59
Guilford
58
60
Alamance
Davidson
0
5
10 mi

– Ex. 4380 –



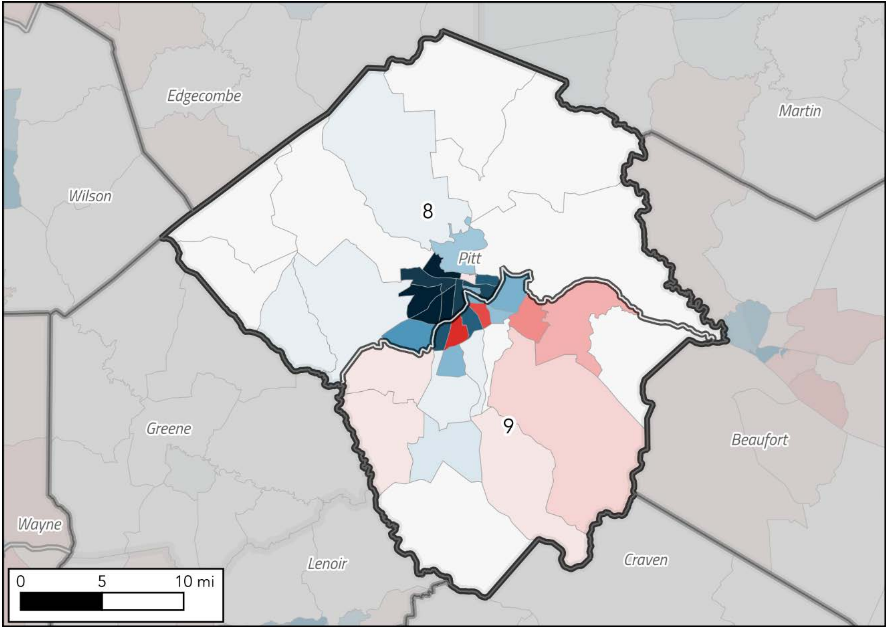
Stokes
Rockingham
Caswell
Stokesdale
Summerfield
Oak Ridge
Forsyth
75
62
57
Kernersville
Greensboro
61
Gibsonville
Whitsett
Sedalia
59
Burlington
Guilford
58
High Point
Jamestown
60
Alamance
Davidson
Pleasant Garden
0
5
10 mi
Randolph

– Ex. 4381 –

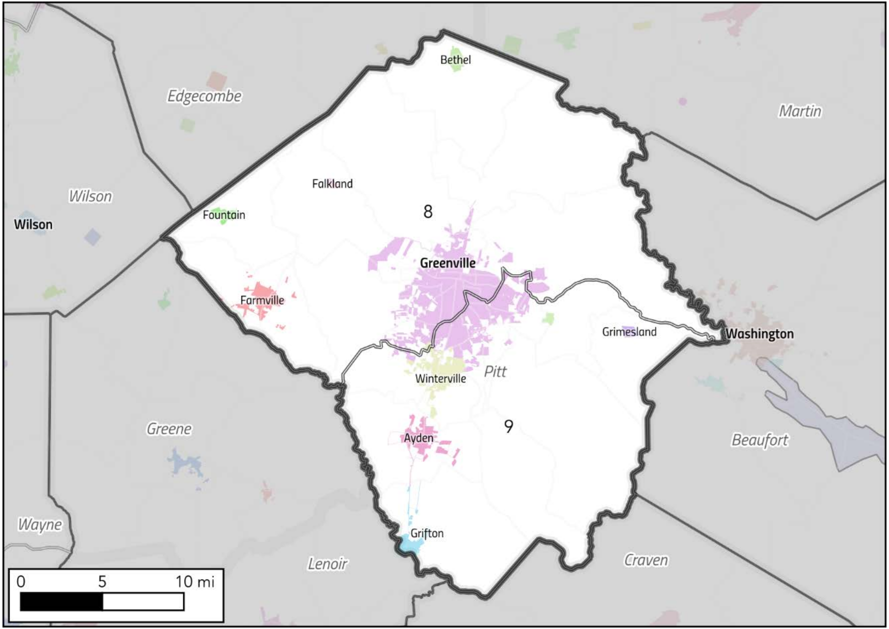


Madison
Yancey
114
McDowell
Buncombe
116
115
Haywood
Henderson
Rutherford
0
5
10 mi

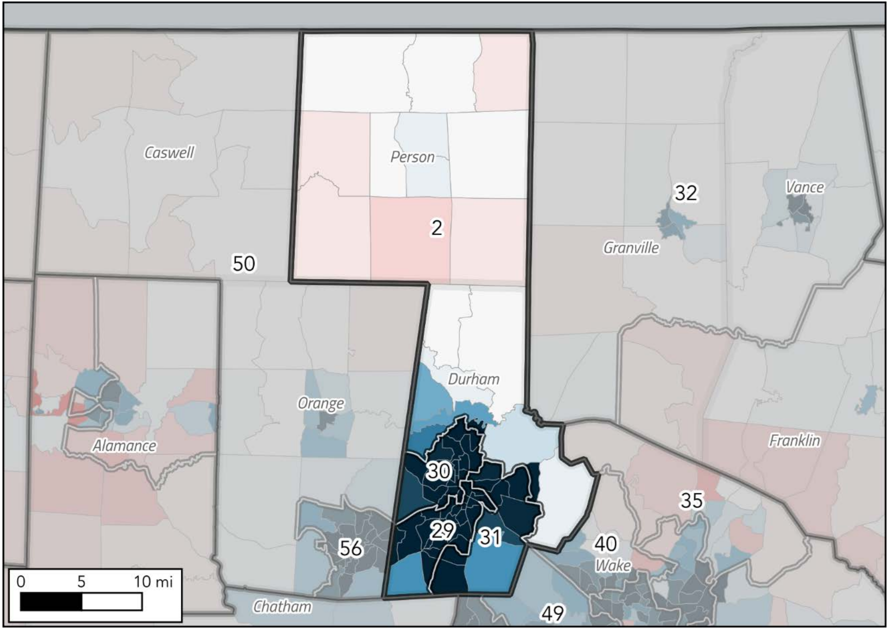
– Ex. 4382 –


Edgecombe
Martin
Wilson
8
Pitt
Greene
9
Beaufort
Wayne
Lenoir
Craven
0
5
10 mi

– Ex. 4383 –

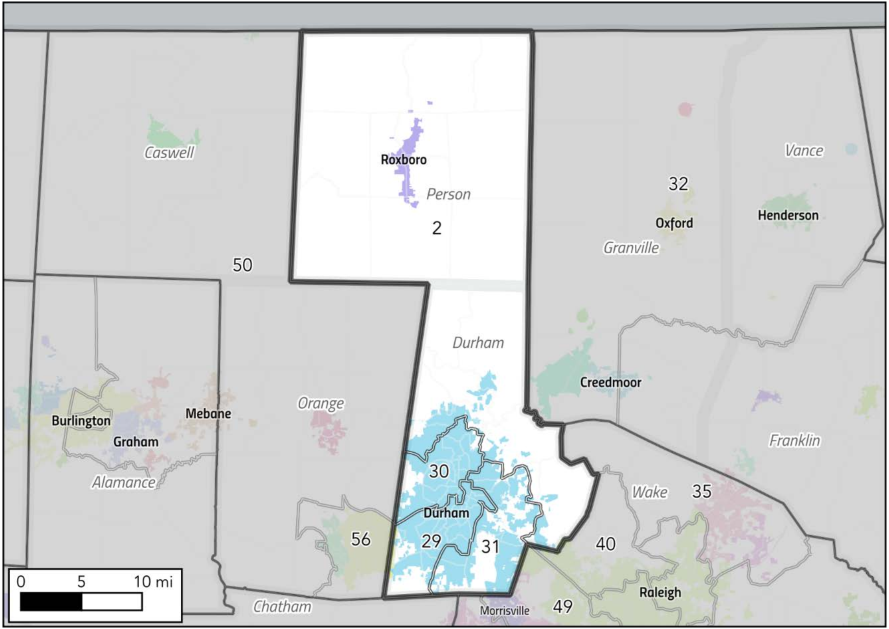

Bethel
Edgecombe
Martin
Falkland
Wilson
8
Fountain
Wilson
Greenville
Farmville
Grimesland
Washington
Pitt
Winterville
9
Greene
Ayden
Beaufort
Wayne
Grifton
Craven
Lenoir
0
5
10 mi

– Ex. 4384 –

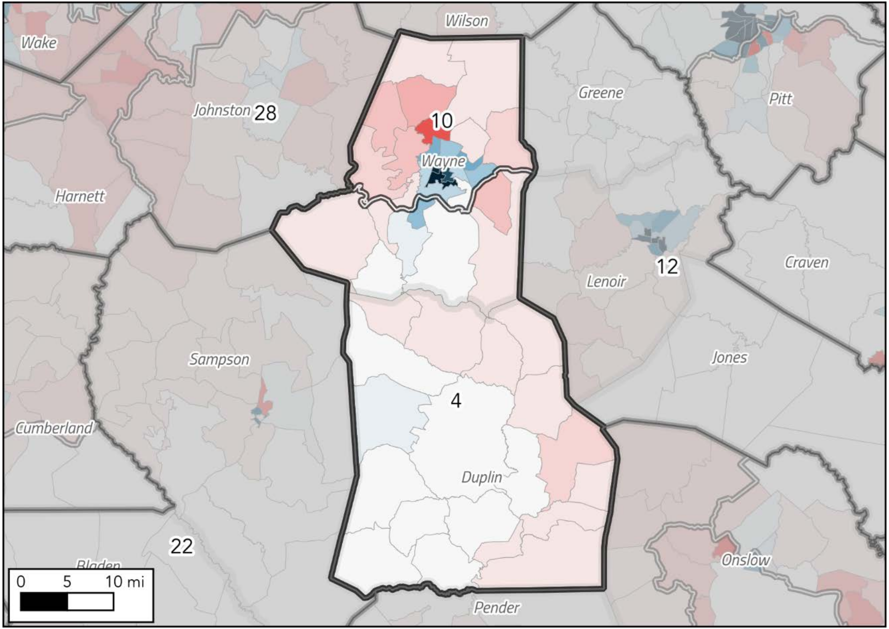


Caswell
Person
2
50
32
Vance
Granville
Durham
Orange
Alamance
Franklin
30
35
29
31
40
56
Wake
0
5
10 mi
Chatham
49

– Ex. 4385 –

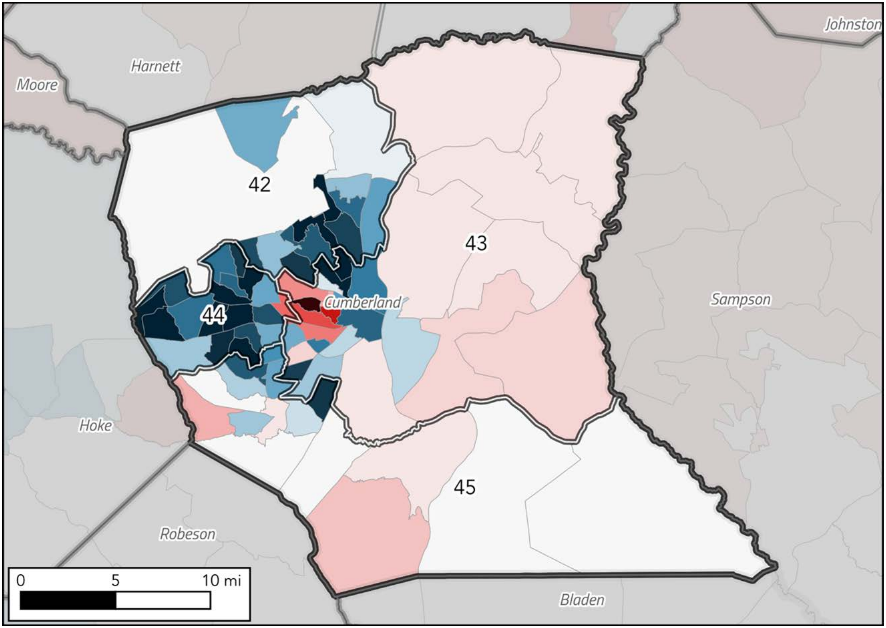

Caswell
Roxboro
Person
2
50
Vance
32
Oxford
Henderson
Granville
Durham
Creedmoor
Orange
Mebane
Burlington
Graham
Alamance
Franklin
30
Wake
35
Durham
56
29
31
40
0
5
10 mi
Chatham
Raleigh
Morrisville
49

– Ex. 4386 –



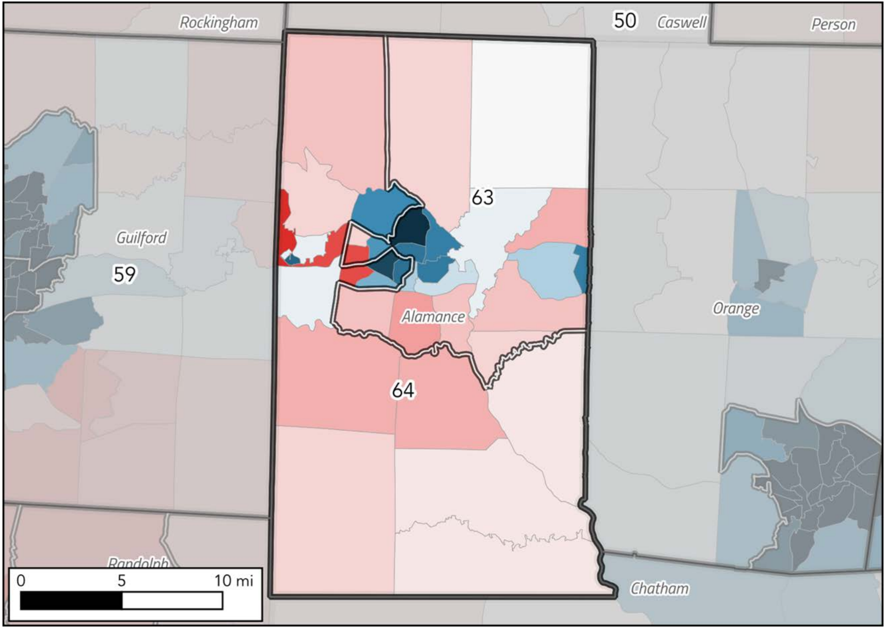
Wilson
Wake
Greene
Pitt
Johnston
28
10
Wayne
Harnett
12
Craven
Lenoir
Sampson
Jones
4
Cumberland
Duplin
22
Bladen
Onslow
0 5 10 mi
Pender

– Ex. 4387 –



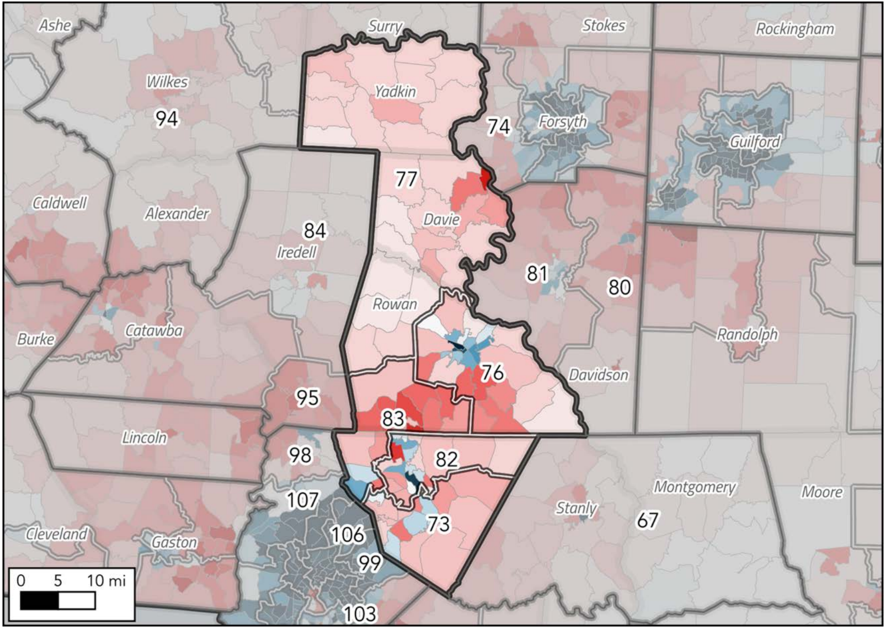
Johnston
Harnett
Moore
42
43
Sampson
Cumberland
44
Hoke
45
Robeson
0
5
10 mi
Bladen

– Ex. 4388 –

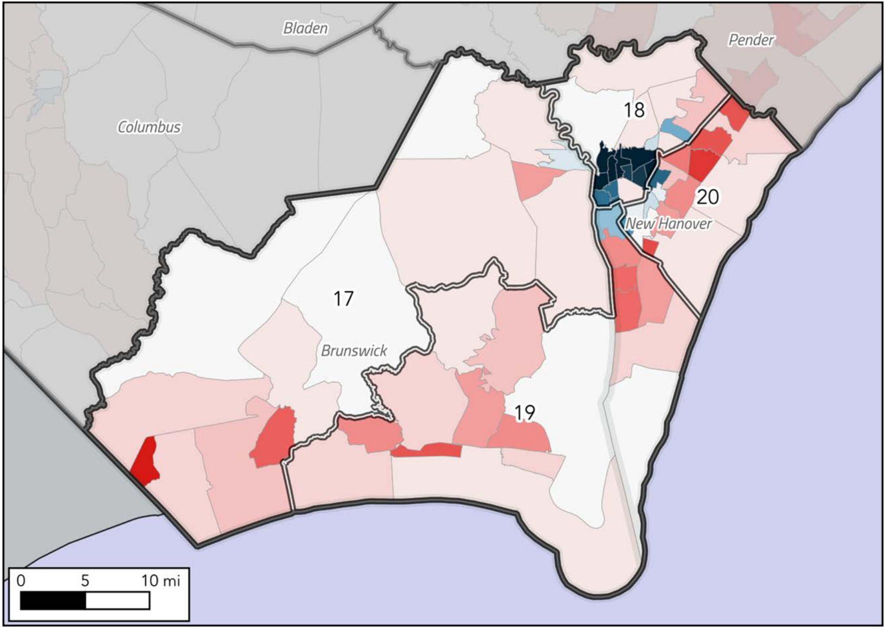


Rockingham
50
Caswell
Person
63
Guilford
59
Orange
Alamance
64
Randolph
0
5
10 mi
Chatham

– Ex. 4389 –


Ashe
Surry
Stokes
Rockingham
Wilkes
Yadkin
94
74
Forsyth
Guilford
77
Caldwell
Alexander
Davie
84
Iredell
81
80
Rowan
Burke
Catawba
Randolph
76
Davidson
95
83
Lincoln
98
82
107
Montgomery
Moore
Stanly
73
67
Cleveland
Gaston
106
99
103
0 5 10 mi

– Ex. 4390 –



Bladen
Pender
Columbus
18
20
New Hanover
17
Brunswick
19
0
5
10 mi

STATE OF NORTH CAROLINA

COUNTY OF WAKE

IN THE GENERAL COURT OF JUSTICE
SUPERIOR COURT DIVISION
No. 21 CVS 015426
No. 21 CVS 500085

NORTH CAROLINA LEAGUE OF CONSERVATION VOTERS, INC., *et al.*,

Plaintiffs,

v.

REPRESENTATIVE DESTIN HALL, IN HIS OFFICIAL CAPACITY AS SENIOR CHAIR OF THE HOUSE STANDING COMMITTEE ON REDISTRICTING, *et al.*,

Defendants.

REBECCA HARPER, *et al.*,

Plaintiffs,

v.

REPRESENTATIVE DESTIN HALL, IN HIS OFFICIAL CAPACITY AS SENIOR CHAIR OF THE HOUSE STANDING COMMITTEE ON REDISTRICTING, *et al.*,

Defendants.

COMMON CAUSE,

Plaintiff,

v.

REPRESENTATIVE DESTIN HALL, IN HIS OFFICIAL CAPACITY AS SENIOR CHAIR OF THE HOUSE STANDING COMMITTEE ON REDISTRICTING, *et al.*,

Defendants.

EXPERT REPORT OF DR. JOWEI CHEN

I, Dr. Jowei Chen, upon my oath, declare and say as follows:

1. I am over the age of eighteen (18) and competent to testify as to the matters set forth herein.

2. I am an Associate Professor in the Department of Political Science at the University of Michigan, Ann Arbor. I am also a Research Associate Professor at the Center for Political Studies of the Institute for Social Research at the University of Michigan and a Research Associate at the Spatial Social Science Laboratory at Stanford University. In 2007, I received a M.S. in Statistics from Stanford University, and in 2009, I received a Ph.D. in Political Science from Stanford University.

3. I have published academic papers on legislative districting and political geography in several political science journals, including The American Journal of Political Science and The American Political Science Review, and Election Law Journal. My academic areas of expertise include legislative elections, spatial statistics, geographic information systems (GIS) data, redistricting, racial politics, legislatures, and political geography. I have expertise in the use of computer simulations of legislative districting and in analyzing political geography, elections, and redistricting.

4. I have authored expert reports in the following redistricting court cases: *The League of Women Voters of Florida v. Detzner* (Fla. 2d Judicial Cir. Leon Cnty. 2012); *Romo v. Detzner* (Fla. 2d Judicial Cir. Leon Cnty. 2013); *Missouri National Association for the Advancement of Colored People v. Ferguson-Florissant School District & St. Louis County Board of Election Commissioners* (E.D. Mo. 2014); *Raleigh Wake Citizens Association v. Wake County Board of Elections* (E.D.N.C. 2015); *Brown v. Detzner* (N.D. Fla. 2015); *City of Greensboro v. Guilford County Board of Elections* (M.D.N.C. 2015); *Common Cause v. Rucho*

(M.D.N.C 2016); *The League of Women Voters of Pennsylvania v. Commonwealth of Pennsylvania* (No. 261 M.D. 2017); *Georgia State Conference of the NAACP v. The State of Georgia* (N.D. Ga. 2017); *The League of Women Voters of Michigan v. Johnson* (E.D. Mich. 2017); *Whitford v. Gill* (W.D. Wis. 2018); *Common Cause v. Lewis* (N.C. Super. 2018); *Harper v. Lewis* (N.C. Super. 2019); *Baroody v. City of Quincy, Florida* (N.D. Fla. 2020*); McConchie v. Illinois State Board of Elections* (N.D. Ill. 2021). I have testified either at deposition or at trial in the following cases: *Romo v. Detzner* (Fla. 2d Judicial Cir. Leon Cnty. 2013); *Missouri National Association for the Advancement of Colored People v. Ferguson-Florissant School District & St. Louis County Board of Election Commissioners* (E.D. Mo. 2014); *Raleigh Wake Citizens Association v. Wake County Board of Elections* (E.D.N.C. 2015); *City of Greensboro v. Guilford County Board of Elections* (M.D.N.C. 2015); *Common Cause v. Rucho* (M.D.N.C. 2016); *The League of Women Voters of Pennsylvania v. Commonwealth of Pennsylvania* (No. 261 M.D. 2017); *Georgia State Conference of the NAACP v. The State of Georgia* (N.D. Ga. 2017); *The League of Women Voters of Michigan v. Johnson* (E.D. Mich. 2017); *Whitford v. Gill* (W.D. Wis. 2018); *Common Cause v. Lewis* (N.C. Super. 2018); *Baroody v. City of Quincy, Florida* (N.D. Fla. 2020); *McConchie v. Illinois State Board of Elections* (N.D. Ill. 2021).

5. I have been retained by Plaintiffs in the above-captioned matter. I am being compensated $550 per hour for my work in this case.

6. Plaintiffs' counsel asked me to analyze the SB 740 districting plan for North Carolina's congressional districts (the "Enacted Plan"), as passed on November 4, 2021. Plaintiffs' counsel asked me to produce a set of computer-simulated plans for North Carolina's congressional districts by following the criteria adopted by the North Carolina General Assembly's Joint Redistricting Committee on August 12, 2021 (the "Adopted Criteria").

Plaintiffs' counsel asked me to compare the district-level partisan attributes of the Enacted Plan to those of the computer-simulated plans and to identify any districts in the Enacted Plan that are partisan outliers. Plaintiffs' counsel also asked me to compare the partisan composition of the individual Plaintiffs' congressional districts under the Enacted Plan to the partisan composition of Plaintiffs' districts under the computer-simulated plans and to identify any Plaintiffs whose Enacted Plan districts are partisan outliers.

7. The Use of Computer-Simulated Districting Plans: In conducting my academic research on legislative districting, partisan and racial gerrymandering, and electoral bias, I have developed various computer simulation programming techniques that allow me to produce a large number of nonpartisan districting plans that adhere to traditional districting criteria using US Census geographies as building blocks. This simulation process ignores all partisan and racial considerations when drawing districts. Instead, the computer simulations are programmed to draw districting plans following various traditional districting goals, such as equalizing population, avoiding county and Voting Tabulation District (VTD) splits, and pursuing geographic compactness. By randomly generating a large number of districting plans that closely adhere to these traditional districting criteria, I am able to assess an enacted plan drawn by a state legislature and determine whether partisan goals motivated the legislature to deviate from these traditional districting criteria. More specifically, by holding constant the application of nonpartisan, traditional districting criteria through the simulations, I am able to determine whether the enacted plan could have been the product of something other than partisan considerations. With respect to North Carolina's 2021 Congressional Enacted Plan, I determined that it could not.

8. I produced a set of 1,000 valid computer-simulated plans for North Carolina's congressional districts using a computer algorithm programmed to strictly follow the required districting criteria enumerated in the August 12, 2021 Adopted Criteria of the General Assembly's Joint Redistricting Committee. In following these Adopted Criteria, the computer algorithm uses the same general approach that I employed in creating the simulated state House and state Senate plans that I analyzed in *Common Cause v. Lewis* (2019) and the simulated congressional plans that I used in *Harper v. Lewis* (2019).

9. By randomly drawing districting plans with a process designed to strictly follow nonpartisan districting criteria, the computer simulation process gives us an indication of the range of districting plans that plausibly and likely emerge when map-drawers are not motivated primarily by partisan goals. By comparing the Enacted Plan against the distribution of simulated plans with respect to partisan measurements, I am able to determine the extent to which a map-drawer's subordination of nonpartisan districting criteria, such as geographic compactness and preserving precinct boundaries, was motivated by partisan goals.

10. These computer simulation methods are widely used by academic scholars to analyze districting maps. For over a decade, political scientists have used such computer-simulated districting techniques to analyze the racial and partisan intent of legislative map-drawers.[1] In recent years, several courts have also relied upon computer simulations to assess partisan bias in enacted districting plans.[2]

---

[1] *E.g.*, Carmen Cirincione, Thomas A. Darling, Timothy G. O'Rourke. "Assessing South Carolina's 1990s Congressional Districting," Political Geography 19 (2000) 189–211; Jowei Chen, "The Impact of Political Geography on Wisconsin Redistricting: An Analysis of Wisconsin's Act 43 Assembly Districting Plan." Election Law Journal.

[2] *See, e.g.*, *League of Women Voters of Pa. v. Commonwealth*, 178 A. 3d 737, 818-21 (Pa. 2018); *Raleigh Wake Citizens Association v. Wake County Board of Elections*, 827 F.3d 333, 344-45 (4th Cir. 2016); *City of Greensboro v. Guilford County Board of Elections*, No. 1:15-CV-599, 2017 WL 1229736 (M.D.N.C. Apr 3, 2017); *Common Cause v. Rucho*, No. 1:16-CV-1164 (M.D.N.C. Jan 11, 2018); *The League of Women Voters of Michigan v. Johnson* (E.D. Mich. 2017); *Common Cause v. David Lewis* (N.C. Super. 2018).

11. Redistricting Criteria: I programmed the computer algorithm to create 1,000 independent simulated plans adhering to the following seven districting criteria, as specified in the Adopted Criteria[3]:

a) Population Equality[4]: Because North Carolina's 2020 Census population was 10,439,388, districts in every 14-member congressional plan have an ideal population of 745,670.6. Accordingly, the computer simulation algorithm populated each districting plan such that precisely six districts have a population of 745,670, while the remaining eight districts have a population of 745,671.

b) Contiguity[5]: The simulation algorithm required districts to be geographically contiguous. Water contiguity is permissible. I also programmed the simulation algorithm to avoid double-traversals within a single county. In other words, for every simulated district, the portion of that district within any given county will be geographically contiguous.

c) Minimizing County Splits[6]: The simulation algorithm avoided splitting any of North Carolina's 100 counties, except when doing so is necessary to avoid violating one of the aforementioned criteria. When a county is divided into two districts, the county is considered to have one split. A county divided into three districts is considered to have two splits. A county divided into four districts is considered to have

---

[3] Since my November 30 report, I made the following changes to the computer simulation algorithm. First, I added additional code at the conclusion of the algorithm that checks for the occurrence of double traversals. The computer is instructed to automatically reject any simulated plan that contains a double traversal. Second, the algorithm now contains several steps that further increase the preservation of municipal boundaries, discussed further below.

[4] The Adopted Criteria state: "The number of persons in each congressional district shall be as nearly as equal as practicable, as determined under the most recent federal decennial census."

[5] The Adopted Criteria state: "No point contiguity shall be permitted in any 2021 Congressional, House, and Senate plan. Congressional, House, and Senate districts shall be compromised of contiguous territory. Contiguity by water is sufficient."

[6] The Adopted Criteria state: "Division of counties in the 2021 Congressional plan shall only be made for reasons of equalizing population and consideration of double bunking."

three splits, and so on. For the purpose of creating equally populated districts, each newly drawn congressional district requires only one county split. But the fourteenth and final district drawn in North Carolina does need not create an additional county split, since this final district should simply be the remaining area unassigned to the first thirteen districts. Therefore, an entire plan of 14 congressional districts requires only 13 county splits. Accordingly, I require that every simulated plan contain only 13 county splits. The 2021 Adopted Criteria do not prohibit splitting a county more than once, so I allow some of these 13 county splits to occur within the same county. As a result, the total number of counties containing one or more splits may be fewer than 13. The algorithm also follows the Adopted Criteria in that it draws a congressional district wholly within Mecklenburg and Wake counties, which each have sufficient population size to contain an entire congressional district within their boundaries.

d) Minimizing VTD Splits[7]: North Carolina is divided into 2,666 VTDs. The computer simulation algorithm attempted to keep these VTDs intact and not split them into multiple districts, except when doing so is necessary for creating equally populated districts. For the purpose of creating equally populated districts, each newly drawn congressional district requires one VTD split. But the fourteenth and final district drawn in North Carolina does need not create an additional VTD split, since this final district should simply be the remaining area unassigned to the first thirteen districts. Therefore, an entire plan of 14 congressional districts requires only 13 VTD splits. I therefore require that every simulated plan split only 13 VTDs in total.

[7] The Adopted Criteria state: "Voting districts ('VTDs') should be split only when necessary."

e) Geographic Compactness[8]: The simulation algorithm prioritized the drawing of geographically compact districts whenever doing so does not violate any of the aforementioned criteria.

f) Avoiding Incumbent Pairings: North Carolina's current congressional delegation includes two incumbents, Representatives Ted Budd and David Price, who announced before the Enacted Plan was adopted that they will not run for reelection in 2022. For the remaining eleven congressional incumbents, the simulation algorithm intentionally avoids pairing multiple incumbents in the same district. Hence, in every computer-simulated plan, each district contains no more than one incumbent's residence.

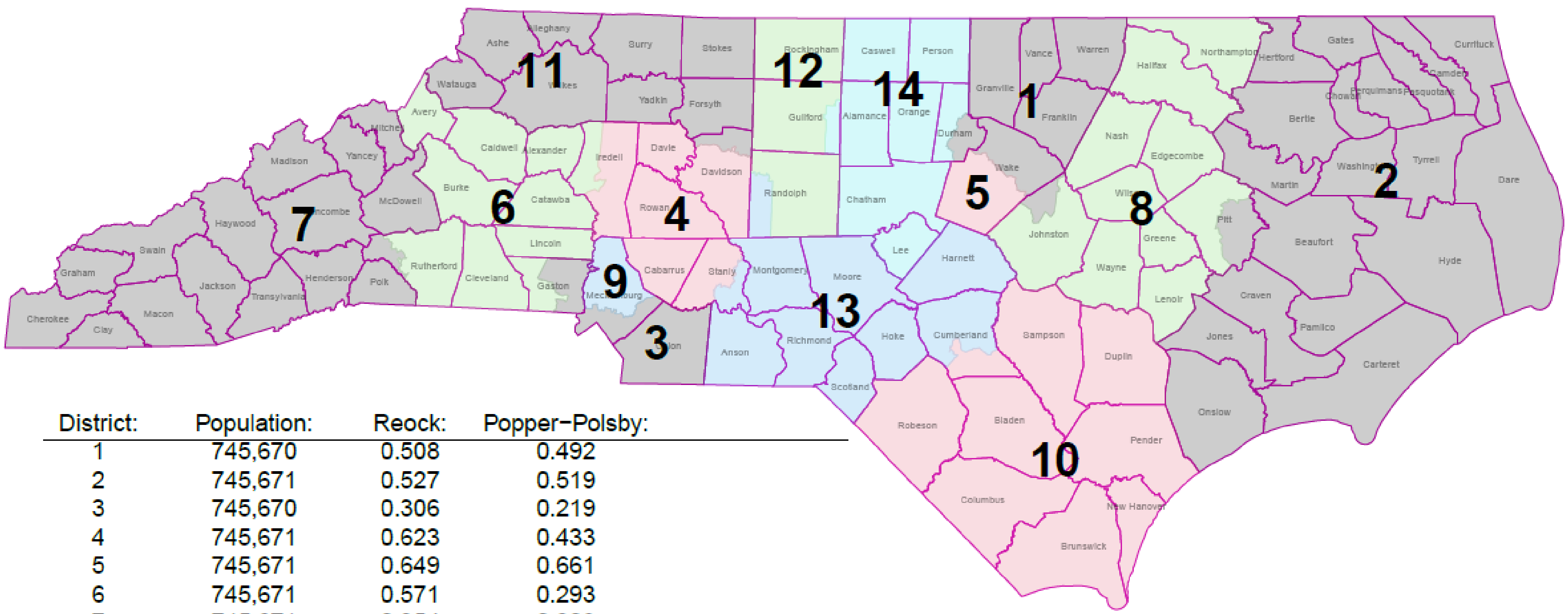
g) Municipal Boundaries[9]: The simulation algorithm generally favors not splitting municipalities. The algorithm contains several steps that favor the preservation of municipal boundaries, so long as other considerations required by the Adopted Criteria are not subordinated. To the extent that the algorithm avoids unnecessary splitting of counties, the municipalities within non-split counties are of course preserved. When the algorithm splits up a county by assigning the county's various VTDs to two different districts, the algorithm only allows one municipality to be split in this process of assigning the county's VTDs to different districts. Finally, as explained earlier, VTDs are only split when doing so is necessary for equalizing district populations. When a single VTD is split for this population equalization purpose, the algorithm attempts to split the VTD in such a way that minimizes the number of municipalities split within the VTD. In

[8] The Adopted Criteria state: "The Committees shall make reasonable efforts to draw legislative districts in the 2021 Congressional, House and Senate plans that are compact."
[9] The Adopted Criteria state: "The Committees may consider municipal boundaries when drawing districts in the 2021 Congressional, House, and Senate plans."

other words, the algorithm attempts to draw the district border within the VTD without crossing municipal boundaries.

12. On the following page of this report, Map 1 displays an example of one of the computer-simulated plans produced by the computer algorithm. The lower half of this Map also reports the population of each district, the compactness scores for each district, and the county splits and VTD splits created by the plan. As with every simulated plan, this plan contains exactly 13 VTD splits and 13 county splits, with 11 counties split into two or more districts.

Map 1:
Example of a Computer−Simulated Congressional Plan Protecting all 11 Incumbents



District:	Population:	Reock:	Popper−Polsby:
1	745,670	0.508	0.492
2	745,671	0.527	0.519
3	745,670	0.306	0.219
4	745,671	0.623	0.433
5	745,671	0.649	0.661
6	745,671	0.571	0.293
7	745,671	0.354	0.303
8	745,670	0.468	0.352
9	745,670	0.576	0.405
10	745,671	0.649	0.534
11	745,670	0.377	0.424
12	745,671	0.4	0.48
13	745,671	0.46	0.301
14	745,670	0.457	0.519
Plan Average:	745,670.6	0.495	0.424

13 Split Counties:
Alamance (Districts 12, 13)
Burke (Districts 10, 3)
Davie (Districts 2, 8)
Granville (Districts 1, 14)
Hoke (Districts 13, 6)
Mecklenburg (Districts 5, 9)
Nash (Districts 1, 11)
Orange (Districts 1, 13)
Pitt (Districts 11, 7)
Rockingham (Districts 12, 2)
Rowan (Districts 10, 8)
Rutherford (Districts 3, 9)
Wake (Districts 14, 4)

13 Split VTD's:
VTD 00008N in Alamance County (Districts 12 and 13)
VTD 000053 in Burke County (Districts 10 and 3)
VTD 000011 in Davie County (Districts 2 and 8)
VTD 00TYHO in Granville County (Districts 1 and 14)
VTD 000063 in Hoke County (Districts 13 and 6)
VTD 000018 in Mecklenburg County (Districts 5 and 9)
VTD 00P09A in Nash County (Districts 1 and 11)
VTD 0000CX in Orange County (Districts 1 and 13)
VTD 001301 in Pitt County (Districts 11 and 7)
VTD 0000LI in Rockingham County (Districts 12 and 2)
VTD 000033 in Rowan County (Districts 10 and 8)
VTD 000018 in Rutherford County (Districts 3 and 9)
VTD 008−03 in Wake County (Districts 14 and 4)

The Enacted Plan's Compliance with the Adopted Criteria

13. Although all seven of the criteria listed above are part of the General Assembly's Adopted Criteria, five of these criteria are ones that the Joint Redistricting Committee "shall" or "should" follow in the process of drawing its Congressional districting plan. These five mandated criteria are equal population, contiguity, minimizing county splits, minimizing VTD splits, and geographic compactness.[10]

14. I assessed whether the 2021 Enacted Plan complies with these five mandated criteria, and I describe my findings in this section. I found that the Enacted Plan does not violate the equal population requirement, nor do any of its districts violate contiguity.

15. However, by comparing the Enacted Plan to the 1,000 computer-simulated plans, I found that the Enacted Plan fails to minimize county splits, fails to minimize VTD splits, and is significantly less geographically compact than is reasonably possible. I describe these findings below in detail.

16. ***Minimizing County Splits:*** In comparing the total number of county splits in the Enacted Plan and in the computer-simulated plans, I counted the total number of times a county is split into more than one district. Specifically, a county fully contained within a single district counts as zero splits. A county split into two full or partial districts counts as one split. And a county split into three full or partial districts counts as two splits. And so on.

17. Using this standard method of accounting for total county splits, I found that the Enacted Plan contains 14 total county splits, which are detailed in Table 1. These 14 total county splits are spread across 11 counties. Eight of these 11 counties are split only once, but Guilford,

[10] In listing these five mandated criteria, I am not including the Adopted Criteria's prohibitions on the use of racial data, partisan considerations, and election results data. I did not assess whether the Enacted Plan complies with theprohibition on racial considerations.

Mecklenburg, and Wake Counties are each split into three districts, thus accounting for two splits each. Thus, the Enacted Plan has 14 total county splits, as listed in Table 1.

Table 1: Total Number of County Splits in the 2021 Enacted Plan

	County:	**Congressional Districts:**	**Total County Splits:**
1	Davidson	7 and 10	1
2	Guilford	7, 10, and 11	2
3	Harnett	4 and 7	1
4	Iredell	10 and 12	1
5	Mecklenburg	8, 9, and 13	2
6	Onslow	1 and 3	1
7	Pitt	1 and 2	1
8	Robeson	3 and 8	1
9	Wake	5, 6, and 7	2
10	Watauga	11 and 14	1
11	Wayne	2 and 4	1
Total County Splits:			**14**

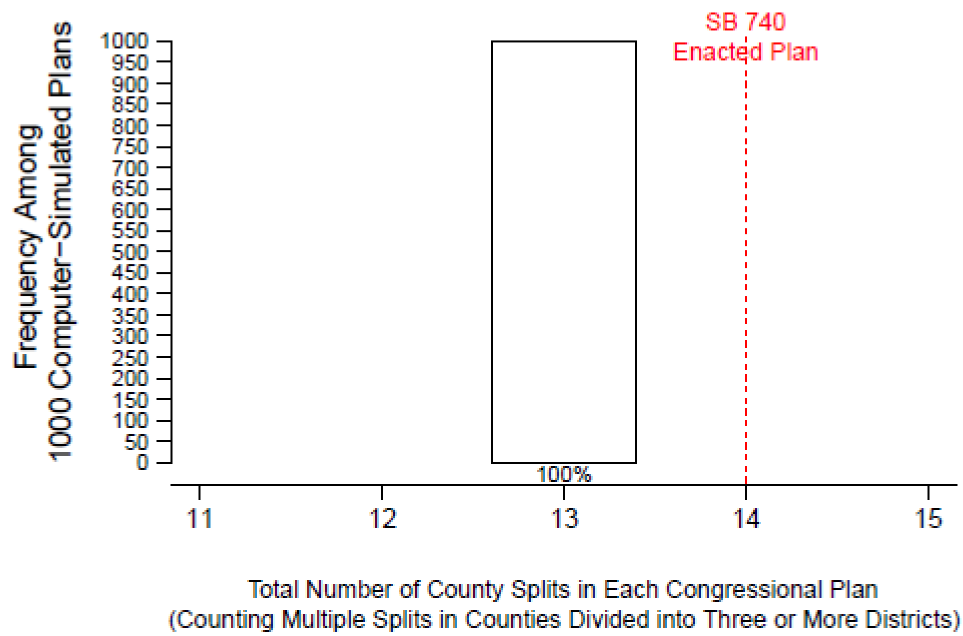
As explained in the previous section, a congressional plan in North Carolina needs to contain only 13 county splits if the map-drawer is attempting to minimize the splitting of counties. The Enacted Plan's 14 county splits is therefore one more split than is necessary. This "extra" split is specifically found at the border between District 7 and District 10. In general, the border between any two congressional districts in North Carolina needs to split only one county, at most. But in the Enacted Plan, the border between Districts 7 and 10 creates two county splits: One split of Davidson County and one split of Guilford County. Creating two county splits of Davidson and Guilford Counties was not necessary for equalizing district populations. Nor was it necessary for protecting incumbents, as no incumbents reside in the portions of Davidson and Guilford Counties within District 7 and District 10. Hence, the "extra" county split in Davidson and Guilford Counties does not appear to be consistent with the 2021 Adopted Criteria, which mandate that "Division of counties in the 2021 Congressional plan shall only be made for reasons of equalizing population and consideration of double bunking."

18. Indeed, I found that the computer simulation algorithm was always able to draw districts complying with the Adopted Criteria without using an "extra" 14th county split. As the upper half of Figure 1 illustrates, all 1,000 computer-simulated plans contain exactly 13 county splits. The Enacted Plan clearly contains more county splits than one would expect from a map-drawing process complying with the Adopted Criteria. Therefore, I conclude that the Enacted Plan does not comply with the Adopted Criteria's rule against unnecessary division of counties.

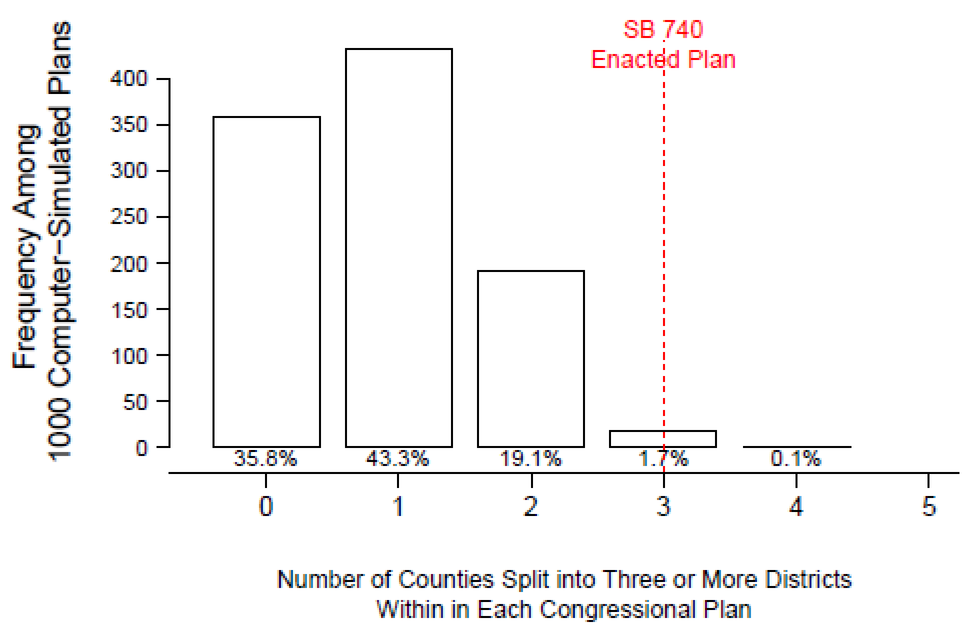
19. The Adopted Criteria do not explicitly limit the number of county splits within any single county. Nevertheless, it is notable that under the Enacted Plan, three different counties (Guilford, Mecklenburg, and Wake) are split multiple times. These three counties are each split into three districts under the Enacted Plan. This is an outcome that rarely occurs under the computer-simulated plans. As the lower half of Figure 1 illustrates, only 1.8% of the computer-simulated plans similarly split three or more counties multiple times. Thus, it is clear that the Enacted Plan's level of concentrating multiple county splits within a single county is an outcome that generally does not occur in a vast majority of the simulated plans drawn according to the Adopted Criteria. Additionally, not once in the small number of simulated plans that split at least three counties three ways are Guilford, Mecklenburg, and Wake Counties all split multiple times.

Figure 1:

Comparison of Total County Splits in Enacted SB 740 Plan and 1,000 Computer–Simulated Plans



Number of Counties Split Multiple Times
in Enacted SB 740 Plan and 1,000 Computer–Simulated Plans



21. **Minimizing VTD Splits:** The Adopted Criteria mandates that "Voting districts ('VTDs') should be split only when necessary." As explained earlier in this report, each newly drawn congressional district needs to create only one VTD split for the purpose of equalizing the district's population. But the fourteenth and final district drawn in North Carolina does need not create an additional VTD split, since this final district should simply be the remaining area unassigned to the first 13 districts. Therefore, an entire plan of 14 congressional districts needs to create only 13 VTD splits.

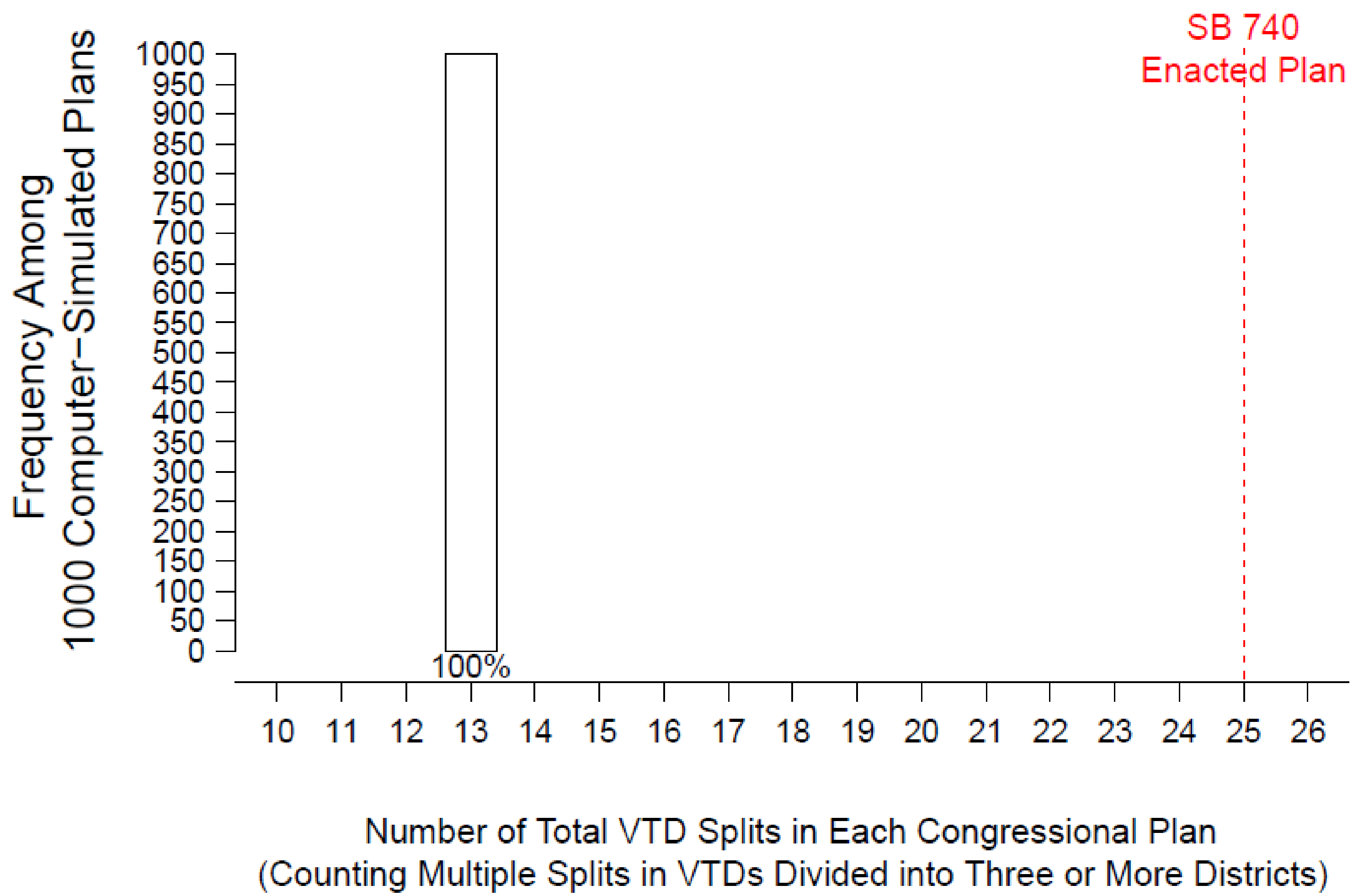
22. However, the Enacted Plan creates far more VTD splits than is necessary. As the General Assembly's "StatPack" Report[11] for the Enacted SB 740 Plan details, the Enacted Plan splits 24 VTDs into multiple districts. Among these 24 split VTDs, 23 VTDs are split into two districts, while one VTD (Wake County VTD 18-02) is split into three districts. Thus, using the same method of accounting for splits described earlier, the Enacted Plan contains 25 total VTD splits, and 24 VTDs are split into two or more districts.

23. The Enacted Plan's 25 total VTD splits is far more than is necessary to comply with the Adopted Criteria' equal population requirement. As explained earlier, only 13 VTD splits are necessary in order to produce an equally populated congressional plan in North Carolina. Thus, as Figure 2 illustrates, every one of the 1,000 computer-simulated plans contains exactly 13 VTD splits, and the Enacted Plan's 25 total VTD splits is clearly not consistent with the Adopted Criteria's requirement that "Voting districts ('VTDs') should be split only when necessary."

[11] Available at: https://webservices.ncleg.gov/ViewBillDocument/2021/53447/0/SL%202021-174%20-%20StatPack%20Report.

Figure 2:

Comparison of Total VTD Splits in Enacted SB 740 Plan and 1,000 Computer−Simulated Plans



24. ***Measuring Geographic Compactness***: The August 12, 2021 Adopted Criteria mandates that the Joint Redistricting Committee "shall" attempt to draw geographically compact congressional districts. The Adopted Criteria also specify two commonly used measures of district compactness: the Reock score and the Polsby-Popper score.

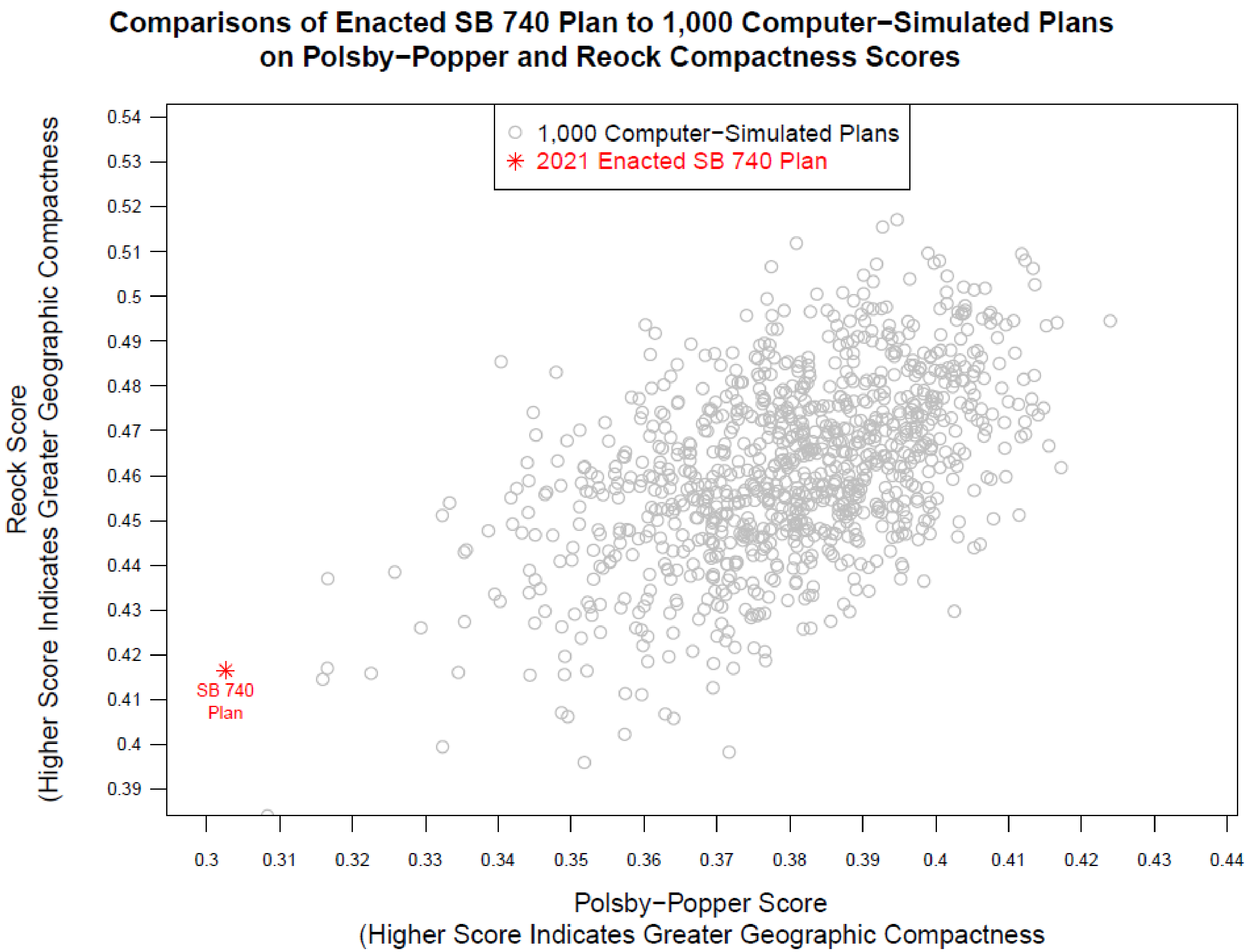
25. In evaluating whether the Enacted Plan follows the compactness requirement of the Adopted Criteria, it is useful to compare the compactness of the Enacted Plan and the 1,000 computer-simulated plans. The computer-simulated plans were produced by a computer algorithm adhering strictly to the traditional districting criteria mandated by the Adopted Criteria and ignoring any partisan or racial considerations. Thus, the compactness scores of these computer-simulated plans illustrate the statistical range of compactness scores that could be reasonably expected to emerge from a districting process that solely seeks to follow the Adopted Criteria while ignoring partisan and racial considerations. I therefore compare the compactness of the simulated plans and the Enacted Plan using the two measures of compactness specified by the 2021 Adopted Criteria.

26. First, I calculate the average Polsby-Popper score of each plan's districts. The Polsby-Popper score for each individual district is calculated as the ratio of the district's area to the area of a hypothetical circle whose circumference is identical to the length of the district's perimeter; thus, higher Polsby-Popper scores indicate greater district compactness. The 2021 Enacted Plan has an average Polsby-Popper score of 0.3026 across its 14 congressional districts. As illustrated in Figure 3, every single one of the 1,000 computer-simulated House plans in this report exhibits a higher Polsby-Popper score than the Enacted Plan. In fact, the middle 50% of these 1,000 computer-simulated plans have an average Polsby-Popper score ranging from 0.37 to 0.39, and the most compact computer-simulated plan has a Polsby-Popper score of 0.42. Hence,

it is clear that the Enacted Plan is significantly less compact, as measured by its Polsby-Popper score, than what could reasonably have been expected from a districting process adhering to the Adopted Criteria.

27. Second, I calculate the average Reock score of the districts within each plan. The Reock score for each individual district is calculated as the ratio of the district's area to the area of the smallest bounding circle that can be drawn to completely contain the district; thus, higher Reock scores indicate more geographically compact districts. The 2021 Enacted Plan has an average Reock score of 0.4165 across its 14 congressional districts. As illustrated in Figure 3, 98.2% of the 1,000 computer-simulated plans exhibit a higher Reock score than the Enacted Plan. In fact, the middle 50% of these 1,000 computer-simulated plans have an average Reock score ranging from 0.45 to 0.46, and the most compact computer-simulated plan has an average Reock score of 0.52. Hence, it is clear that the Enacted Plan is significantly less compact, as measured by its Reock score, than what could reasonably have been expected from a districting process adhering to the Adopted Criteria.

Figure 3:


Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
on Polsby−Popper and Reock Compactness Scores
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
SB 740
Plan
Reock Score
(Higher Score Indicates Greater Geographic Compactness
0.54
0.53
0.52
0.51
0.5
0.49
0.48
0.47
0.46
0.45
0.44
0.43
0.42
0.41
0.4
0.39
0.3
0.31
0.32
0.33
0.34
0.35
0.36
0.37
0.38
0.39
0.4
0.41
0.42
0.43
0.44
Polsby−Popper Score
(Higher Score Indicates Greater Geographic Compactness

Measuring the Partisanship of Districting Plans

28. In general, I use actual election results from recent, statewide election races in North Carolina to assess the partisan performance of the Enacted Plan and the computer-simulated plans analyzed in this report. Overlaying these past election results onto a districting plan enables me to calculate the Republican (or Democratic) share of the votes cast from within each district in the Enacted Plan and in each simulated plan. I am also able to count the total number of Republican and Democratic-leaning districts within each simulated plan and within the Enacted Plan. All of these calculations thus allow me to directly compare the partisanship of the Enacted Plan and the simulated plans. These partisan comparisons allow me to determine whether or not the partisanship of individual districts and the partisan distribution of seats in the Enacted Plan could reasonably have arisen from a districting process adhering to the Adopted Criteria and its explicit prohibition on partisan considerations. Past voting history in federal and statewide elections is a strong predictor of future voting history. Mapmakers thus can and do use past voting history to identify the class of voters, at a precinct-by-precinct level, who are likely to vote for Republican or Democratic congressional candidates.

29. In the 2011, 2016, and 2017 rounds of state legislative and congressional redistricting last decade, the North Carolina General Assembly publicly disclosed that it was relying solely on recent statewide elections in measuring the partisanship of the districting plans being created. I therefore follow the General Assembly's past practice from last decade by using results from a similar set of recent statewide elections in order to measure the partisanship of districts in the Enacted Plan and in the computer-simulated plans.

30. ***The 2016-2020 Statewide Election Composite:*** During the General Assembly's 2017 legislative redistricting process, Representative David Lewis announced at the Joint Redistricting Committee's August 10, 2017 meeting that the General Assembly would measure

the partisanship of legislative districts using the results from some of the most recent elections held in North Carolina for the following five offices: US President, US Senator, Governor, Lieutenant Governor, and Attorney General.

31. To measure the partisanship of all districts in the computer-simulated plans and the 2021 Enacted Plan, I used the two most-recent election contests held in North Carolina for these same five offices during 2016-2020. In other words, I used the results of the following ten elections: 2016 US President, 2016 US Senator, 2016 Governor, 2016 Lieutenant Governor, 2016 Attorney General, 2020 US President, 2020 US Senator, 2020 Governor, 2020 Lieutenant Governor, and 2020 Attorney General. I use these election results because these are the same state and federal offices whose election results were used by the General Assembly during its 2017 legislative redistricting process, and the 2017 redistricting process was the most recent one in which the leadership of the General Assembly's redistricting committees publicly announced how the General Assembly would evaluate the partisanship of its own districting plans.

32. I obtained precinct-level results for these ten elections, and I disaggregated these election results down to the census block level. I then aggregated these block-level election results to the district level within each computer-simulated plan and the Enacted Plan, and I calculated the number of districts within each plan that cast more votes for Republican than Democratic candidates. I use these calculations to measure the partisan performance of each simulated plan analyzed in this report and of the Enacted Plan. In other words, I look at the census blocks that would comprise a particular district in a given simulation and, using the actual election results from those census blocks, I calculate whether voters in that simulated district collectively cast more votes for Republican or Democratic candidates in the 2016-2020 statewide election contests. I performed such calculations for each district under each simulated plan to

measure the number of districts Democrats or Republicans would win under that particular simulated districting map.

33. I refer to the aggregated election results from these ten statewide elections as the "2016-2020 Statewide Election Composite." For the Enacted Plan districts and for all districts in each of the 1,000 computer-simulated plans, I calculate the percentage of total two-party votes across these ten elections that were cast in favor of Republican candidates in order to measure the average Republican vote share of the district. In the following section, I present district-level comparisons of the Enacted Plan and simulated plan districts in order to identify whether any individual districts in the Enacted Plan are partisan outliers. I also present plan-wide comparisons of the Enacted Plan and the simulated plans in order to identify the extent to which the Enacted Plan is a statistical outlier in terms of common measures of districting plan partisanship.

District-Level and Plan-Wide Partisan Comparisons of the Enacted Plan and Simulated Plans

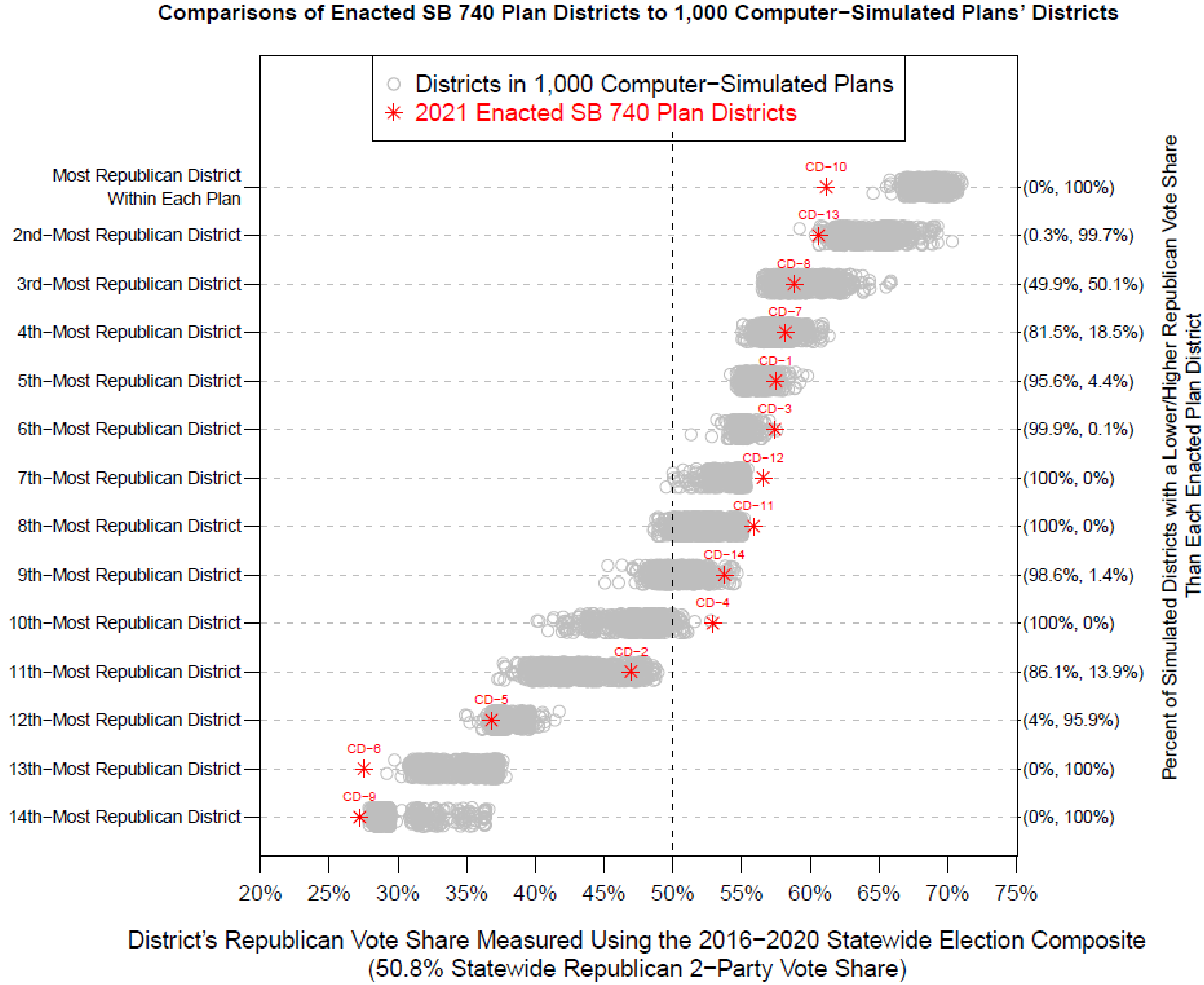
34. In this section, I present partisan comparisons of the Enacted Plan to the computer-simulated plans at both a district-by-district level as well as a plan-wide level using several common measures of districting plan partisanship. First, I compare the district-level Republican vote share of the Enacted Plan's districts and the districts in the computer-simulatedplans. Next, I compare the number of Republican-favoring districts in the Enacted Plan and in the computer-simulated plans. Finally, I use several common measures of partisan bias to compare the Enacted Plan to the computer-simulated plans. Overall, I find that the several individual districts in the Enacted Plan are statistical outliers, exhibiting extreme partisan characteristics that are rarely or never observed in the computer-simulated plan districts drawn with strict adherence to the Adopted Criteria. Moreover, I find that at the plan-wide level, the

Enacted Plan creates a degree of partisan bias favoring Republicans that is more extreme than the vast majority of the computer-simulated plans. I describe these findings in detail below:

35. ***Partisan Outlier Districts in the Enacted Plan***: In Figure 4, I directly compare the partisan distribution of districts in the Enacted Plan to the partisan distribution of districts in the 1,000 computer-simulated plans. I first order the Enacted Plan's districts from the most to theleast-Republican district, as measured by Republican vote share using the 2016-2020 Statewide Election Composite. The most-Republican district appears on the top row, and the least- Republican district appears on the bottom row of Figure 4. Next, I analyze each of the 1,000 computer-simulated plans and similarly order each simulated plan's districts from the most- to the least-Republican district. I then directly compare the most-Republican Enacted Plan district (CD-10) to the most-Republican simulated district from each of the 1,000 computer-simulated plans. In other words, I compare one district from the Enacted Plan to 1,000 computer-simulated

districts, and I compare these districts based on their Republican vote share. I then directly compare the second-most-Republican district in the Enacted Plan to the second-most-Republican district from each of the 1,000 simulated plans. I conduct the same comparison for each district in the Enacted Plan, comparing the Enacted Plan district to its computer-simulated counterparts from each of the 1,000 simulated plans.

Figure 4:


Comparisons of Enacted SB 740 Plan Districts to 1,000 Computer−Simulated Plans' Districts
Districts in 1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan Districts
Most Republican District Within Each Plan
2nd−Most Republican District
3rd−Most Republican District
4th−Most Republican District
5th−Most Republican District
6th−Most Republican District
7th−Most Republican District
8th−Most Republican District
9th−Most Republican District
10th−Most Republican District
11th−Most Republican District
12th−Most Republican District
13th−Most Republican District
14th−Most Republican District
CD−10
CD−13
CD−8
CD−7
CD−1
CD−3
CD−12
CD−11
CD−14
CD−4
CD−2
CD−5
CD−6
CD−9
(0%, 100%)
(0.3%, 99.7%)
(49.9%, 50.1%)
(81.5%, 18.5%)
(95.6%, 4.4%)
(99.9%, 0.1%)
(100%, 0%)
(100%, 0%)
(98.6%, 1.4%)
(100%, 0%)
(86.1%, 13.9%)
(4%, 95.9%)
(0%, 100%)
(0%, 100%)
Percent of Simulated Districts with a Lower/Higher Republican Vote Share Than Each Enacted Plan District
20%
25%
30%
35%
40%
45%
50%
55%
60%
65%
70%
75%
District's Republican Vote Share Measured Using the 2016−2020 Statewide Election Composite
(50.8% Statewide Republican 2−Party Vote Share)

36. Thus, the top row of Figure 4 directly compares the partisanship of the most-Republican Enacted Plan district (CD-10) to the partisanship of the most-Republican district from each of the 1,000 simulated plans. The two percentages (in parentheses) in the right margin of this Figure report the percentage of these 1,000 simulated districts that are less Republican than, and more Republican than, the Enacted Plan district. Similarly, the second row of this Figure compares the second-most-Republican district from each plan, the third row compares the third-most-Republican district from each plan, and so on. In each row of this Figure, the Enacted Plan's district is depicted with a red star and labeled in red with its district number; meanwhile, the 1,000 computer-simulated districts are depicted with 1,000 gray circles on each row.

37. As the bottom row of Figure 4 illustrates, the most-Democratic district in the Enacted Plan (CD-9) is more heavily Democratic than 100% of the most-Democratic districts in each of the 1,000 computer-simulated plans. This calculation is numerically reported in the right margin of the Figure. Every single one of the computer-simulated counterpart districts would have been more politically moderate than CD-9 in terms of partisanship: CD-9 exhibits a Republican vote share of 27.2%, while all 1,000 of the most-Democratic districts in the computer-simulated plans would have exhibited a higher Republican vote share and would therefore have been more politically moderate. It is thus clear that CD-9 packs together Democratic voters to a more extreme extent than the most-Democratic district in 100% of the computer-simulated plans. I therefore identify CD-9 as an extreme partisan outlier when compared to its 1,000 computer-simulated counterparts, using a standard threshold test of 95% for statistical significance.

38. The next-to-bottom row of Figure 4 reveals a similar finding regarding CD-6 in the Enacted Plan. This row illustrates that the second-most-Democratic district in the Enacted

Plan (CD-6) is more heavily Democratic than 100% of the second-most-Democratic districts in each of the 1,000 computer-simulated plans. Every single one of its computer-simulated counterpart districts would have been more politically moderate than CD-6 in terms of partisanship: CD-6 exhibits a Republican vote share of 27.5%, while 100% of the second-most-Democratic districts in the computer-simulated plans would have exhibited a higher Republican vote share and would therefore have been more politically moderate. In other words, CD-6 packs together Democratic voters to a more extreme extent than the second-most-Democratic district in 100% of the computer-simulated plans. I therefore identify CD-6 as an extreme partisan outlier when compared to its 1,000 computer-simulated counterparts, using a standard threshold test of 95% for statistical significance.

39. Meanwhile, the top two rows of Figure 4 reveal a similar finding: As the top row illustrates, the most-Republican district in the Enacted Plan (CD-10) is less heavily Republican than 100% of the most-Republican districts in each of the 1,000 computer-simulated plans. A similar pattern appears in the second-to-top row of Figure 4, which illustrates that the second-most-Republican district in the Enacted Plan (CD-13) is less heavily Republican than 99.7% of the second-most-Republican districts in each of the 1,000 computer-simulated plans.

40. It is especially notable that these four aforementioned Enacted Plan districts – the two most Republican districts (CD-10 and CD-13) and the two most Democratic districts (CD-9 and CD-6) in the Enacted Plan – were drawn to include more Democratic voters than virtually allof their counterpart districts in the 1,000 computer-simulated plans. These "extra" Democratic voters in the four most partisan-extreme districts in the Enacted Plan had to come from the remaining ten more moderate districts in the Enacted Plan. Having fewer Democratic voters in these more moderate districts enhances Republican candidate performance in these districts.

41. Indeed, the middle six rows in Figure 4 (i.e., rows 5 through 10) confirm this precise effect. The middle six rows in Figure 4 compare the partisanship of districts in the fifth, sixth, seventh, eighth, ninth, and tenth-most Republican districts within the Enacted Plan and the 1,000 computer-simulated plans. In all six of these rows, the Enacted Plan district is a partisan outlier. In each of these six rows, the Enacted Plan's district is more heavily Republican than over 95% of its counterpart districts in the 1,000 computer-simulated plans. Three of these six rows illustrate Enacted Plan districts that are more heavily Republican than 100% of their counterpart districts in the computer-simulated plans. The six Enacted Plan districts in these six middle rows (CD-1, 3, 4, 11, 12, and 14) are more heavily Republican than nearly all of their counterpart computer-simulated plan districts because the four most partisan-extreme districts inthe Enacted Plan (CD-6, 9, 10, and 13) are more heavily Democratic than nearly all of their counterpart districts in the computer-simulated plans.

42. I therefore identify the six Enacted Plan districts in the six middle rows (CD-1, 3,4, 11, 12, and 14) of Figure 4 as partisan statistical outliers. Each of these six districts has a Republican vote share that is higher than over 95% of the computer-simulated districts in its respective row in Figure 4. I also identify the four Enacted Plan districts in the top rows and the bottom two rows (CD-6, 9, 10, and 13) of Figure 4 as partisan statistical outliers. Each of these four districts has a Republican vote share that is lower than at least 99.7% of the computer-simulated districts in its respective row in Figure 4.

43. In summary, Figure 4 illustrates that 10 of the 14 districts in the Enacted Plan are partisan outliers: Six districts (CD-1, 3, 4, 11, 12, and 14) in the Enacted Plan are more heavily Republican than over 95% of their counterpart computer-simulated plan districts, while four

districts (CD-6, 9, 10, and 13) are more heavily Democratic than at least 99.7% of their counterpart districts in the computer-simulated plans.

44. The Appendix of this report contains ten additional Figures (Figures A1 through A10) that each contain a similar analysis of the Enacted Plan districts and the computer- simulated plan districts. Each of these ten Figures in the Appendix measures the partisanship ofdistricts using one of the individual ten elections included in the 2016-2020 Statewide ElectionComposite. These ten Figures generally demonstrate that the same extreme partisan outlier patterns observed in Figure 4 are also present when district partisanship is measured using any one of the ten statewide elections held in North Carolina during 2016-2020.

45. ***"Mid-Range" Republican Districts:*** Collectively, the upper ten rows in Figure 4 illustrate that the Enacted Plan's ten most-Republican districts exhibit a significantly narrower range of partisanship than is exhibited by the ten most-Republican districts in each of the computer-simulated plans. Specifically, the Enacted Plan's ten most-Republican districts all have Republican vote shares within the narrow range of 52.9% to 61.2%. As explained earlier, this narrow range is the product of two distinct dynamics: In the top two rows of Figure 4, the Enacted Plan's districts are significantly less Republican than nearly all of the simulated plans' districts in these rows. But in the fifth to tenth rows of Figure 4, the Enacted Plan's districts are more safely Republican-leaning than over 95% of the computer-simulated districts within each of these six rows. The overall result of these two distinct dynamics is that the Enacted Plan contains ten districts that all have Republican vote shares within the narrow range of 52.9% to 61.2%. I label any districts within this narrow range of partisanship as "mid-range" Republican-leaning districts, reflecting the fact that these districts have generally favored Republican candidates, but not by overwhelmingly large margins.

46. Is the Enacted Plan's creation of ten such "mid-range" Republican-leaning districts an outcome that ever occurs in the 1,000 computer-simulated plans? I analyzed the simulated plans and counted the number of districts within each plan that are similarly "mid- range" with a Republican vote share between 52.9% and 61.2%. As Figure 5 illustrates, the Enacted Plan's creation of ten "mid-range" Republican districts is an extreme statistical outlier. None of the 1,000 simulated plans comes close to creating ten such districts. Virtually all of the simulated plans contain from two to six "mid-range" Republican districts, and the most commonoutcome among the simulations is four such districts. Hence, the Enacted Plan is clearly an extreme partisan outlier in terms of its peculiar focus on maximizing the number of "mid-range"Republican districts, and the Enacted Plan did so to an extreme degree far beyond any of the 1,000 simulated plans created using a partisan-blind computer algorithm that follows the Adopted Criteria.

47. ***Competitive Districts:*** The Enacted Plan's maximization of "mid-range" Republican districts necessarily comes at the expense of creating more competitive districts. As Figure 4 illustrates, the Enacted Plan contains zero districts whose Republican vote share is higher than 47.0% and lower than 52.9%, as measured using the 2016-2020 Statewide ElectionComposite. In other words, there are zero districts in which the Republican vote share is within5% of the Democratic vote share.

48. I label districts with a Republican vote share from 47.5% to 52.5% as "competitive" districts to reflect the fact that such districts have a nearly even share of Republican and Democratic voters, and election outcomes in the district could therefore swing in favor of either party. The Enacted Plan contains zero "competitive" districts, as measured using the 2016-2020 Statewide Election Composite.

Figure 5:

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans On Number of Mid−Range Republican Districts

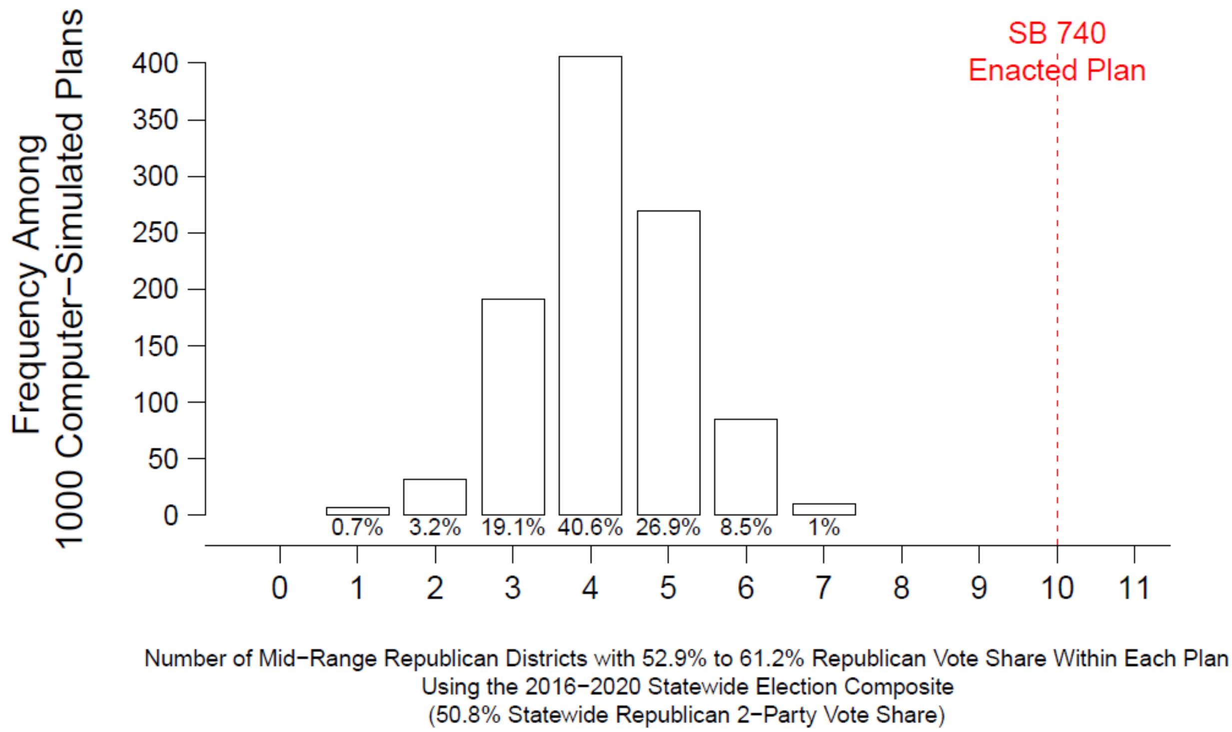
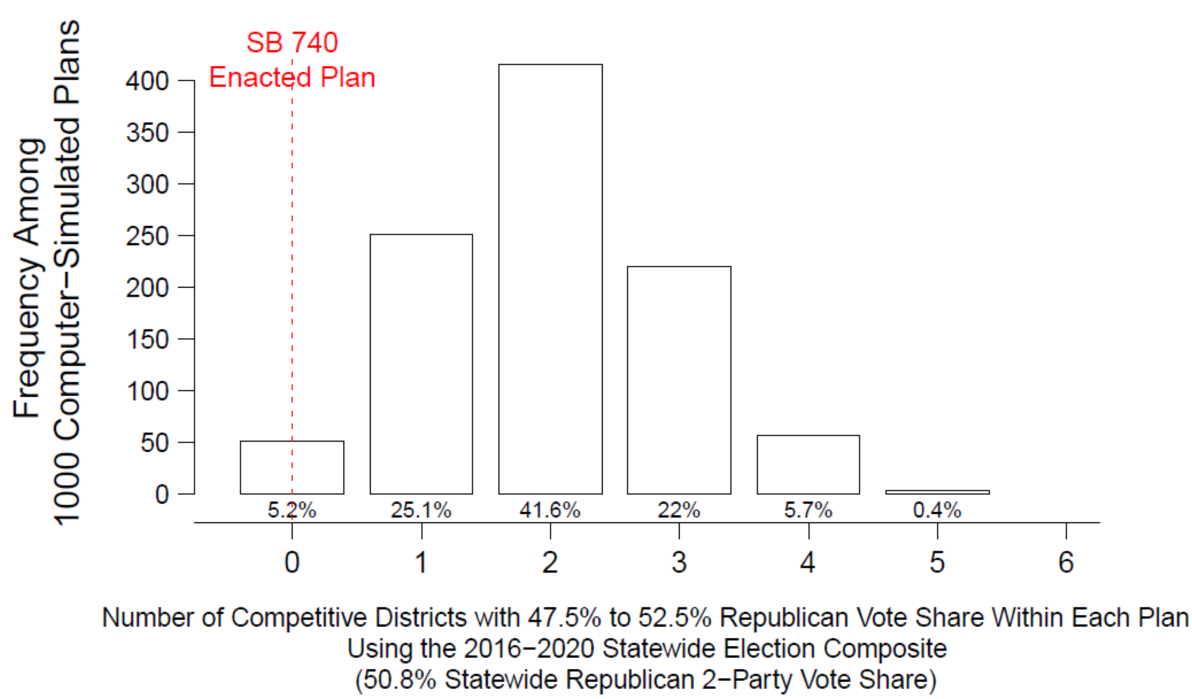


Figure 6:

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans On Number of Competitive Districts



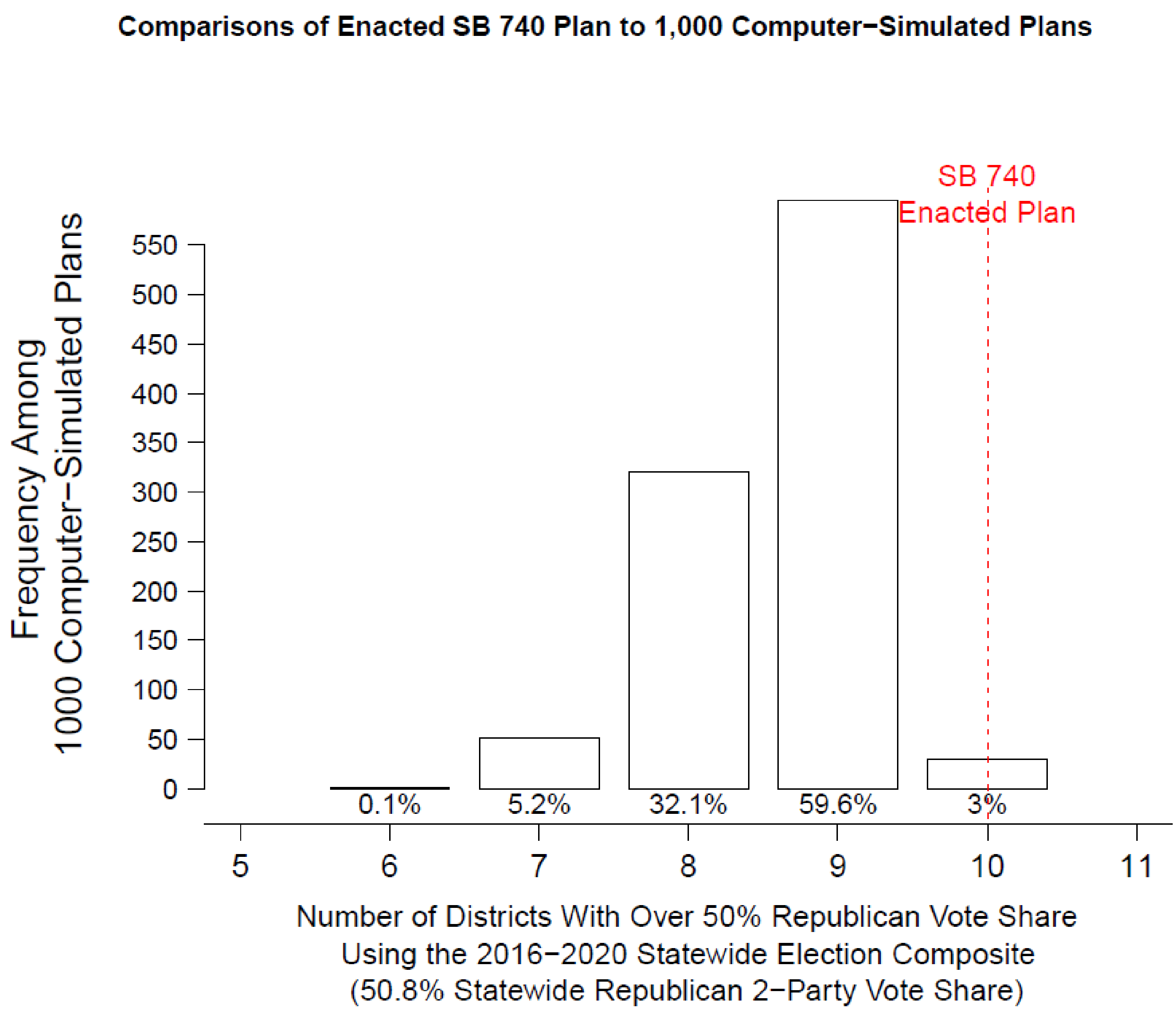
49. Is the Enacted Plan's failure to create any "competitive" districts an outcome that ever occurs in the 1,000 computer-simulated plans? I analyzed the simulated plans and counted the number of districts within each plan that are "competitive" districts with a Republican vote share between 47.5% and 52.5%. As Figure 6 illustrates, the Enacted Plan's creation of zero "competitive" districts is almost a statistical outlier: Only 5.2% of the 1,000 simulated plans similarly fail to have a single "competitive" district. The vast majority of the computer-simulated plans contain two or more "competitive" districts. Almost 95% of the computer-simulated plans create more "competitive" districts than the Enacted Plan does.

50. ***Number of Democratic and Republican Districts:*** Figure 7 compares the partisan breakdown of the computer-simulated plans to the partisanship of the Enacted Plan. Specifically, Figure 7 uses the 2016-2020 Statewide Election Composite to measure the number of Republican-favoring districts created in each of the 1,000 simulated plans. Across the entire state, Republican candidates collectively won a 50.8% share of the votes in the ten elections in the 2016-2020 Statewide Election Composite. But within the 14 districts in the Enacted Plan, Republicans have over a 50% vote share in 10 out of 14 districts. In other words, the Enacted Plan created 10 Republican-favoring districts, as measured using the 2016-2020 Statewide Election Composite. By contrast, only 3% of the computer-simulated plans create 10 Republican-favoring districts, and no computer-simulated plan ever creates more than 10 Republican districts.

51. Hence, in terms of the total number of Republican-favoring districts created by the plan, the 2021 Enacted Plan is a statistical outlier when compared to the 1,000 computer- simulated plans. The Enacted Plan creates the maximum number of Republican districts that everoccurs in any computer-simulated plan, and the Enacted Plan creates more Republican districts

than 97% of the computer-simulated plans, which were drawn using a non-partisan districting process adhering to the General Assembly's 2021 Adopted Criteria. I characterize the Enacted Plan's creation of 10 Republican districts as a statistical outlier among the computer-simulated plans because the Enacted Plan exhibits an outcome that is more favorable to Republicans than over 95% of the simulated plans.

Figure 7:



52. Notably, the ten elections included in the Statewide Election Composite all occurred in two election years and in electoral environments that were relatively favorable to Republicans across the country (November 2016 and November 2020). North Carolina did not hold any statewide elections for non-judicial offices in November 2018, which was an electoral environment more favorable to Democrats across the country.

53. Hence, the projected number of Republican seats would be even lower in the computer-simulated plans if one measured district partisanship using a statewide election whose outcome was more partisan-balanced or even favorable to Democrats. In the Appendix, I presentten histograms (labeled as Figures B1 to B10), each presenting the projected number of Republican seats across all of the simulated plans and the Enacted Plan using only one of the tenelections in the Statewide Election Composite.

54. The ten histograms in Figures B1 to B10 illustrate how the partisanship of the Enacted Plan compares to the partisanship of the 1,000 computer-simulated plans under a range of different electoral environments, as reflected by the ten elections in the Statewide Election Composite. Most notably, under all ten of these elections, the Enacted Plan always contains exactly 10 Republican-favoring districts and 4 Democrat-favoring districts. Hence, it is clear thatthe Enacted Plan creates a 10-to-4 distribution of seats in favor of Republican candidates that is durable across a range of different electoral conditions.

55. Moreover, the histograms in Figures B1 to B10 demonstrate that the Enacted Plan becomes a more extreme partisan outlier relative to the computer-simulated plans under electoral conditions that are slightly to moderately favorable to the Democratic candidate. For example, Figure B1 compares the Enacted Plan to the computer-simulated plan using the results of the 2016 Attorney General election, which was a near-tied statewide contest in which Democrat Josh

Stein defeated Republican Buck Newton by a very slim margin. Using the 2016 Attorney General election to measure district partisanship, the 2021 Enacted Plan contains 10 Republican-favoring districts out of 14. The Enacted Plan's creation of 10 districts favoring Republican BuckNewton over Democrat Josh Stein is an outcome that never occurs in the 1,000 computer-simulated plans, indicating that the Enacted Plan is a partisan statistical outlier under electoral conditions that are more favorable for Democrats (and thus relatively more unfavorable for Republicans) than is normal in North Carolina.

56. An even more favorable election for the Democratic candidate was the 2020 gubernatorial contest, in which Democrat Roy Cooper defeated Republican Dan Forest by a 4.5% margin. Figure B7 compares the Enacted Plan to the computer-simulated plans using the resultsof this 2020 gubernatorial election. Using the results from this election, the 2021 Enacted Plan contains 10 Republican-favoring districts out of 14. None of the 1,000 simulated plans ever contain 10 districts favoring the Republican candidate. The Enacted Plan's creation of 10 Republican-favoring districts is therefore an extreme partisan outlier that is durable even in Democratic-favorable electoral conditions. In fact, the 10-to-4 Republican partisan advantage under the Enacted Plan appears to become even more of an extreme partisan outlier under Democratic-favorable elections.

57. ***The Mean-Median Difference:*** I also calculate each districting plan's mean-median difference, which is another accepted method that redistricting scholars commonly use to compare the relative partisan bias of different districting plans. The mean-median difference for any given plan is calculated as the mean district-level Republican vote share, minus the median district-level Republican vote share. For any congressional districting plan, the mean is calculated as the average of the Republican vote shares in each of the 14 districts. The median, in

turn, is the Republican vote share in the district where Republican performed the middle-best, which is the district that Republican would need to win to secure a majority of the congressional delegation. For a congressional plan containing 14 districts, the median district is calculated as the average of the Republican vote share in the districts where Republican performed the 7th and8th-best across the state.

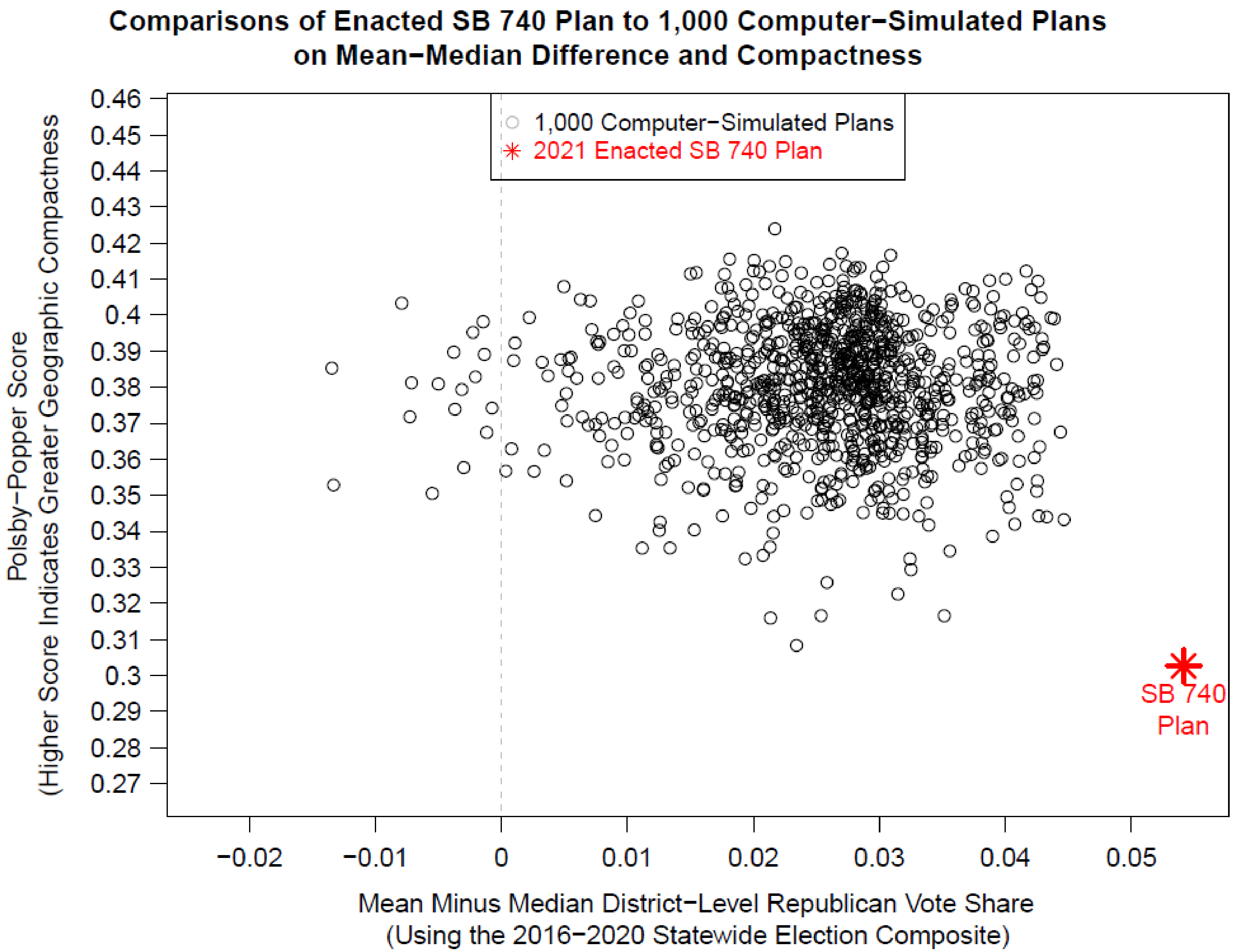
58. Using the 2016-2020 Statewide Election Composite to measure partisanship, the districts in the 2021 Enacted Plan have a mean Republican vote share of 50.8%, while the median district has a Republican vote share of 56.2%. Thus, the Enacted Plan has a mean-mediandifference of +5.4%, indicating that the median district is skewed significantly more Republican than the plan's average district. The mean-median difference thus indicates that the Enacted Plandistributes voters across districts in such a way that most districts are significantly more Republican-leaning than the average North Carolina congressional district, while Democratic voters are more heavily concentrated in a minority of the Enacted Plan's districts.

59. I perform this same mean-median difference calculation on all computer-simulated plans in order to determine whether this partisan skew in the median congressional districts could have resulted naturally from North Carolina's political geography and the application of the Adopted Criteria. Figure 8 compares the mean-median difference of the Enacted Plan to the mean-median difference for each the 1,000 computer-simulated plans.

60. Figure 8 contains 1,000 gray circles, representing the 1,000 computer-simulated plans, as well as a red star, representing the 2021 Enacted Plan. The horizontal axis in this Figure measures the mean-median difference of the 2021 Enacted Plan and each simulated plan using the 2016-2020 Statewide Election Composite, while the vertical axis measures the average Polsby-Popper compactness score of the districts within each plan, with higher Polsby-Popper

scores indicating more compact districts. Figure 8 illustrates that the Enacted Plan’s mean-median difference is +5.4%, indicating that the median district is skewed significantly more Republican than the plan’s average district. Figure 8 further indicates that this difference is an extreme statistical outlier compared to the 1,000 computer-simulated plans. Indeed, the Enacted Plan's +5.4% mean-median difference is an outcome never observed across these 1,000 simulated plans. The 1,000 simulated plans all exhibit mean-median differences that range from -0.1% to +4.6%. In fact, the middle 50% of these computer-simulated plans have mean-median differences ranging from +2.1% to +3.1%, indicating a much smaller degree of skew in the median district than occurs under the 2021 Enacted Plan. These results confirm that the Enacted Plan creates an extreme partisan outcome that cannot be explained by North Carolina’s voter geography or by strict adherence to the required districting criteria set forth in the General Assembly’s Adopted Criteria.

Figure 8:


Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
on Mean−Median Difference and Compactness
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
SB 740
Plan
Polsby−Popper Score
(Higher Score Indicates Greater Geographic Compactness
0.46
0.45
0.44
0.43
0.42
0.41
0.4
0.39
0.38
0.37
0.36
0.35
0.34
0.33
0.32
0.31
0.3
0.29
0.28
0.27
−0.02
−0.01
0
0.01
0.02
0.03
0.04
0.05
Mean Minus Median District−Level Republican Vote Share
(Using the 2016−2020 Statewide Election Composite)

61. Figure 8 illustrates that the Enacted Plan is less geographically compact than every single one of the computer-simulated plans, as measured by each plan's average Polsby-Popper score. The simulated plans have Polsby-Popper scores ranging from 0.31 to 0.42. In fact, the middle 50% of these computer-simulated plans have Polsby-Popper scores ranging from 0.37 to 0.39. Meanwhile, the Enacted Plan exhibits a Polsby-Popper score of only 0.30, which is lower than all 1,000 of the computer-simulated plans. Hence, it is clear that the Enacted Plan did not seek to draw districts that were as geographically compact as reasonably possible. Instead, the Enacted Plan subordinated geographic compactness, which enabled the Enacted Plan to create a partisan skew in North Carolina's congressional districts favoring Republican candidates.

62. ***The Efficiency Gap:*** Another commonly used measure of a districting plan's partisan bias is the efficiency gap.[12] To calculate the efficiency gap of the Enacted Plan and every computer-simulated plan, I first measure the number of Republican and Democratic votes within each Enacted Plan district and each computer-simulated district, as measured using the 2016- 2020 Statewide Election Composite. Using this measure of district-level partisanship, I then calculate each districting plan's efficiency gap using the method outlined in *Partisan Gerrymandering and the Efficiency Gap.*[13] Districts are classified as Democratic victories if, using the 2016-2020 Statewide Election Composite, the sum total of Democratic votes in the district during these elections exceeds the sum total of Republican votes; otherwise, the district is classified as Republican. For each party, I then calculate the total sum of surplus votes in districts the party won and lost votes in districts where the party lost. Specifically, in a district lost by a

[12] Eric McGhee, "Measuring Partisan Bias in Single-Member District Electoral Systems." Legislative Studies Quarterly Vol. 39, No. 1: 55–85 (2014).
[13] Nicholas O. Stephanopoulos & Eric M. McGhee, *Partisan Gerrymandering and the Efficiency Gap*, 82 University of Chicago Law Review 831 (2015).

given party, all of the party's votes are considered lost votes; in a district won by a party, only the party's votes exceeding the 50% threshold necessary for victory are considered surplus votes. A party's total wasted votes for an entire districting plan is the sum of its surplus votes in districts won by the party and its lost votes in districts lost by the party. The efficiency gap is then calculated as total wasted Democratic votes minus total wasted Republican votes, divided by the total number of two-party votes cast statewide across all seven elections.

63. Thus, the theoretical importance of the efficiency gap is that it tells us the degree to which more Democratic or Republican votes are wasted across an entire districting plan. A significantly positive efficiency gap indicates far more Democratic wasted votes, while a significantly negative efficiency gap indicates far more Republican wasted votes.

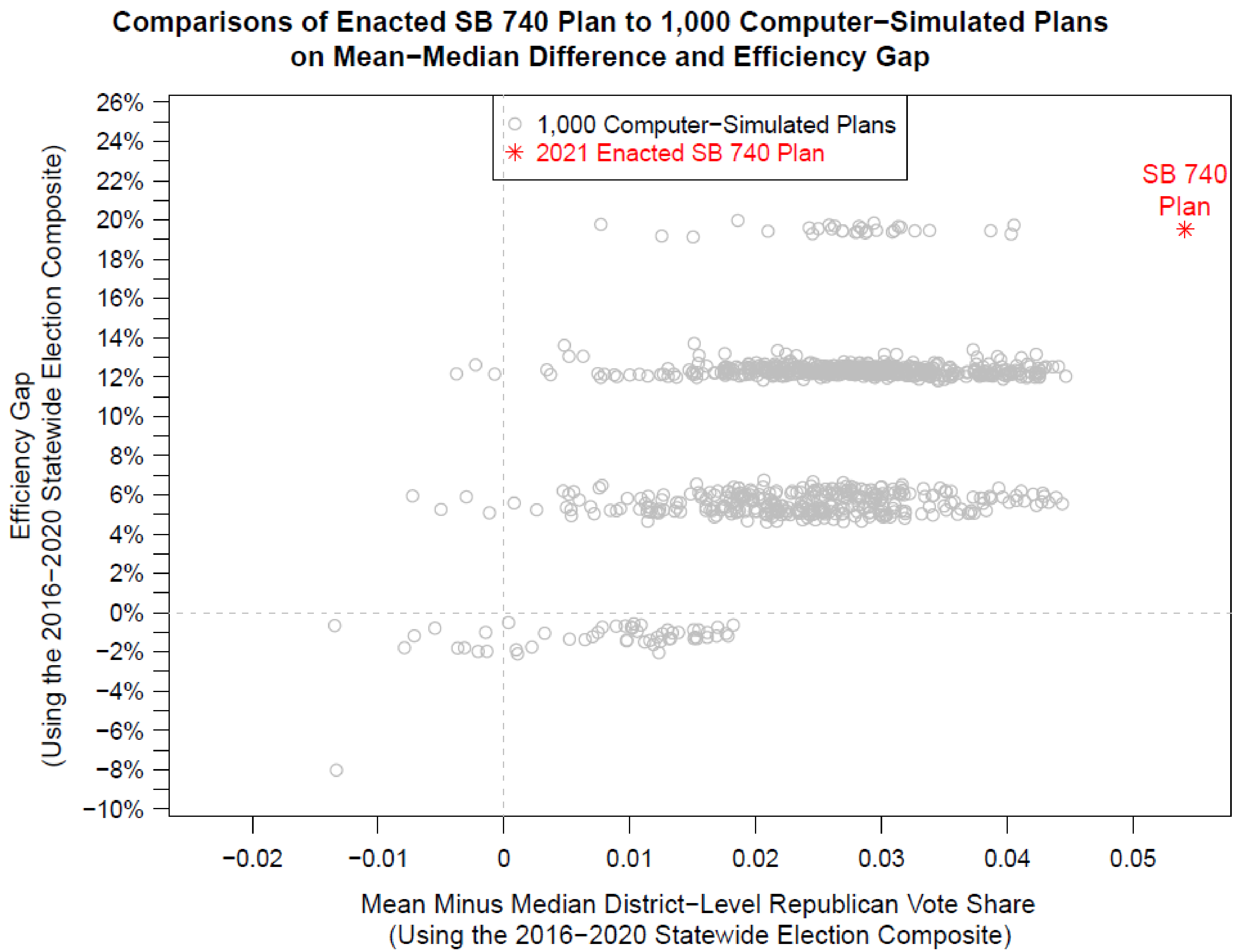
64. I analyze whether the Enacted Plan's efficiency gap arises naturally from a map-drawing process strictly adhering to the mandated criteria in the General Assembly's Adopted Criteria, or rather, whether the skew in the Enacted Plan's efficiency gap is explainable only as the product of a map-drawing process that intentionally favored one party over the other. By comparing the efficiency gap of the Enacted Plan to that of the computer-simulated plans, I am able to evaluate whether or not such the Enacted Plan's efficiency gap could have realistically resulted from adherence to the Adopted Criteria.

65. Figure 9 compares the efficiency gaps of the Enacted Plan and of the 1,000 computer-simulated plans. As before, the 1,000 circles in this Figure represent the 1,000 computer-simulated plans, while the red star in the upper right corner represents the Enacted Plan. Each plan is plotted along the vertical axis according to its efficiency gap, while each plan is plotted along the horizontal axis according to its mean-median difference.

66. The results in Figure 9 illustrate that the Enacted Plan exhibits an efficiency gap

of +19.5%, indicating that the plan results in far more wasted Democratic votes than wasted Republican votes. Specifically, the difference between the total number of wasted Democratic votes and wasted Republican votes amounts to 19.5% of the total number of votes statewide. The Enacted Plan's efficiency gap is larger than the efficiency gaps exhibited by 98.7% of the computer-simulated plans. This comparison reveals that the significant level of Republican bias exhibited by the Enacted Plan cannot be explained by North Carolina's political geography or the Adopted Criteria alone.

Figure 9:


Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
on Mean−Median Difference and Efficiency Gap
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
SB 740
Plan
26%
24%
22%
20%
18%
16%
14%
12%
10%
8%
6%
4%
2%
0%
−2%
−4%
−6%
−8%
−10%
Efficiency Gap
(Using the 2016−2020 Statewide Election Composite)
−0.02
−0.01
0
0.01
0.02
0.03
0.04
0.05
Mean Minus Median District−Level Republican Vote Share
(Using the 2016−2020 Statewide Election Composite)

67. ***The Lopsided Margins Measure:*** Another measure of partisan bias in districting plans is the "lopsided margins" test. The basic premise captured by this measure is that a partisan-motivated map-drawer may attempt to pack the opposing party's voters into a small number of extreme districts that are won by a lopsided margin. Thus, for example, a map-drawer attempting to favor Party A may pack Party B's voters into a small number of districts that very heavily favor Party B. This packing would then allow Party A to win all the remaining districts with relatively smaller margins. This sort of partisan manipulation in districting would result in Party B winning its districts by extremely large margins, while Party A would win its districts by relatively small margins.

68. Hence, the lopsided margins test is performed by calculating the difference between the average margin of victory in Republican-favoring districts and the average margin of victory in Democratic-favoring districts. The 2021 Enacted Plan contains four Democratic-favoring districts (CD-2, 5, 6, and 9), and these four districts have an average Democratic vote share of 65.4%, as measured using the 2016-2020 Statewide Election Composite. By contrast, the Enacted Plan contains ten Republican-favoring districts (CD-1, 3, 4, 7, 8, 10, 11, 12, 13, and 14), and these ten districts have an average Republican vote share of 57.3%. Hence, the difference between the average Democratic margin of victory in Democratic-favoring districts and the average Republican margin of victory in Republican-favoring districts is +8.1%, which is calculated as 65.4% - 57.3%. I refer to this calculation of +8.1% as the Enacted Plan's lopsided margins measure.

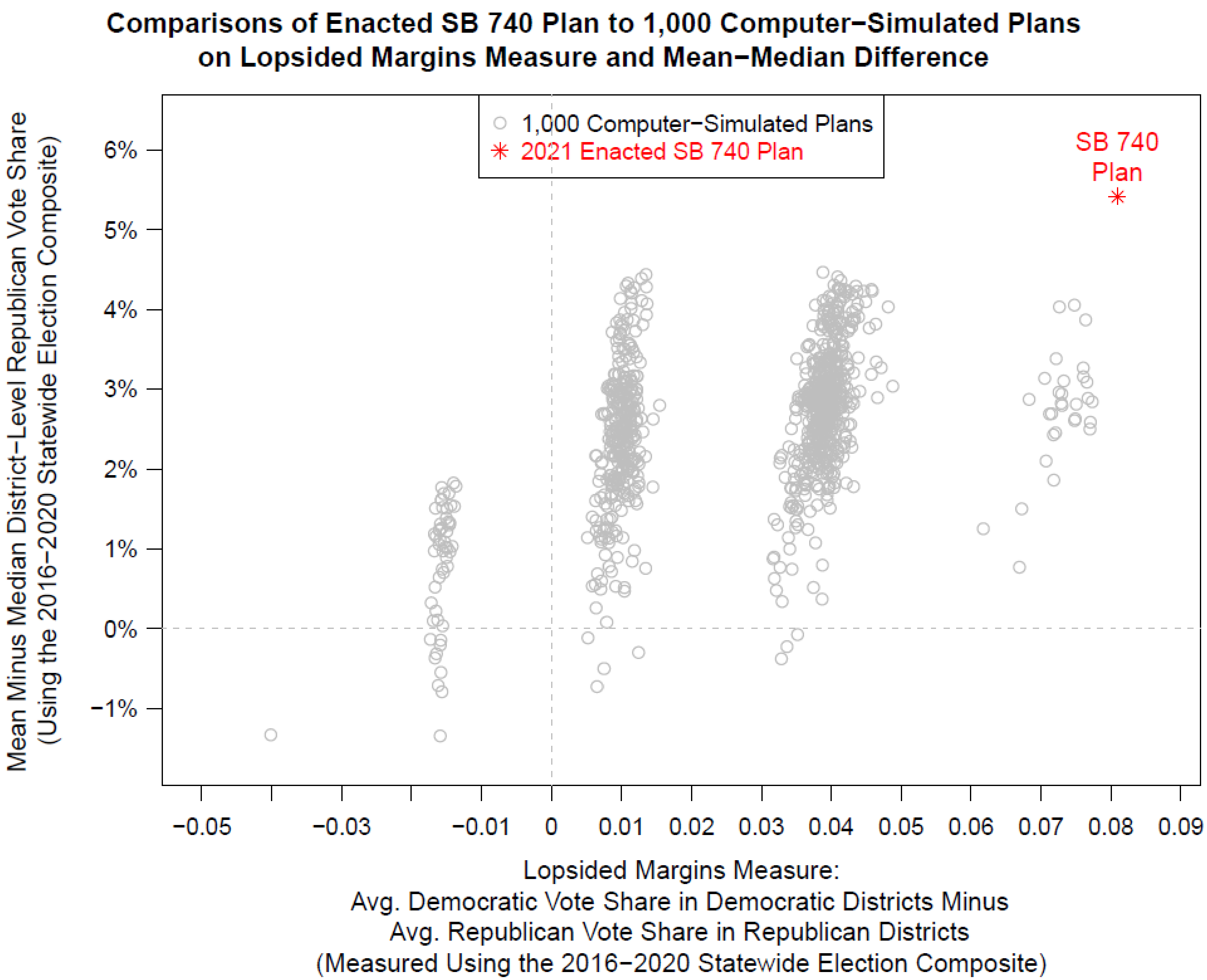
69. How does the 8.1% lopsided margins measure of the Enacted Plan compare to the same calculation for the 1,000 computer-simulated plans? Figure 10 reports the lopsided margins calculations for the Enacted Plan and for the simulated plans. In Figure 10, each plan is plotted

along the horizontal axis according to its lopsided margins measure and along the vertical axis according to its mean-median difference.

70. Figure 10 reveals that the Enacted Plan's +8.1% lopsided margins measure is an extreme outlier compared to the lopsided margins measures of the 1,000 computer-simulated plans. All 1,000 of the simulated plans have a smaller lopsided margins measure than the Enacted Plan. In fact, a significant minority (37.3%) of the 1,000 simulated plans have a lopsided margins measure of between -2% to +2%, indicating a plan in which Democrats and Republicans win their respective districts by similar average margins.

71. By contrast, the Enacted Plan's lopsided margins measure of +8.1% indicates that the Enacted Plan creates districts in which Democrats are extremely packed into their districts, while the margin of victory in Republican districts is significantly smaller. The "lopsidedness" of the two parties' average margin of victory is extreme when compared to the computer-simulated plans. The finding that all 1,000 simulated plans have a smaller lopsided margins measure indicates that the Enacted Plan's extreme packing of Democrats into Democratic-favoring districts was not simply the result of North Carolina's political geography, combined with adherence to the Adopted Criteria.

Figure 10:


Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
on Lopsided Margins Measure and Mean−Median Difference
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
SB 740
Plan
Mean Minus Median District−Level Republican Vote Share
(Using the 2016−2020 Statewide Election Composite)
6%
5%
4%
3%
2%
1%
0%
−1%
−0.05
−0.03
−0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
Lopsided Margins Measure:
Avg. Democratic Vote Share in Democratic Districts Minus
Avg. Republican Vote Share in Republican Districts
(Measured Using the 2016−2020 Statewide Election Composite)

72. ***Partisan Symmetry Based on Uniform Swing:*** Another common measure of partisan bias is based on the concept of partisan symmetry and asks the following question: Under a given districting plan and given a particular election-based measure of district partisanship, what share of seats would each party win in a hypothetical tied election (i.e., 50% vote share for each of two parties). To approximate the district-level outcomes in a hypothetical tied election, one normally uses a uniform swing in order to simulate a tied statewide election. We then calculate whether each party would receive more than or less than 50% of the seats under this hypothetical tied election in a given districting plan. This particular measure is often referred to in the academic literature as "partisan bias." In order to avoid confusion with other measures of partisan bias described in this report, I will refer to this measure as "Partisan Symmetry Based on Uniform Swing."

73. Specifically, I use the 2016-2020 Statewide Election Composite to calculate the Partisan Symmetry measure for both the Enacted Plan and for the computer-simulated plans. The 2016-2020 Statewide Election Composite produces a statewide Republican vote share of 50.8%. Therefore, I use a uniform swing of -0.8% in order to estimate the partisanship of districts under a hypothetical tied election in which each party wins exactly 50% of the statewide vote. In other words, this uniform swing subtracts 0.8% from the Republican vote share in every district, both in the Enacted Plan and in all simulated plans.

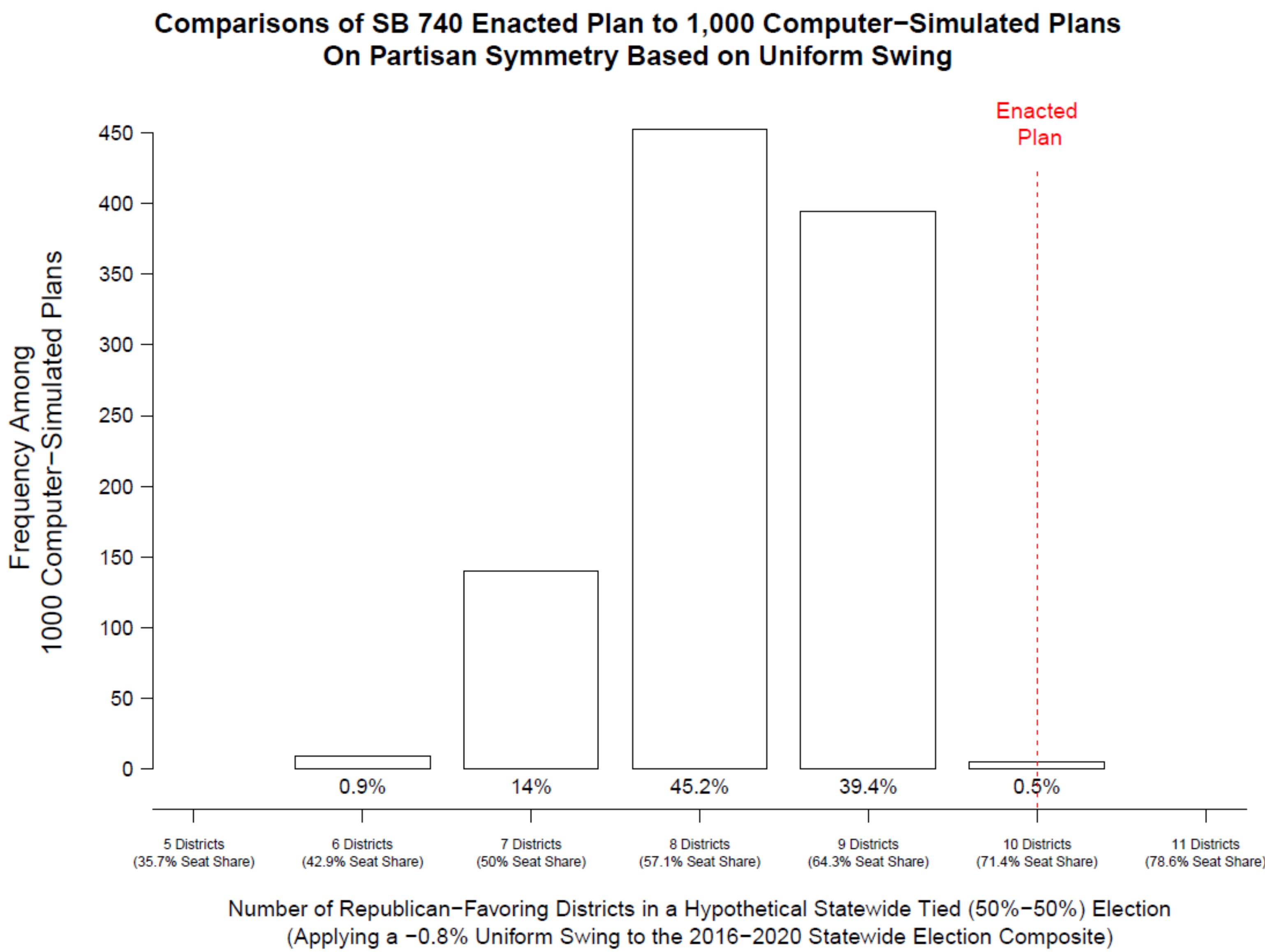
74. After applying this -0.8% uniform swing, I compare the number of Republican-favoring districts in the Enacted Plan and the simulated plans. In the Enacted Plan, 71.4% of the districts (10 out of 14) are Republican-favoring after applying the uniform swing. I then report the Republicans' seat share (71.4%) under this hypothetical tied election in Figure 11 as the "Partisan

Symmetry Based on Uniform Swing" measure for the Enacted Plan. Figure 11 also reports the calculations for all 1,000 simulated plans using this identical method.

75. Figure 11 reveals 99.5% of the 1,000 simulated plans have a "Partisan Symmetry Based on Uniform Swing" measure that is closer to 50% than the Enacted Plan's measure. In fact, 14% of the simulated plans have a measure that is exactly 50% (7 out of 14 districts), while over 60% of the simulated plans are between 40% and 60%.

76. By contrast, the Enacted Plan's measure of 71.4% in Figure 11 would be a statistical outlier and is more favorable to Republicans than in 99.5% of the simulated plans. Substantively, this 71.4% measure reflects the Enacted Plan's creation of a durable Republican majority for North Carolina's congressional delegation, such that even when Democrats win 50% of the statewide vote, Republicans will still be favored in 10 out of 14 (71.4%) of the congressional districts, while Democrats will only be favored in only 4 out of the 14 (28.6%) districts.

Figure 11:


Comparisons of SB 740 Enacted Plan to 1,000 Computer−Simulated Plans
On Partisan Symmetry Based on Uniform Swing
Enacted Plan
Frequency Among 1000 Computer−Simulated Plans
450
400
350
300
250
200
150
100
50
0
0.9%
14%
45.2%
39.4%
0.5%
5 Districts (35.7% Seat Share)
6 Districts (42.9% Seat Share)
7 Districts (50% Seat Share)
8 Districts (57.1% Seat Share)
9 Districts (64.3% Seat Share)
10 Districts (71.4% Seat Share)
11 Districts (78.6% Seat Share)
Number of Republican−Favoring Districts in a Hypothetical Statewide Tied (50%−50%) Election
(Applying a −0.8% Uniform Swing to the 2016−2020 Statewide Election Composite)

Conclusions Regarding Partisanship and Traditional Districting Criteria

77. The analysis described thus far in this report lead me to reach two main findings: First, among the five traditional districting criteria mandated by the General Assembly's 2021 Adopted Criteria, the Enacted Plan fails to minimize county splits, fails to minimize VTD splits, and is significantly less geographically compact than is reasonably possible under a districting process that follows the Adopted Criteria. Second, I found that the Enacted Plan is an extreme partisan outlier when compared to computer-simulated plans produced by a process following the Adopted Criteria. The Enacted Plan contains 10 districts that are partisan outliers when compared to the simulated plans' districts, and using several different common measures of partisan bias, the Enacted Plan creates a level of pro-Republican bias more extreme than in over 95% of the computer-simulated plans. In particular, the Enacted Plan creates more "mid-range" Republican districts than is created in 100% of the computer-simulated plans (Paragraphs 45-46).

78. Based on these two main findings, I conclude that partisanship predominated in the drawing of the 2021 Enacted Plan and subordinated the traditional districting principles of avoiding county splits, avoiding VTD splits, and geographic compactness. Because the Enacted Plan fails to follow three of the Adopted Criteria's mandated districting principles while simultaneously creating an extreme level of partisan bias, I therefore conclude that the partisan bias of the Enacted Plan did not naturally arise by chance from a districting process adhering to the Adopted Criteria. Instead, I conclude that partisan goals predominated in the drawing of the Enacted Plan. By subordinating traditional districting criteria, the General Assembly's Enacted Plan was able to achieve partisan goals that could not otherwise have been achieved under a partisan-neutral districting process that follows the Adopted Criteria.

Regional Comparisons of Enacted Plan and Simulated Plan Districts

79. I have thus far compared the Enacted Plan to the simulated plans at a statewide level using several common measures of partisan bias and by identifying individual districts that are partisan outliers. However, I also analyzed the extent to which partisan bias affected the map-drawing process within specific cities and geographic regions of North Carolina. I found that the Enacted Plan's individual districts in certain regions exhibit extreme political bias when compared to the computer-simulated districts in the same regions. Below, I describe my findings regarding the partisan bias caused by the Enacted Plan's district boundaries in the Piedmont Triad area, in the Research Triangle, and in Mecklenburg County.

80. ***The Piedmont Triad Area:*** The Enacted Plan splits Guilford County into three different districts: CD-7, 10, and 11. These three fragments of Guilford County, which has voted solidly Democratic in recent statewide elections, are each combined with more Republican areas in surrounding counties across the Piedmont Triad area. This three-way splitting of Guilford County results in CD-7, 10, and 11 being safely Republican, each with a Republican vote share between 55.9% and 61.2%, as measured using the 2016-2020 Statewide Election Composite.

81. Is this three-way splitting of Guilford County, and the resulting creation of three safe Republican districts, a districting outcome that could have resulted naturally from the region's political geography, combined with the districting principles required by the Adopted Criteria? A comparison of the Enacted Plan's districts to the simulated districts in the Piedmont Triad area reveals that the Enacted Plan managed to crack Democratic voters in the region to a more extreme extent than in virtually all of the computer-simulated plans. Moreover, the Enacted Plan achieved this extreme cracking of Democrats by creating districts that are significantly less compact than virtually all of the Guilford County districts in the computer-simulated plans.

82. Figure 12 directly compares the partisanship of the Enacted Plan's districts to the simulated plans' districts in the Piedmont Triad area at a local level. Specifically, the top row of Figure 12 describes the district within each plan that contains the most amount of Greensboro's population. In the Enacted Plan, this district is CD-11, and Figure 12 directly compares the Republican vote share of CD-11 to the Republican vote shares of all simulated districts that contain the largest portion of Greensboro residents among all districts in their respective simulated plans. The Figure reveals that the Enacted Plan's CD-11 is more safely Republican than 99.6% of the computer-simulated Greensboro districts. In fact, although CD-11 exhibits a 55.9% Republican vote share, 96.1% of the simulated districts containing Greensboro are Democratic-favoring districts. Hence, it is clear that the Enacted Plan created a safe Republican district for Greensboro, even though a partisan-neutral districting process following the Adopted Criteria would almost always have placed Greensboro in a Democratic-favoring district.

83. The second row of Figure 12 illustrates a similar finding regarding the city of High Point in Guilford County. The Enacted Plan places High Point into CD-10, which has a Republican vote share of 61.2%. CD-10 is more heavily Republican than 99.6% of the High Point-based district in the 1,000 computer-simulated plans. Once again, nearly all of the simulated plans place High Point into a Democratic-favoring district, but the Enacted Plan managed to place High Point into an anomalously Republican district.

84. The third row of Figure 12 reveals a similar finding regarding CD-7, the third district containing a fragment of Guilford County. The city of Burlington (Alamance and Guilford Counties) is assigned to the Enacted Plan's CD-7, which exhibits a 58.2% Republican vote share. CD-7 is more heavily Republican than 99.7% of the Burlington-based districts in the 1,000 computer-simulated plans. In fact, 95.5% of the Burlington districts in the simulated plans

favor the Democrats, often by an extremely wide margin. Thus, it is clear that the Enacted Plan created a far more Republican-favorable district for Burlington than could be reasonably expected from a partisan-blind districting process.

85. Of course, the creation of three safe Republican districts (CD-7, 10, and 11) in the Guilford County area required bringing in Republican voters from other, surrounding districts. One such district was CD-12, a safely Republican district covering areas in the Piedmont Triad region to the west of Guilford County. The fourth row of Figure 12 compares the partisanship of the Enacted Plan's district containing Winston-Salem (CD-12) to the simulated plans' districts containing Winston-Salem. The simulated plan results on this row illustrate that under a partisan-blind districting process, Winston-Salem would normally be placed into an even more heavily Republican district than the Enacted Plan's CD-12. The Enacted Plan's CD-12 is a safe Republican seat with a Republican vote share of 56.6%, but it is less heavily Republican than 91.4% of the computer-simulated districts containing the most of Winston-Salem's population. This finding suggests that CD-12 was drawn to be less extremely Republican than should be expected, given the political geography of the Piedmont Triad area. As a result, more Republican voters could be placed in the surrounding districts, particularly CD-10 and CD-11, that split up Guilford County.


Figure 12: Piedmont Triad Area:
Comparison of Individual Districts' Republican Vote Shares
in the SB 740 Plan and in 1,000 Computer−Simulated Plans
Legend:
1,000 Computer−Simulated Plans
SB 740 Enacted Plan
The District in Each Plan Containing the Most of Greensboro's Population:
CD−11
(99.6%, 0.4%)
The District in Each Plan Containing the Most of High Point's Population:
CD−10
(99.6%, 0.4%)
The District in Each Plan Containing the Most of Burlington's Population:
CD−7
(99.7%, 0.3%)
The District in Each Plan Containing the Most of Winston−Salem's Population:
CD−12
(8.6%, 91.4%)
30%
35%
40%
45%
50%
55%
60%
65%
District's Republican Vote Share Measured Using the 2016−2020 Statewide Election Composite
Percent of Simulated Districts with a Lower/Higher Republican Vote Share than the Enacted Plan's District

86. Could the Enacted Plan's cracking of Guilford County Democrats into three districts (CD-7, 10, and 11) have resulted from a mapdrawing process attempting to follow the Adopted Criteria? The geographic characteristics of these three districts illustrate the opposite conclusion: The General Assembly managed to split Guilford County into three safe Republican districts by subordinating the districting principles required by the Adopted Criteria. Although the Adopted Criteria do not explicitly prohibit dividing Guilford County into three districts, doing so was not necessary to comply with the Adopted Criteria. Guilford County's population is well under that of an equally populated congressional district. In fact, the vast majority (75.6%) of the computer-simulated plans do not split Guilford County a single time. When Guilford County is split, the simulated plans usually split it only once.

87. Moreover, the compactness scores of the Enacted Plan's CD-7, 10, and 11 reveal that the General Assembly subordinated geographic compactness considerations in the process of cracking Democrats in Guilford County. The first row of Figure 13 illustrates that the Enacted Plan's CD-11 has a lower Polsby-Popper score than all 1,000 of the Greensboro-based districts in the computer-simulated plans. The second and third rows of Figure 13 reveal a nearly identical conclusion regarding the other two districts covering Guilford County (CD-7 and CD-10). In fact, there is a vast disparity between the compactness of the Enacted Plan's Guilford County districts and the simulated plans' districts in Guilford County. CD-7, 10, and 11 have Polsby-Popper scores of 0.197, 0.199, and 0.207. Meanwhile, over half of the simulated districts displayed in these upper three rows of Figure 13 have a Polsby-Popper score over 0.5. It is therefore clear that the Enacted Plan subordinated geographic compactness in the pursuit of Republican partisan advantage in the drawing of district boundaries in the Piedmont Triad area.

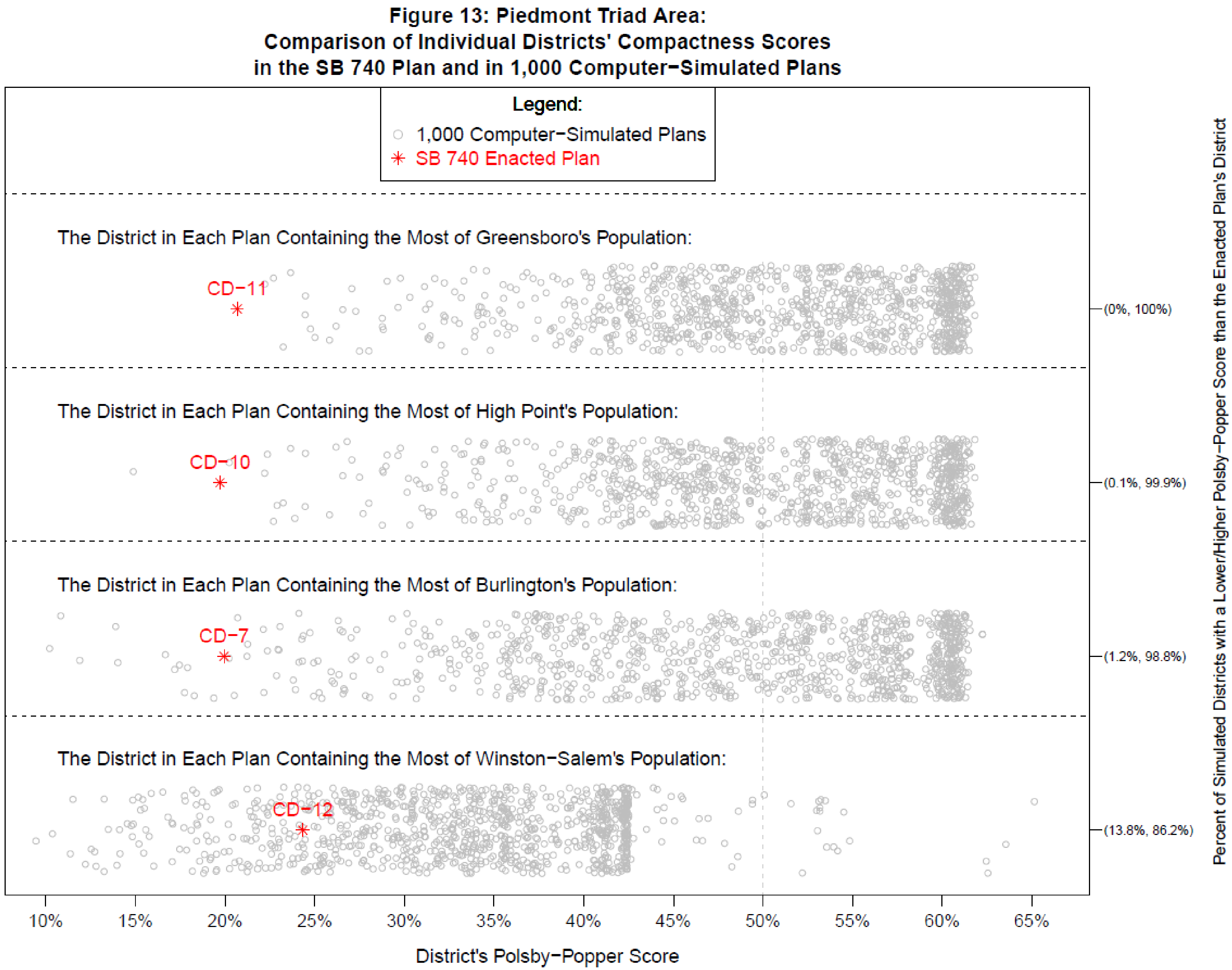


Figure 13: Piedmont Triad Area:
Comparison of Individual Districts' Compactness Scores
in the SB 740 Plan and in 1,000 Computer−Simulated Plans
Legend:
1,000 Computer−Simulated Plans
SB 740 Enacted Plan
The District in Each Plan Containing the Most of Greensboro's Population:
CD−11
The District in Each Plan Containing the Most of High Point's Population:
CD−10
The District in Each Plan Containing the Most of Burlington's Population:
CD−7
The District in Each Plan Containing the Most of Winston−Salem's Population:
CD−12
(0%, 100%)
(0.1%, 99.9%)
(1.2%, 98.8%)
(13.8%, 86.2%)
10%
15%
20%
25%
30%
35%
40%
45%
50%
55%
60%
65%
District's Polsby−Popper Score
Percent of Simulated Districts with a Lower/Higher Polsby−Popper Score than the Enacted Plan's District

88. ***The Research Triangle:*** Figures 14 and 15 present a similar analysis of the districts in the Research Triangle. The top row of Figure 14 compares the Republican vote shares of the Enacted Plan's and each computer-simulated plan's district containing the most of Raleigh's population. The second row of Figure 14 is a similar comparison of the Enacted Plan's and each simulated plan's district containing the most of Durham's population. Overall, these two rows illustrate that the Enacted Plan's Raleigh-based district (CD-5) and Durham-based district (CD-6) are more heavily packed with Democrats than almost 100% of the computer-simulated districts containing Raleigh and Durham.

89. The top two rows of Figure 15 illustrate that extreme degree of Democratic voter packing in CD-5 and CD-6 is not the result of the Research Triangle's political geography or the Adopted Criteria. Instead, Figure 15 reveals that CD-5 and CD-6 are less geographically compact than nearly 100% of the computer-simulated districts containing Raleigh and Durham. Thus, the General Assembly managed to unnaturally pack Democrats in its Raleigh-based and Durham-based districts by subordinating geographic compactness in the drawing of these districts.

90. As a result of this packing of Democratic voters in CD-5 and CD-6, the surrounding districts in the Enacted Plan are more safely Republican than they would have been in the absence of such packing of Democrats. One example of these surrounding Republican districts in the Enacted Plan is CD-7, which combines Southern Wake County with various counties west of the Research Triangle. Southern Wake County is more politically moderate than the heavily Democratic cores of Raleigh and Durham. The third row of Figure 14 compares the partisanship of the Enacted Plan's district and each simulated plan's district containing the most of Holly Springs's and Fuquay-Varina's populations in Southern Wake County. The results on

this row illustrate that in the computer-simulated plans drawn according to the Adopted Criteria, Southern Wake County is generally placed into a heavily-Democratic district because it is generally placed into the same district with part of Raleigh. But the Enacted Plan packed Democrats into CD-5 (Raleigh) and CD-6 (Durham), so the General Assembly was able to create a safe Republican district by combining Southern Wake County with other Republican-favoring counties to the west of the Research Triangle. As the third row of Figure 14 illustrates, this outcome is an extreme statistical outlier compared to the computer-simulated districts in Southern Wake County. 99.2% of the simulated plans place Southern Wake County into a Democratic-favoring district, and 100% of the simulated districts containing Southern Wake County are less extremely Republican than CD-7. Hence, it is clear that CD-7 is a partisan outlier that was enabled by the packing of Democratic voters in CD-5 (Raleigh) and CD-6 (Durham).


Figure 14: Research Triangle Area:
Comparison of Individual Districts' Republican Vote Shares
in the SB 740 Plan and in 1,000 Computer−Simulated Plans
Legend:
1,000 Computer−Simulated Plans
SB 740 Enacted Plan
The District in Each Plan Containing the Most of Raleigh's Population:
CD−5
(0.4%, 99.6%)
The District in Each Plan Containing the Most of Durham's Population:
CD−6
(0%, 100%)
The District in Each Plan Containing the Most of Holly Springs and Fuquay−Varina's (Southern Wake County) Population:
CD−7
(100%, 0%)
25%
30%
35%
40%
45%
50%
55%
60%
65%
District's Republican Vote Share Measured Using the 2016−2020 Statewide Election Composite
Percent of Simulated Districts with a Lower/Higher Republican Vote Share than the Enacted Plan's District

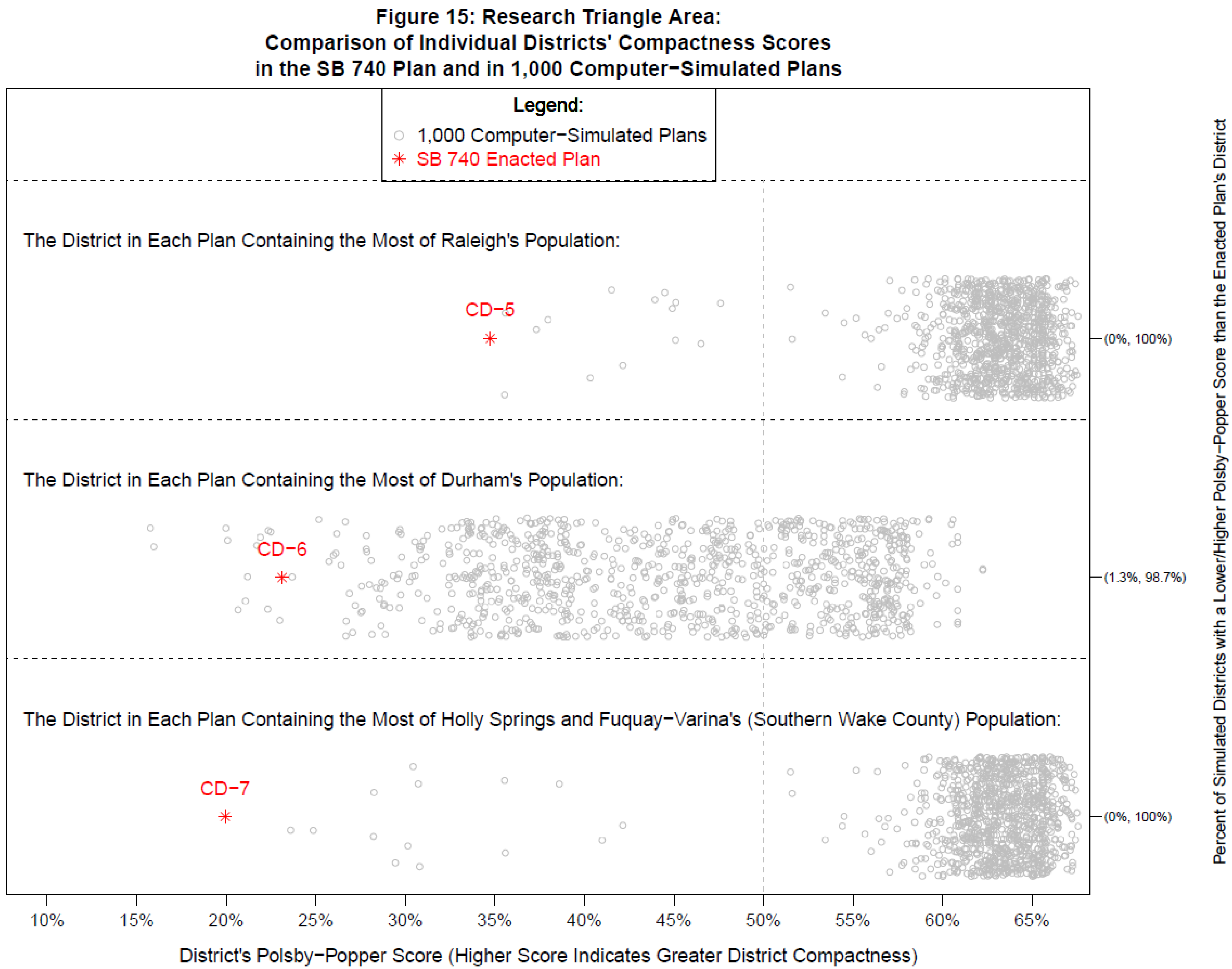


Figure 15: Research Triangle Area:
Comparison of Individual Districts' Compactness Scores
in the SB 740 Plan and in 1,000 Computer−Simulated Plans
Legend:
1,000 Computer−Simulated Plans
SB 740 Enacted Plan
The District in Each Plan Containing the Most of Raleigh's Population:
CD−5
(0%, 100%)
The District in Each Plan Containing the Most of Durham's Population:
CD−6
(1.3%, 98.7%)
The District in Each Plan Containing the Most of Holly Springs and Fuquay−Varina's (Southern Wake County) Population:
CD−7
(0%, 100%)
10%
15%
20%
25%
30%
35%
40%
45%
50%
55%
60%
65%
District's Polsby−Popper Score (Higher Score Indicates Greater District Compactness)
Percent of Simulated Districts with a Lower/Higher Polsby−Popper Score than the Enacted Plan's District

91. ***Mecklenburg County Districts:*** Figure 16 illustrates a similar finding regarding Mecklenburg County. The top row of Figure 16 compares the partisanship of the Enacted Plan's district and each simulated plan's district containing the most of Charlotte's population. The results in this row illustrate that the Enacted Plan's CD-9 is more heavily Democratic than 100% of the simulated plans' primary Charlotte districts.

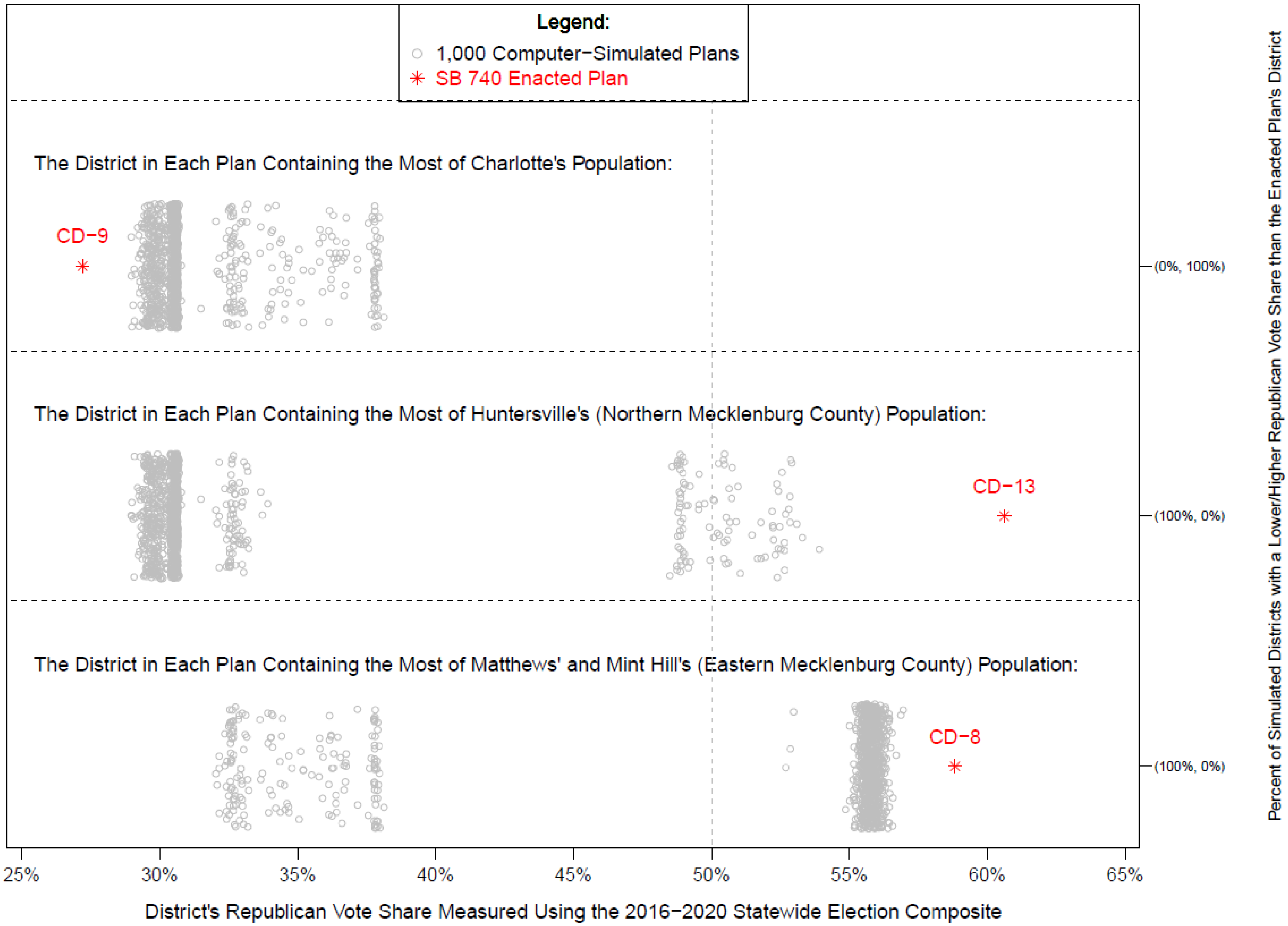
92. As a result, the second and third rows of Figure 16 reveal that the surrounding suburban districts in the Enacted Plan are more safely Republican than their geographic counterparts in all of the computer-simulated plans. Specifically, the second row of Figure 16 compares the partisanship of the Enacted Plan's district and each simulated plan's district containing the most of Huntersville's (Northern Mecklenburg County) population. In the simulated plans, Huntersville is either placed into the same district as most of Charlotte, resulting in a heavily Democratic district, or it is grouped with other counties outside of Mecklenburg, thus forming a politically competitive district with a Republican vote share close to 50%. But the Enacted Plan places Huntersville into a district (CD-13) that is much more strongly Republican than all 100% of the simulated districts containing Huntersville.

93. The third row of Figure 16 reveals a similar finding regarding Eastern Mecklenburg County. Specifically, this row compares the partisanship of the Enacted Plan's district and each simulated plan's district containing the most of Mint Hill's and Matthews' (Eastern Mecklenburg County) population. Once again, the results reveal that the Enacted Plan places Eastern Mecklenburg County into a district (CD-8) that is more strongly Republican than all 100% of the computer-simulated districts containing Mint Hill and Matthews.

94. Thus, it is clear that the Enacted Plan packed Democrats in Mecklenburg County to an extent greater than what naturally occurs as a result of the area's political geography.

Democratic voters are residentially concentrated in Charlotte, and this political geography tends to cause a clustering of Democratic voters in Mecklenburg County districts, as reflected in the simulation results in Figure 16. But the Enacted Plan's packing of Democratic voters in Mecklenburg goes beyond what is caused by political geography, resulting in a Charlotte district that is even more heavily Democratic than what could be expected from a partisan-blind map-drawing process.

Figure 16: Mecklenburg County:
Comparison of Individual Districts' Republican Vote Shares
in the SB 740 Plan and in 1,000 Computer−Simulated Plans



North Carolina's Political Geography Did Not Cause the Enacted Plan's Extreme Partisan Bias

95. How does North Carolina's political geography affect the partisan characteristics of the 2021 Enacted Plan? Democratic voters tend to be geographically concentrated in the urban cores of several of the state's largest cities, including Charlotte, Raleigh, and Greensboro. As I have explained in my prior academic research,[14] these large urban clusters of Democratic voters, combined with the common districting principle of drawing geographically compact districts, can sometimes result in urban districts that "naturally" pack together Democratic voters, thus boosting the Republican vote share of other surrounding suburban and rural districts.

96. More importantly, my prior academic research explained how I can estimate the precise level of electoral bias in districting caused by a state's unique political geography: I programmed a computer algorithm that draws districting plans using North Carolina's unique political geography, including the state's census population data and political subdivision boundaries. In this report, I have also programmed the algorithm to follow North Carolina's Adopted Criteria. I then analyzed the partisan characteristics of the simulated districting plans using North Carolina's precinct-level voting data from past elections (past elections that were themselves skewed towards Republicans). Hence, the entire premise of conducting districting simulations is to fully account for North Carolina's unique political geography, its political subdivision boundaries, and its districting criteria, as mandated by the General Assembly's Adopted Criteria.

97. This districting simulation analysis allowed me to identify how much of the

[14] Jowei Chen and Jonathan Rodden, 2013. "Unintentional Gerrymandering: Political Geography and Electoral Bias in Legislatures" Quarterly Journal of Political Science, 8(3): 239-269; Jowei Chen and David Cottrell, 2016. "Evaluating Partisan Gains from Congressional Gerrymandering: Using Computer Simulations to Estimate the Effect of Gerrymandering in the U.S. House." Electoral Studies, Vol. 44, No. 4: 329-430.

electoral bias in the 2021 Enacted Plan is caused by North Carolina's political geography and how much is caused by the map-drawer's intentional efforts to favor one political party over the other. North Carolina's natural political geography, combined with the Adopted Criteria, almost never resulted in simulated congressional plans containing 10 Republican-favoring districts out of 14 total districts.

98. The 2021 Enacted Plan's creation of 10 electorally safe Republican districts, which persists across a range of electoral outcomes, goes beyond any "natural" level of electoral bias caused by North Carolina's political geography or the political composition of the state's voters. The Enacted Plan is a statistical outlier in terms of its partisan characteristics when compared to the 1,000 computer-simulated plans and cannot be explained by North Carolina's natural political geography.

99. The two most Republican districts (CD-10 and CD-13) and the two most Democratic districts (CD-9 and CD-6) in the Enacted Plan were drawn to include more Democratic voters than virtually all of their counterpart districts in the 1,000 computer-simulated plans. Six other districts (CD-1, 3, 4, 11, 12, and 14) were drawn to be more heavily Republican than over 95% of their counterpart computer-simulated plan districts. Ten districts were drawn precisely to have Republican vote shares within the narrow range of 52.9% to 61.2%—an outcome that never arises in the computer-simulated plans.

100. This extreme, additional level of partisan bias in the 2021 Enacted Plan can be directly attributed to the map-drawer's clear efforts to favor the Republican Party. This level of partisan bias was not caused by North Carolina's political geography.

The Effect of the Enacted Plan Districts on Plaintiffs

101. I evaluated the congressional districts in which each Plaintiff would reside under the 1,000 computer-simulated maps using a list of geocoded residential addresses for the Plaintiffs that counsel for the Plaintiffs provided me. I used these geocoded addresses to identify the specific district in which each Plaintiff would be located under each computer-simulated plan, as well as under the Enacted Plan. I then compared the partisanship of each individual Plaintiff's Enacted Plan district to the partisanship of the Plaintiff's 1,000 districts from the 1,000 computer-simulated plans. Using this approach, I identify whether each Plaintiff's district is a partisan outlier when compared to the Plaintiff's 1,000 computer-simulated districts.

102. Figures 17a and 17b present the results of this analysis. These Figures list the individual Plaintiffs and describes the partisanship of each Plaintiff's district of residence in the Enacted Plan, as well as the partisanship of the district the Plaintiff would have resided in under each of the 1,000 simulated congressional plans. The first half of the plaintiffs are analyzed in Figure 17a, while the second half of the plaintiffs appear in Figure 17b.

103. To explain these analyses with an example, each row in Figure 17a corresponds to a particular individual Plaintiff. In the first row, describing Plaintiff Bobby Jones, the red star depicts the partisanship of the Plaintiff's Enacted Plan district (CD-2), as measured by its Republican vote share using the 2016-2020 Statewide Election Composite. The 1,000 gray circles on this row depict the Republican vote share of each of the 1,000 simulated districts in which the Plaintiff would reside in each of the 1,000 computer-simulated plans, based on that Plaintiff's residential address. In the margin to the right of each row, I list in parentheses how many of the 1,000 simulated plans would place the plaintiff in a more Democratic-leaning district (on the left) and how many of the 1,000 simulations would place the plaintiff in a more

Republican-leaning district (on the right) than the Plaintiff's Enacted Plan district. Thus, for example, the first row of Figure 17a reports that 99% of the 1,000 computer-simulated plans would place Plaintiff Bobby Jones in a more Republican-leaning district than his actual Enacted Plan district (CD-2). Therefore, I can conclude that Plaintiff Bobby Jones' Enacted Plan district is a partisan statistical outlier when compared to his district under the 1,000 simulated plans.

Figure 17a:
Plaintiffs' Districts in the SB 740 Plan and in 1,000 Computer−Simulated Plans

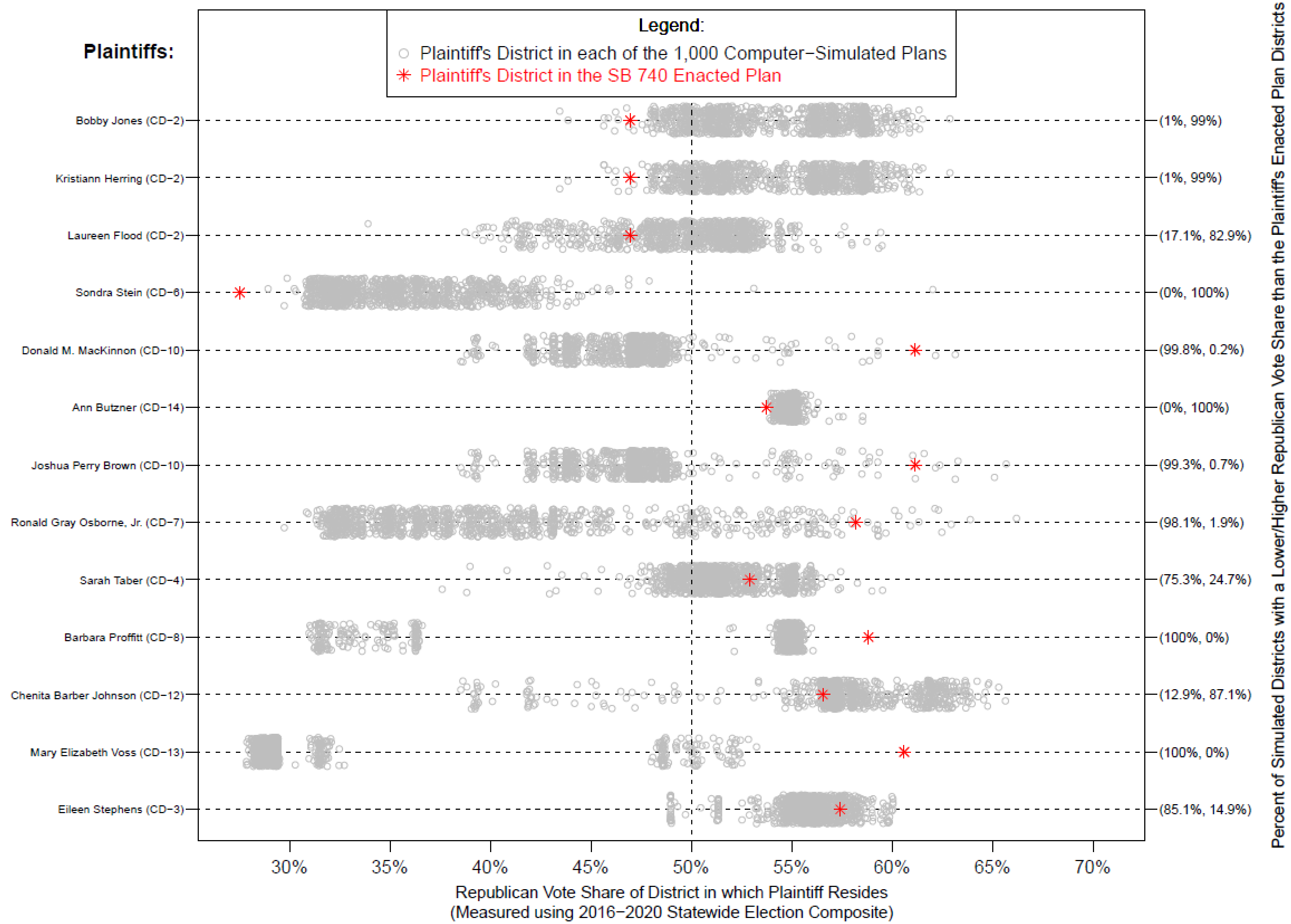


Figure 17b:
Plaintiffs' Districts in the SB 740 Plan and in 1,000 Computer−Simulated Plans

104. Figures 17a and 17b show that seven Plaintiffs residing in Republican-leaning districts under the Enacted Plan would be placed in a more Democratic-leaning district in over 95% of the computer-simulated plans: Donald M. MacKinnon (CD-10), Joshua Perry Brown (CD-10), Ronald Gray Osborne, Jr. (CD-7), Barbara Proffitt (CD-8), Mary Elizabeth Voss (CD-13); David Brown (CD-11) and Lily Nicole Quick (CD-7). Additionally, six Plaintiffs residing in Democratic-leaning districts under the Enacted Plan would be placed in a more Republican-leaning district in over 95% of the computer-simulated plans: Bobby Jones (CD-2), Kristiann Herring (CD-2), Sondra Stein (CD-6), Virginia Brien (CD-9), Jackson Dunn (CD-9), and Rebecca Harper (CD-6). Additionally, six Plaintiffs would be placed in a more Republican district in 99.9% or more of the simulated plans relative to their districts under the Enacted Plan: Ann Butzner (CD-14), Virginia Brien (CD-9), Jackson Dunn (CD-9), Mark Peters (CD-14), Kathleen Barnes (CD-14), Richard R. Crews (CD-14), and Rebecca Harper (CD-6).

I declare under penalty of perjury that the foregoing is true and correct to the best of my knowledge.

This 23rd day of December, 2021.

Dr. Jowei Chen

Jowei Chen
Curriculum Vitae

Department of Political Science
University of Michigan
5700 Haven Hall
505 South State Street
Ann Arbor, MI 48109-1045
Phone: 917-861-7712, Email: jowei@umich.edu
Website: http://www.umich.edu/~jowei

Academic Positions:
Associate Professor (2015-present), Assistant Professor (2009-2015), Department of Political Science, University of Michigan.
Research Associate Professor (2016-present), Faculty Associate (2009-2015), Center for Political Studies, University of Michigan.
W. Glenn Campbell and Rita Ricardo-Campbell National Fellow, Hoover Institution, Stanford University, 2013.
Principal Investigator and Senior Research Fellow, Center for Governance and Public Policy Research, Willamette University, 2013 – Present.

Education:
Ph.D., Political Science, Stanford University (June 2009)
M.S., Statistics, Stanford University (January 2007)
B.A., Ethics, Politics, and Economics, Yale University (May 2004)

Publications:

Chen, Jowei and Neil Malhotra. 2007. "The Law of k/n: The Effect of Chamber Size on Government Spending in Bicameral Legislatures."
    ***American Political Science Review*. 101(4): 657-676.**

Chen, Jowei, 2010. "The Effect of Electoral Geography on Pork Barreling in Bicameral Legislatures."
    ***American Journal of Political Science*. 54(2): 301-322.**

Chen, Jowei, 2013. "Voter Partisanship and the Effect of Distributive Spending on Political Participation."
    ***American Journal of Political Science*. 57(1): 200-217.**

Chen, Jowei and Jonathan Rodden, 2013. "Unintentional Gerrymandering: Political Geography and Electoral Bias in Legislatures"
    ***Quarterly Journal of Political Science*, 8(3): 239-269.**

Bradley, Katharine and Jowei Chen, 2014. “Participation Without Representation? Senior Opinion, Legislative Behavior, and Federal Health Reform.”
Journal of Health Politics, Policy and Law. **39(2), 263-293.**

Chen, Jowei and Tim Johnson, 2015. “Federal Employee Unionization and Presidential Control of the Bureaucracy: Estimating and Explaining Ideological Change in Executive Agencies.”
Journal of Theoretical Politics, **Volume 27, No. 1: 151-174.**

Bonica, Adam, Jowei Chen, and Tim Johnson, 2015. “Senate Gate-Keeping, Presidential Staffing of ‘Inferior Offices’ and the Ideological Composition of Appointments to the Public Bureaucracy.”
Quarterly Journal of Political Science. **Volume 10, No. 1: 5-40.**

Chen, Jowei and Jonathan Rodden, 2015. “Cutting Through the Thicket: Redistricting Simulations and the Detection of Partisan Gerrymanders.”
Election Law Journal. **Volume 14, Number 4: 331-345.**

Chen, Jowei and David Cottrell, 2016. “Evaluating Partisan Gains from Congressional Gerrymandering: Using Computer Simulations to Estimate the Effect of Gerrymandering in the U.S. House.”
Electoral Studies. **Volume 44 (December 2016): 329-340.**

Chen, Jowei, 2017. “Analysis of Computer-Simulated Districting Maps for the Wisconsin State Assembly.”
Election Law Journal. **Volume 16, Number 4 (December 2017): 417-442.**

Chen, Jowei and Nicholas Stephanopoulos, 2020. “The Race-Blind Future of Voting Rights.”
Yale Law Journal, Forthcoming. Volume 130, Number 4: 778-1049.

Kim, Yunsieg and Jowei Chen, 2021. "Gerrymandered by Definition: The Distortion of 'Traditional' Districting Principles and a Proposal for an Empirical Redefinition."
Wisconsin Law Review, Forthcoming, Volume 2021, Number 1.

Chen, Jowei and Nicholas Stephanopoulos, 2021. "Democracy's Denominator."
California Law Review, Accepted for Publication, Volume 109.

Non-Peer-Reviewed Publication:

Chen, Jowei and Tim Johnson. 2017. “Political Ideology in the Bureaucracy.”
***Global Encyclopedia of Public Administration, Public Policy, and Governance*.**

Research Grants:

"How Citizenship-Based Redistricting Systemically Disadvantages Voters of Color". 2020 ($18,225). Combating and Confronting Racism Grant. University of Michigan Center for Social Solutions and Poverty Solutions.

Principal Investigator. National Science Foundation Grant SES-1459459, September 2015 – August 2018 ($165,008). "The Political Control of U.S. Federal Agencies and Bureaucratic Political Behavior."

"Economic Disparity and Federal Investments in Detroit," (with Brian Min) 2011. Graham Institute, University of Michigan ($30,000).

"The Partisan Effect of OSHA Enforcement on Workplace Injuries," (with Connor Raso) 2009. John M. Olin Law and Economics Research Grant ($4,410).

Invited Talks:

September, 2011. University of Virginia, American Politics Workshop.
October 2011. Massachusetts Institute of Technology, American Politics Conference.
January 2012. University of Chicago, Political Economy/American Politics Seminar.
February 2012. Harvard University, Positive Political Economy Seminar.
September 2012. Emory University, Political Institutions and Methodology Colloquium.
November 2012. University of Wisconsin, Madison, American Politics Workshop.
September 2013. Stanford University, Graduate School of Business, Political Economy Workshop.
February 2014. Princeton University, Center for the Study of Democratic Politics Workshop.
November 2014. Yale University, American Politics and Public Policy Workshop.
December 2014. American Constitution Society for Law & Policy Conference: Building the Evidence to Win Voting Rights Cases.
February 2015. University of Rochester, American Politics Working Group.
March 2015. Harvard University, Voting Rights Act Workshop.
May 2015. Harvard University, Conference on Political Geography.
Octoer 2015. George Washington University School of Law, Conference on Redistricting Reform.
September 2016. Harvard University Center for Governmental and International Studies, Voting Rights Institute Conference.
March 2017. Duke University, Sanford School of Public Policy, Redistricting Reform Conference.
October 2017. Willamette University, Center for Governance and Public Policy Research
October 2017, University of Wisconsin, Madison. Geometry of Redistricting Conference.
February 2018: University of Georgia Law School
September 2018. Willamette University.
November 2018. Yale University, Redistricting Workshop.

November 2018. University of Washington, Severyns Ravenholt Seminar in Comparative Politics.
January 2019. Duke University, Reason, Reform & Redistricting Conference.
February 2019. Ohio State University, Department of Political Science. Departmental speaker series.
March 2019. Wayne State University Law School, Gerrymandering Symposium.
November 2019. Big Data Ignite Conference.
November 2019. Calvin College, Department of Mathematics and Statistics.
September 2020 (Virtual). Yale University, Yale Law Journal Scholarship Workshop

Conference Service:

Section Chair, 2017 APSA (San Francisco, CA), Political Methodology Section
Discussant, 2014 Political Methodology Conference (University of Georgia)
Section Chair, 2012 MPSA (Chicago, IL), Political Geography Section.
Discussant, 2011 MPSA (Chicago, IL) “Presidential-Congressional Interaction.”
Discussant, 2008 APSA (Boston, MA) “Congressional Appropriations.”
Chair and Discussant, 2008 MPSA (Chicago, IL) “Distributive Politics: Parties and Pork.”

Conference Presentations and Working Papers:

“Ideological Representation of Geographic Constituencies in the U.S. Bureaucracy,” (with Tim Johnson). 2017 APSA.

“Incentives for Political versus Technical Expertise in the Public Bureaucracy,” (with Tim Johnson). 2016 APSA.

“Black Electoral Geography and Congressional Districting: The Effect of Racial Redistricting on Partisan Gerrymandering”. 2016 Annual Meeting of the Society for Political Methodology (Rice University)

“Racial Gerrymandering and Electoral Geography.” Working Paper, 2016.

“Does Deserved Spending Win More Votes? Evidence from Individual-Level Disaster Assistance,” (with Andrew Healy). 2014 APSA.

“The Geographic Link Between Votes and Seats: How the Geographic Distribution of Partisans Determines the Electoral Responsiveness and Bias of Legislative Elections,” (with David Cottrell). 2014 APSA.

“Gerrymandering for Money: Drawing districts with respect to donors rather than voters.” 2014 MPSA.

“Constituent Age and Legislator Responsiveness: The Effect of Constituent Opinion on the Vote for Federal Health Reform.” (with Katharine Bradley) 2012 MPSA.

“Voter Partisanship and the Mobilizing Effect of Presidential Advertising.” (with Kyle Dropp) 2012 MPSA.

“Recency Bias in Retrospective Voting: The Effect of Distributive Benefits on Voting Behavior.” (with Andrew Feher) 2012 MPSA.

“Estimating the Political Ideologies of Appointed Public Bureaucrats,” (with Adam Bonica and Tim Johnson) 2012 Annual Meeting of the Society for Political Methodology (University of North Carolina)

“Tobler’s Law, Urbanization, and Electoral Bias in Florida.” (with Jonathan Rodden) 2010 Annual Meeting of the Society for Political Methodology (University of Iowa)

“Unionization and Presidential Control of the Bureaucracy” (with Tim Johnson) 2011 MPSA.

“Estimating Bureaucratic Ideal Points with Federal Campaign Contributions” 2010 APSA. (Washington, DC).

“The Effect of Electoral Geography on Pork Spending in Bicameral Legislatures,” Vanderbilt University Conference on Bicameralism, 2009.

“When Do Government Benefits Influence Voters’ Behavior? The Effect of FEMA Disaster Awards on US Presidential Votes,” 2009 APSA (Toronto, Canada).

“Are Poor Voters Easier to Buy Off?” 2009 APSA (Toronto, Canada).

“Credit Sharing Among Legislators: Electoral Geography’s Effect on Pork Barreling in Legislatures,” 2008 APSA (Boston, MA).

“Buying Votes with Public Funds in the US Presidential Election,” Poster Presentation at the 2008 Annual Meeting of the Society for Political Methodology (University of Michigan).

“The Effect of Electoral Geography on Pork Spending in Bicameral Legislatures,” 2008 MPSA.

“Legislative Free-Riding and Spending on Pure Public Goods,” 2007 MPSA (Chicago, IL).

“Free Riding in Multi-Member Legislatures,” (with Neil Malhotra) 2007 MPSA (Chicago, IL).

“The Effect of Legislature Size, Bicameralism, and Geography on Government Spending: Evidence from the American States,” (with Neil Malhotra) 2006 APSA (Philadelphia, PA).

Figure A1: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 Attorney General Election Results

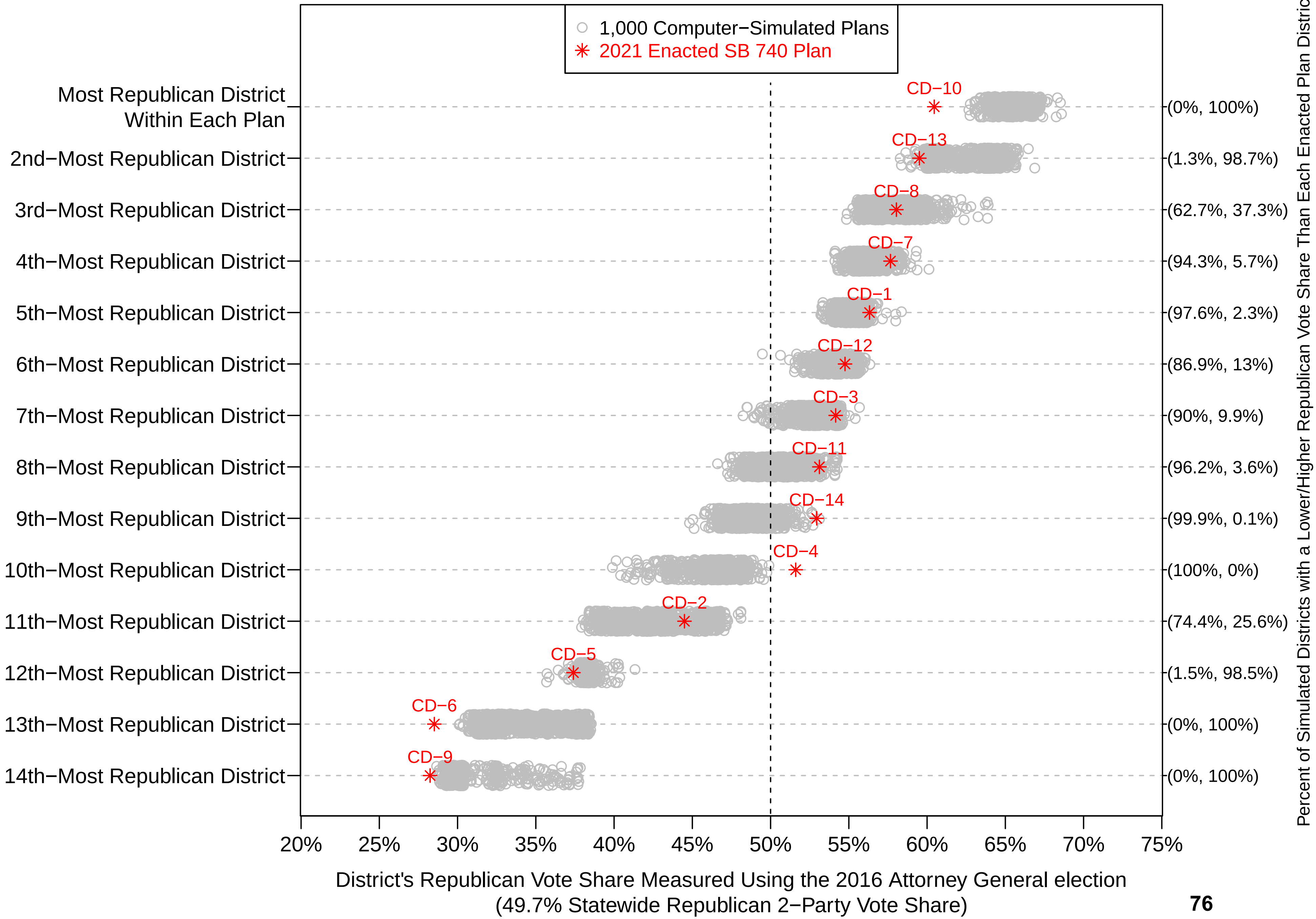


Figure A2: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 Governor Election Results

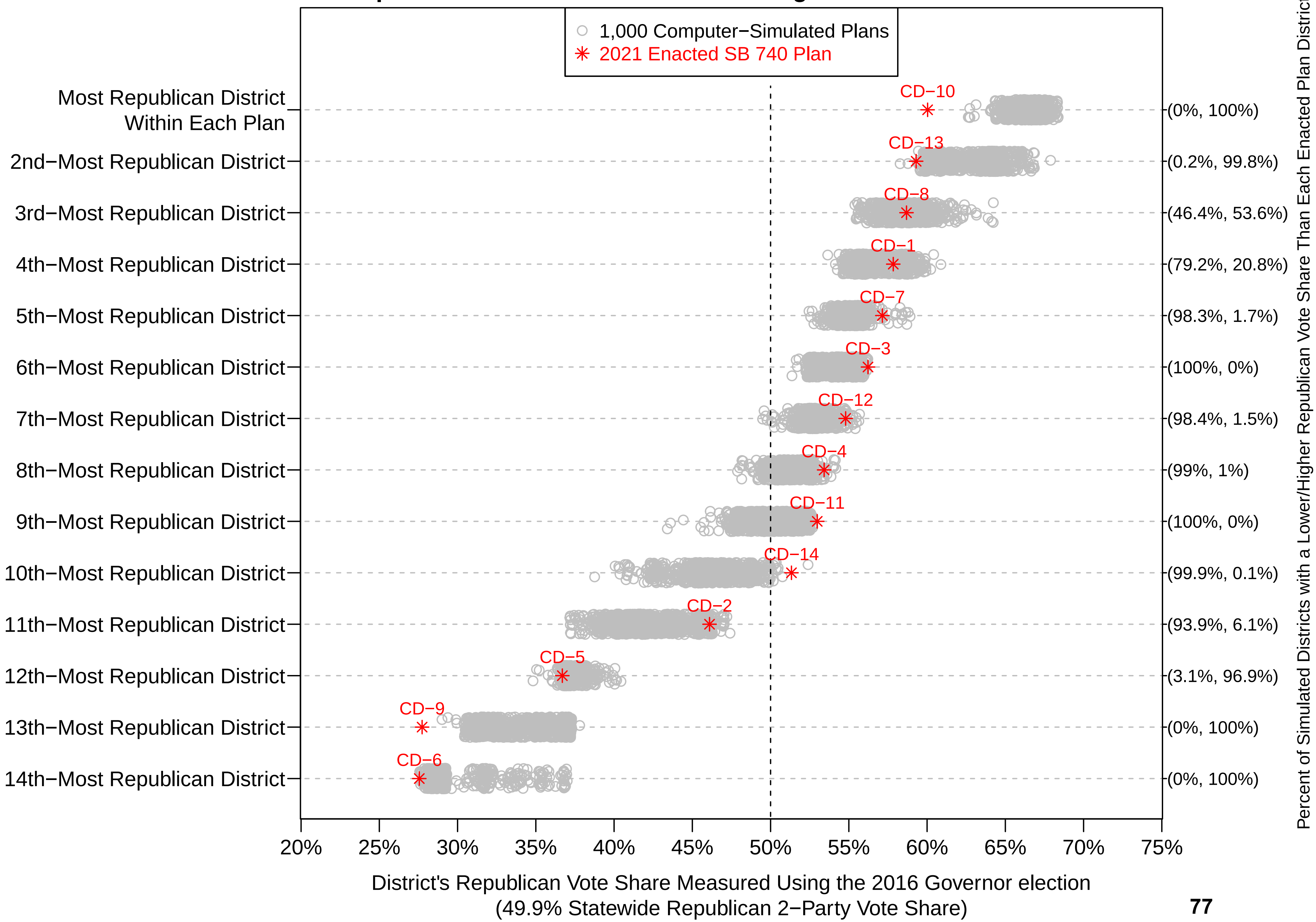


Figure A3: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 Lieutenant Governor Election Results

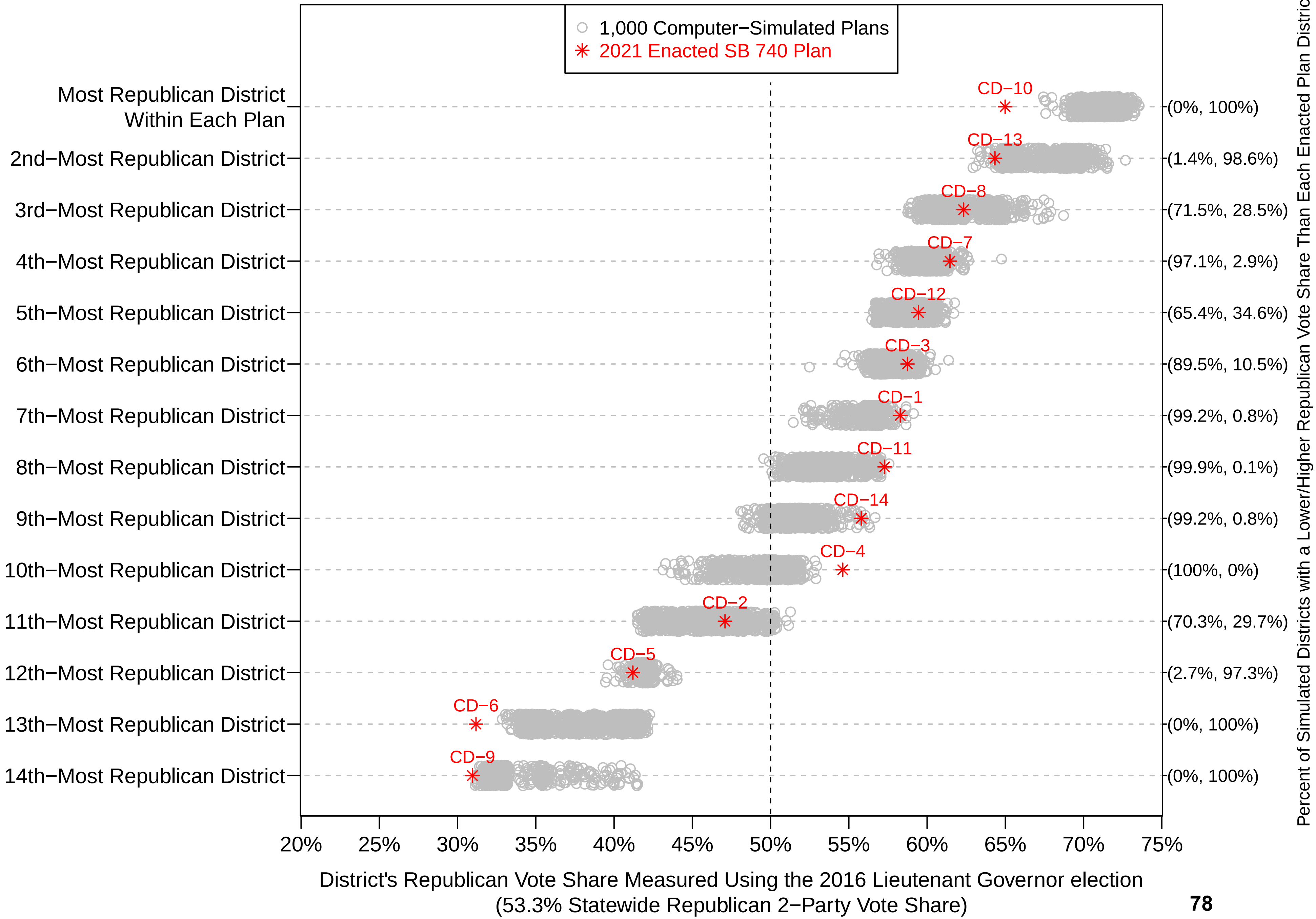


Figure A4: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 US President Election Results

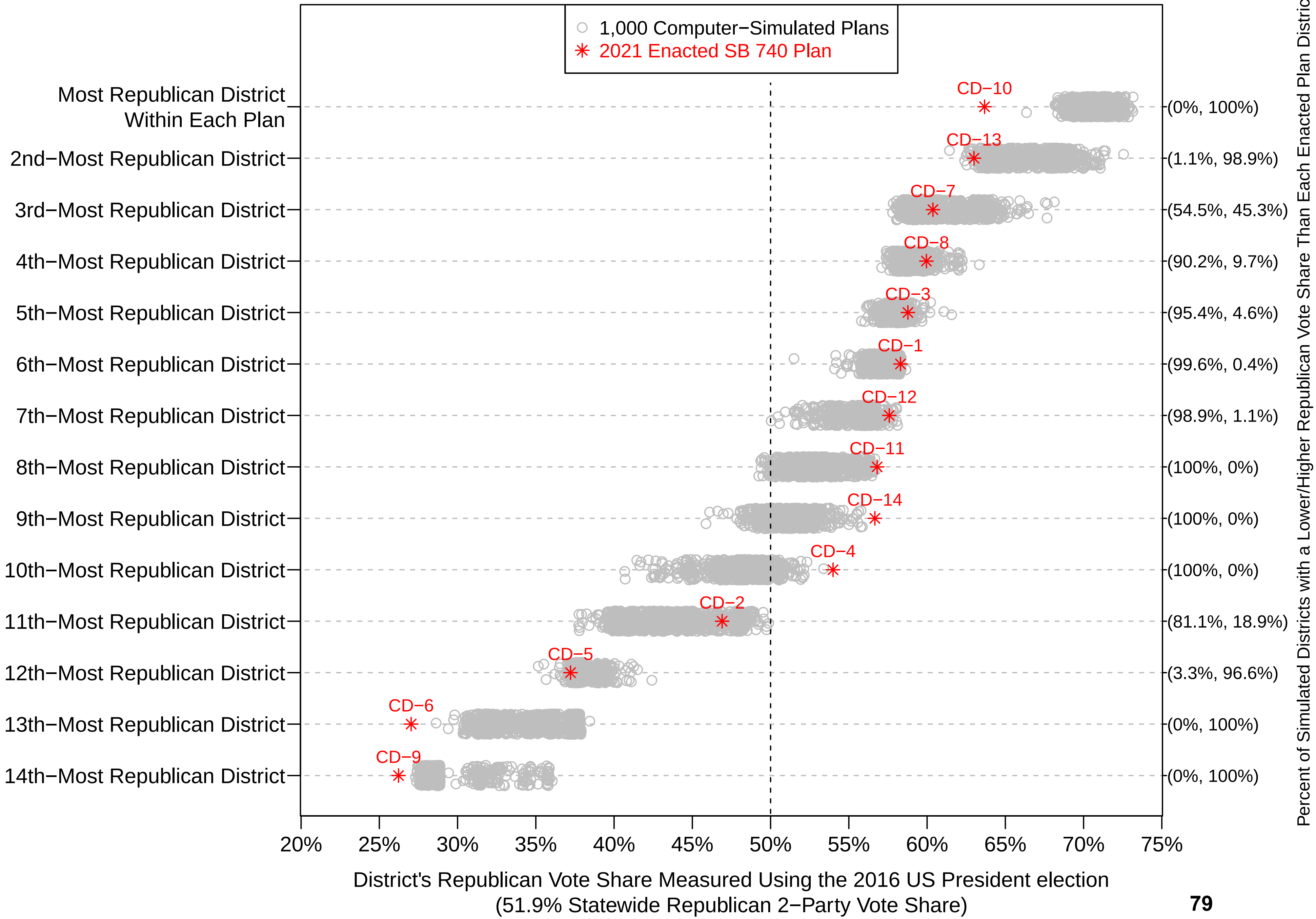


Figure A5: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 US Senator Election Results

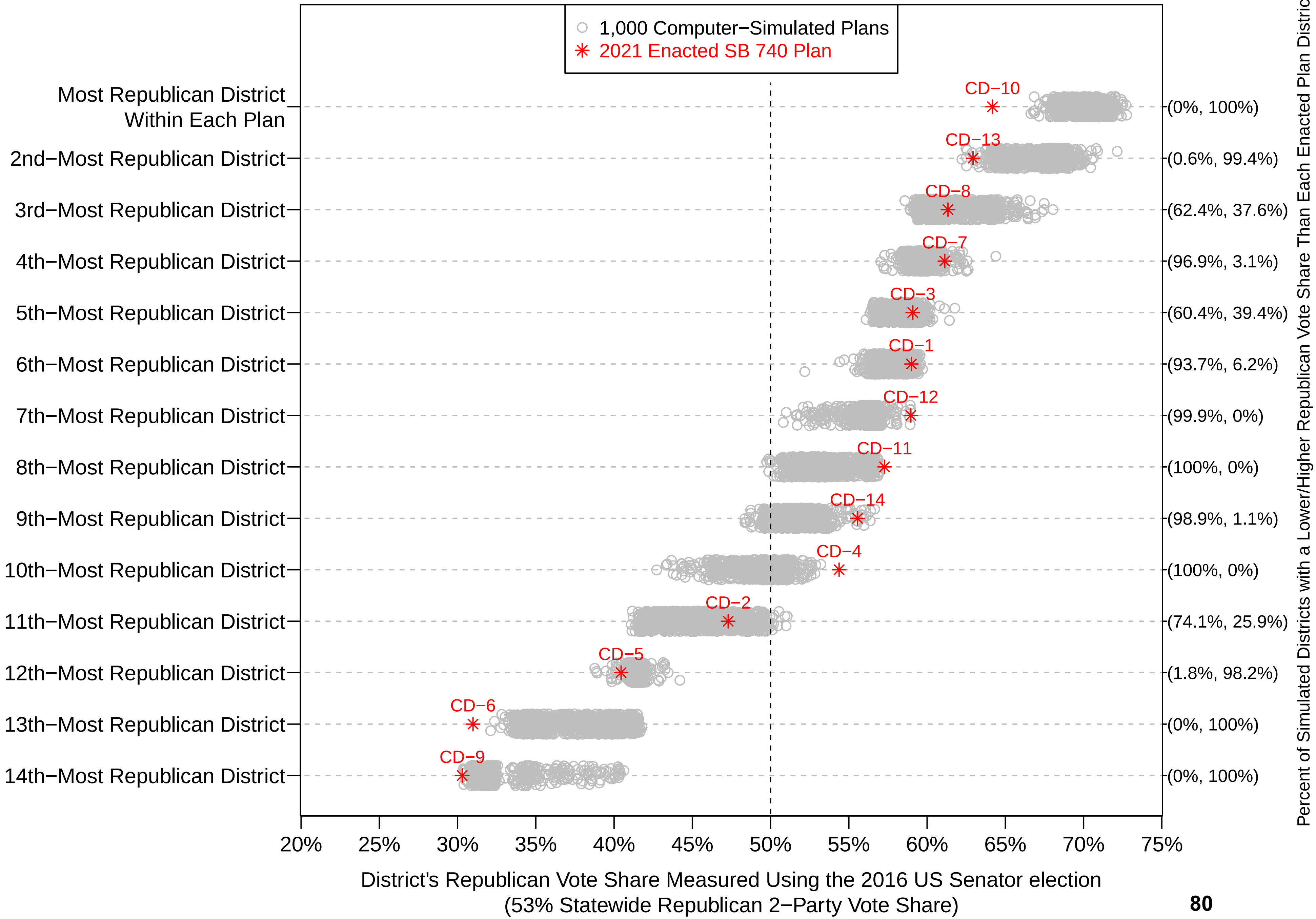
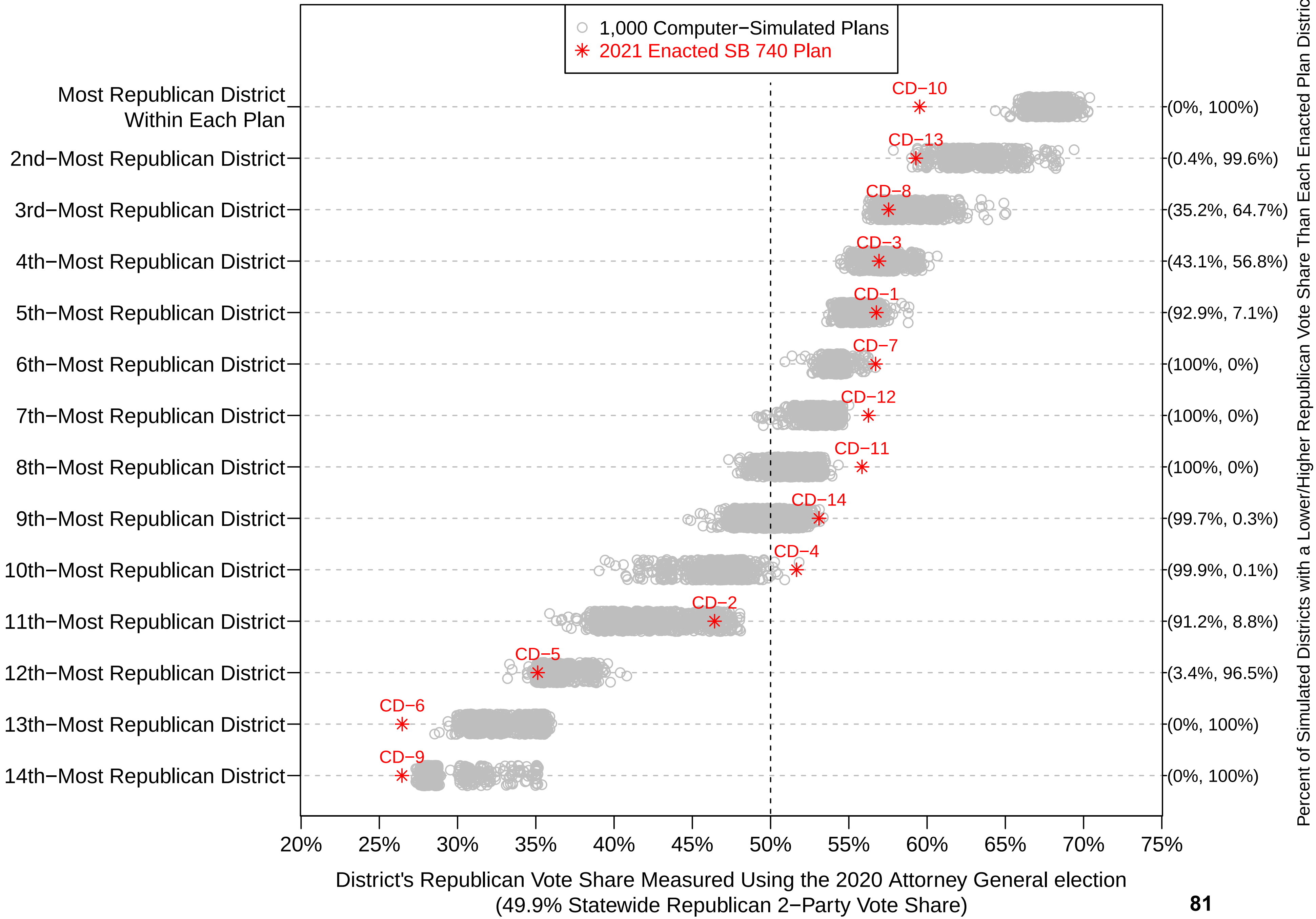


Figure A6: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 Attorney General Election Results



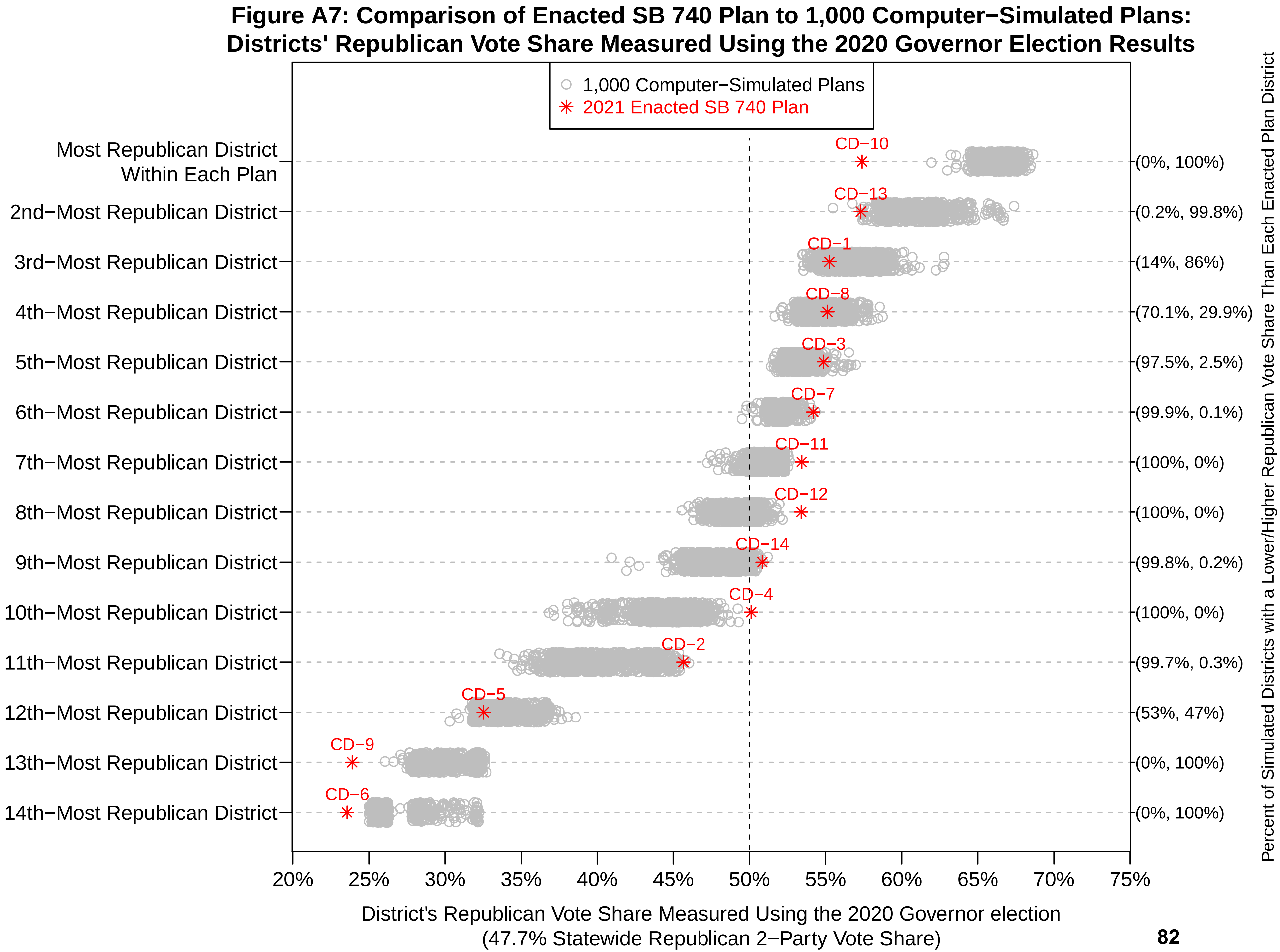


Figure A7: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans:
Districts' Republican Vote Share Measured Using the 2020 Governor Election Results
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
Most Republican District Within Each Plan
2nd−Most Republican District
3rd−Most Republican District
4th−Most Republican District
5th−Most Republican District
6th−Most Republican District
7th−Most Republican District
8th−Most Republican District
9th−Most Republican District
10th−Most Republican District
11th−Most Republican District
12th−Most Republican District
13th−Most Republican District
14th−Most Republican District
CD−10
CD−13
CD−1
CD−8
CD−3
CD−7
CD−11
CD−12
CD−14
CD−4
CD−2
CD−5
CD−9
CD−6
(0%, 100%)
(0.2%, 99.8%)
(14%, 86%)
(70.1%, 29.9%)
(97.5%, 2.5%)
(99.9%, 0.1%)
(100%, 0%)
(100%, 0%)
(99.8%, 0.2%)
(100%, 0%)
(99.7%, 0.3%)
(53%, 47%)
(0%, 100%)
(0%, 100%)
Percent of Simulated Districts with a Lower/Higher Republican Vote Share Than Each Enacted Plan District
20%
25%
30%
35%
40%
45%
50%
55%
60%
65%
70%
75%
District's Republican Vote Share Measured Using the 2020 Governor election
(47.7% Statewide Republican 2−Party Vote Share)

Figure A8: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 Lieutenant Governor Election Results

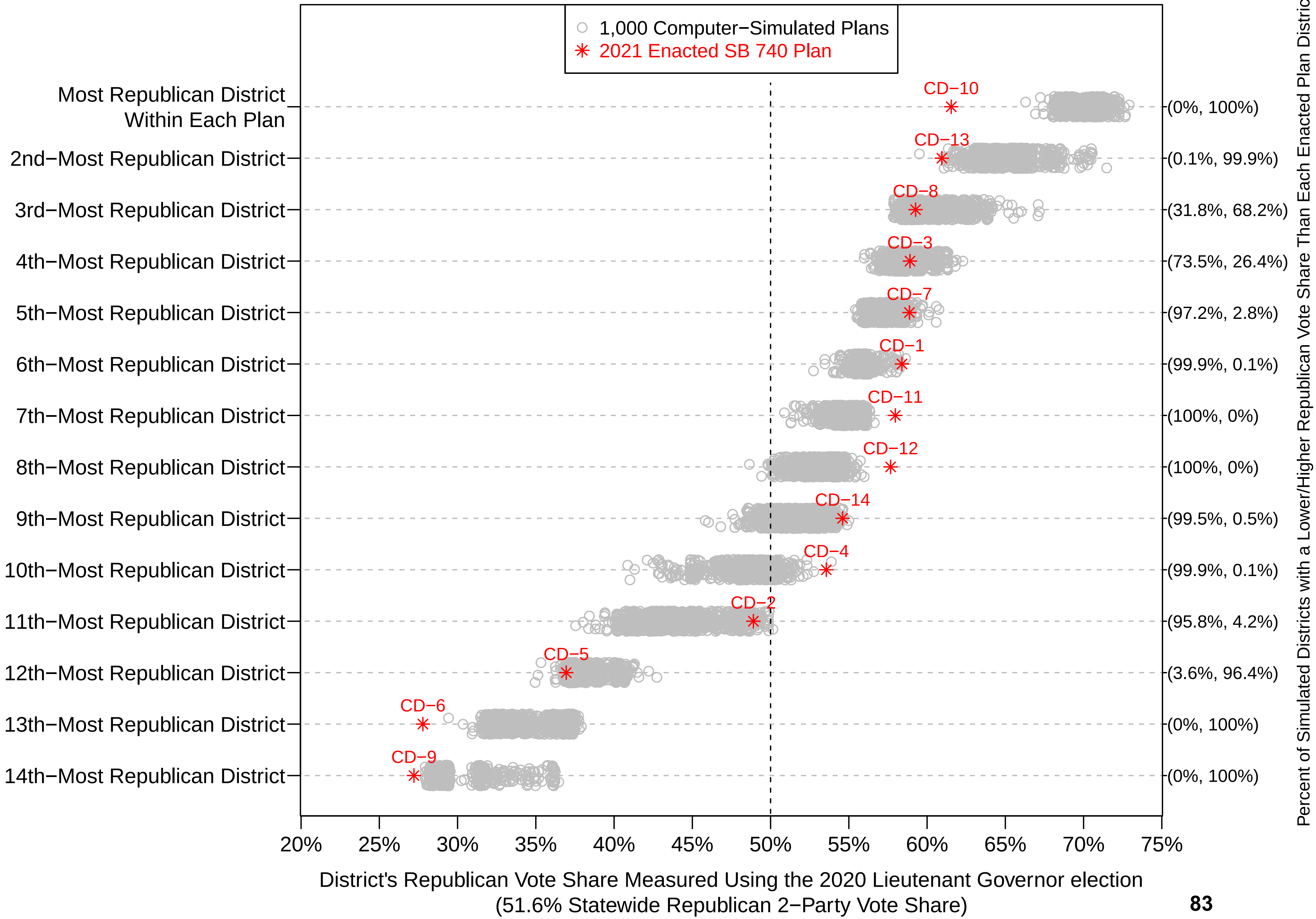


Figure A9: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 US President Election Results

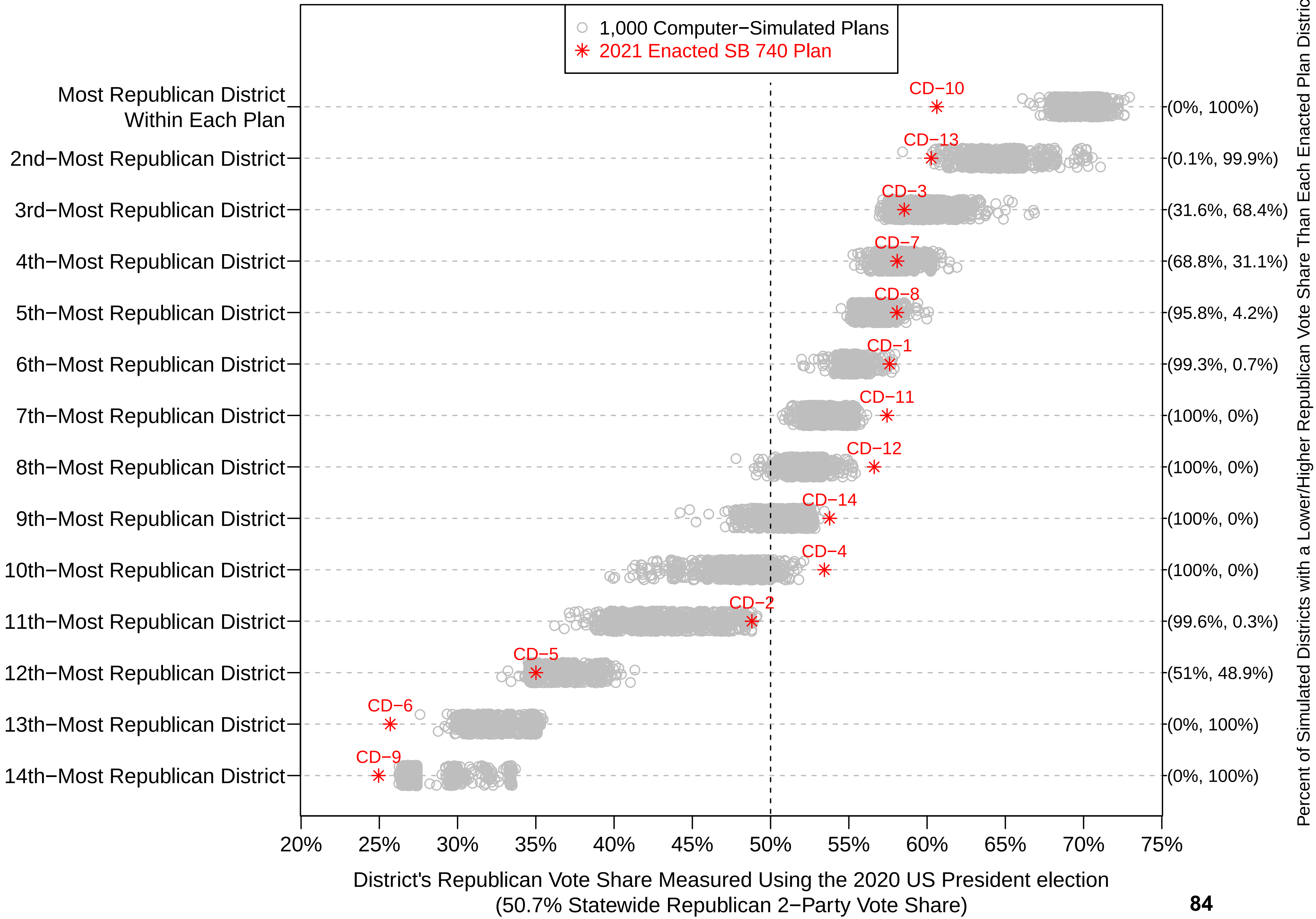


Figure A10: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 US Senator Election Results

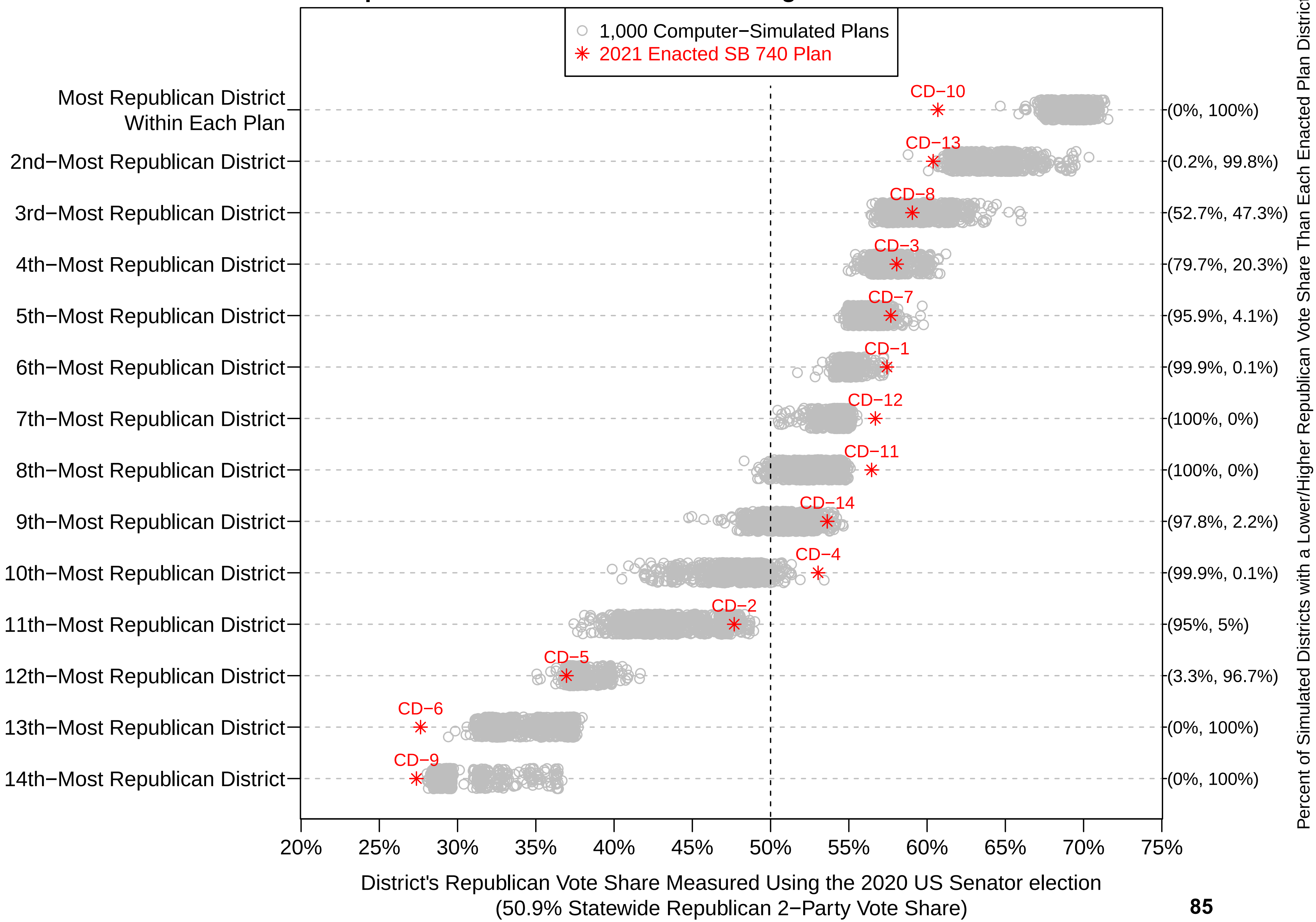


Figure B1: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 Attorney General election
(49.7% Statewide Republican 2−Party Vote Share)

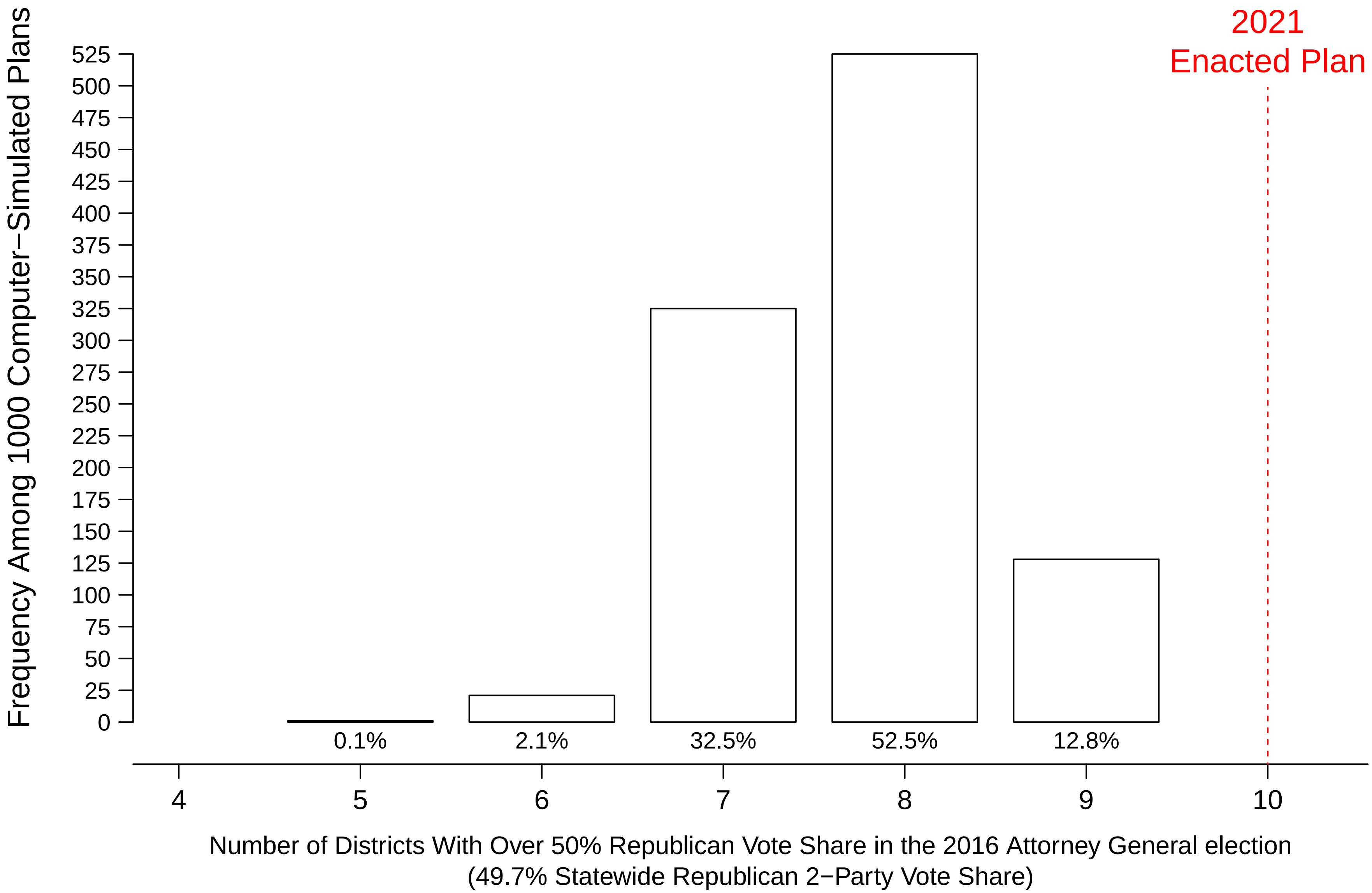


Figure B2: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 Governor election
(49.9% Statewide Republican 2−Party Vote Share)

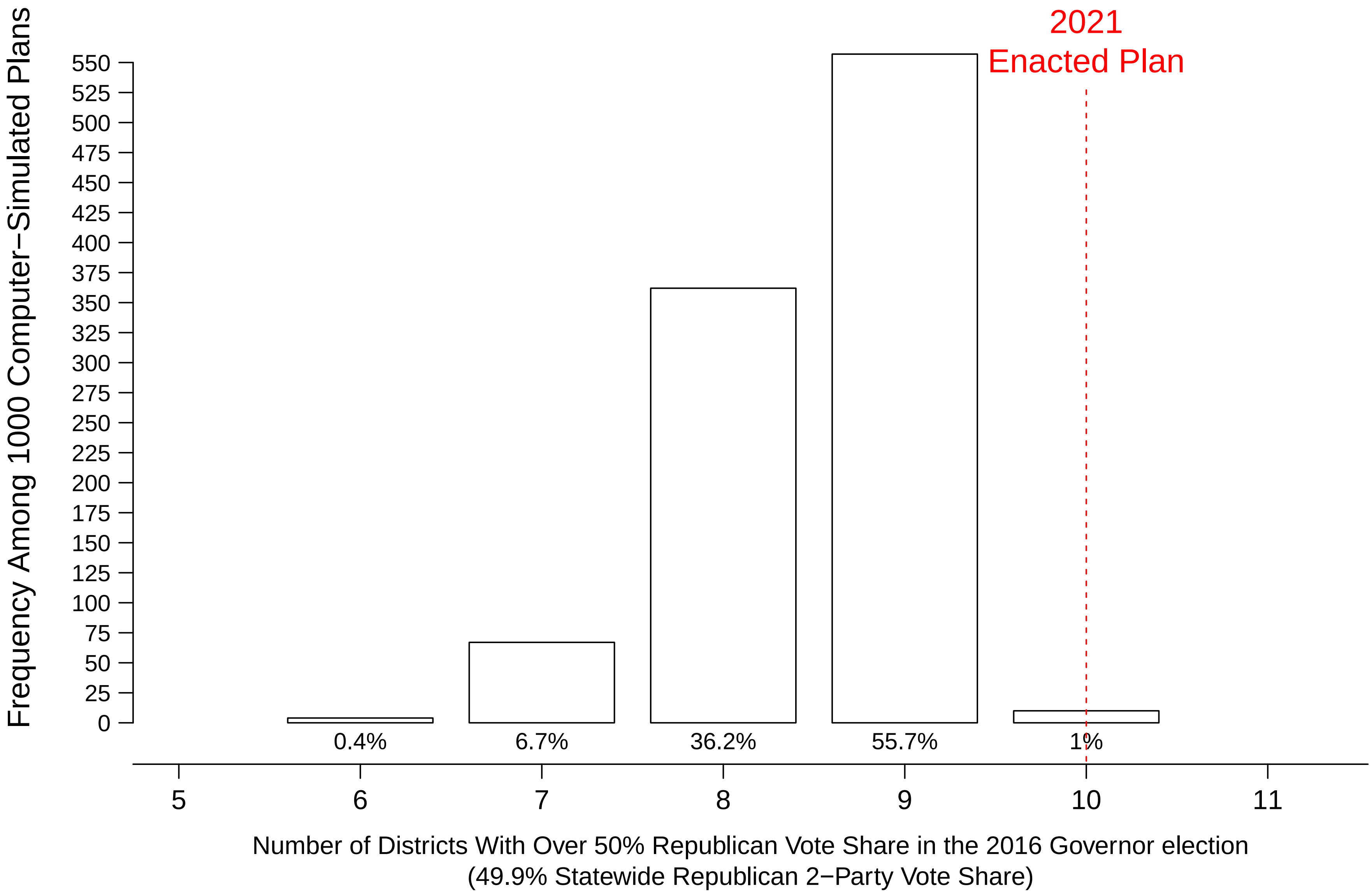


Figure B3: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 Lieutenant Governor election
(53.3% Statewide Republican 2−Party Vote Share)

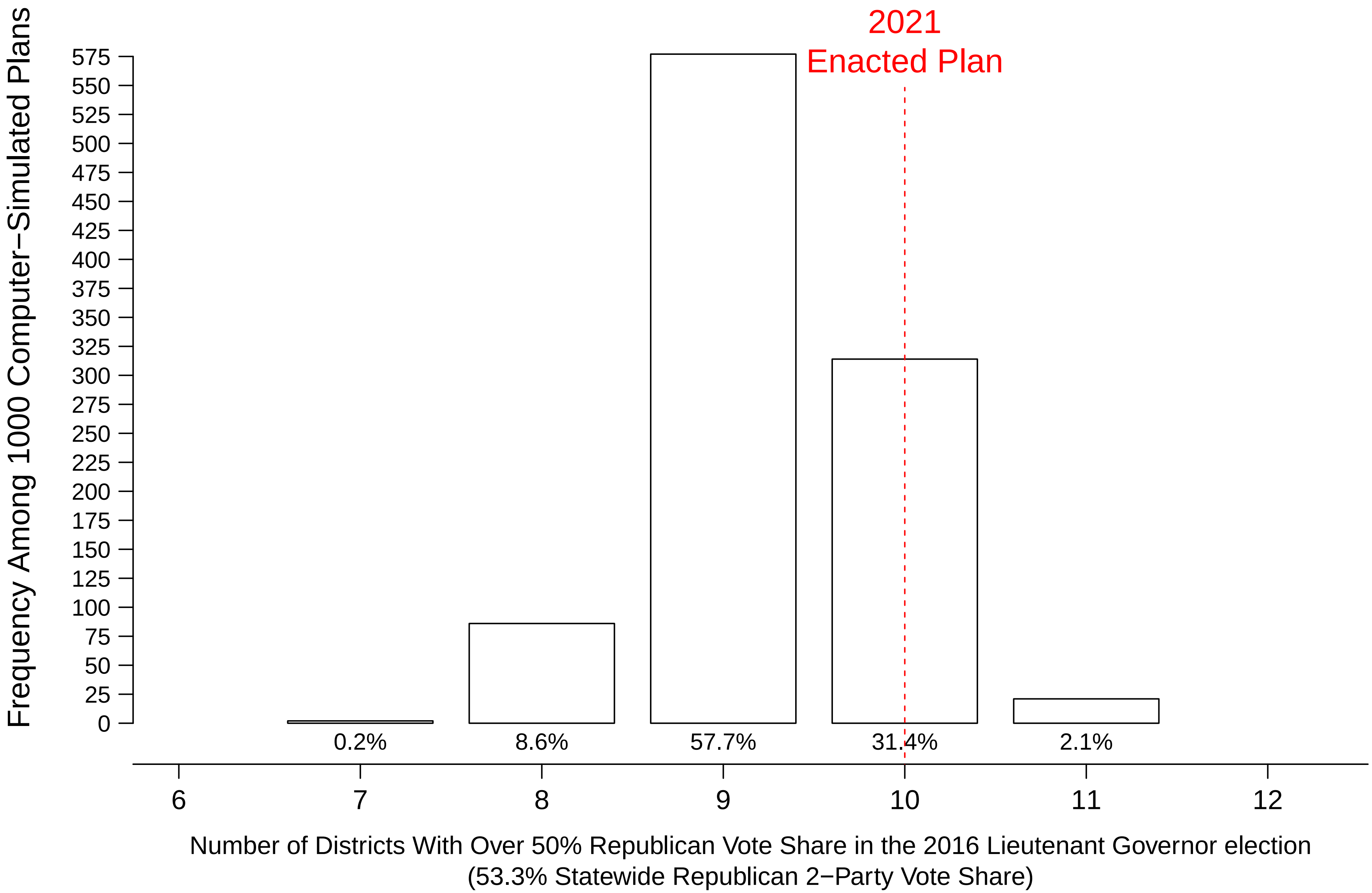


Figure B4: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 US President election
(51.9% Statewide Republican 2−Party Vote Share)

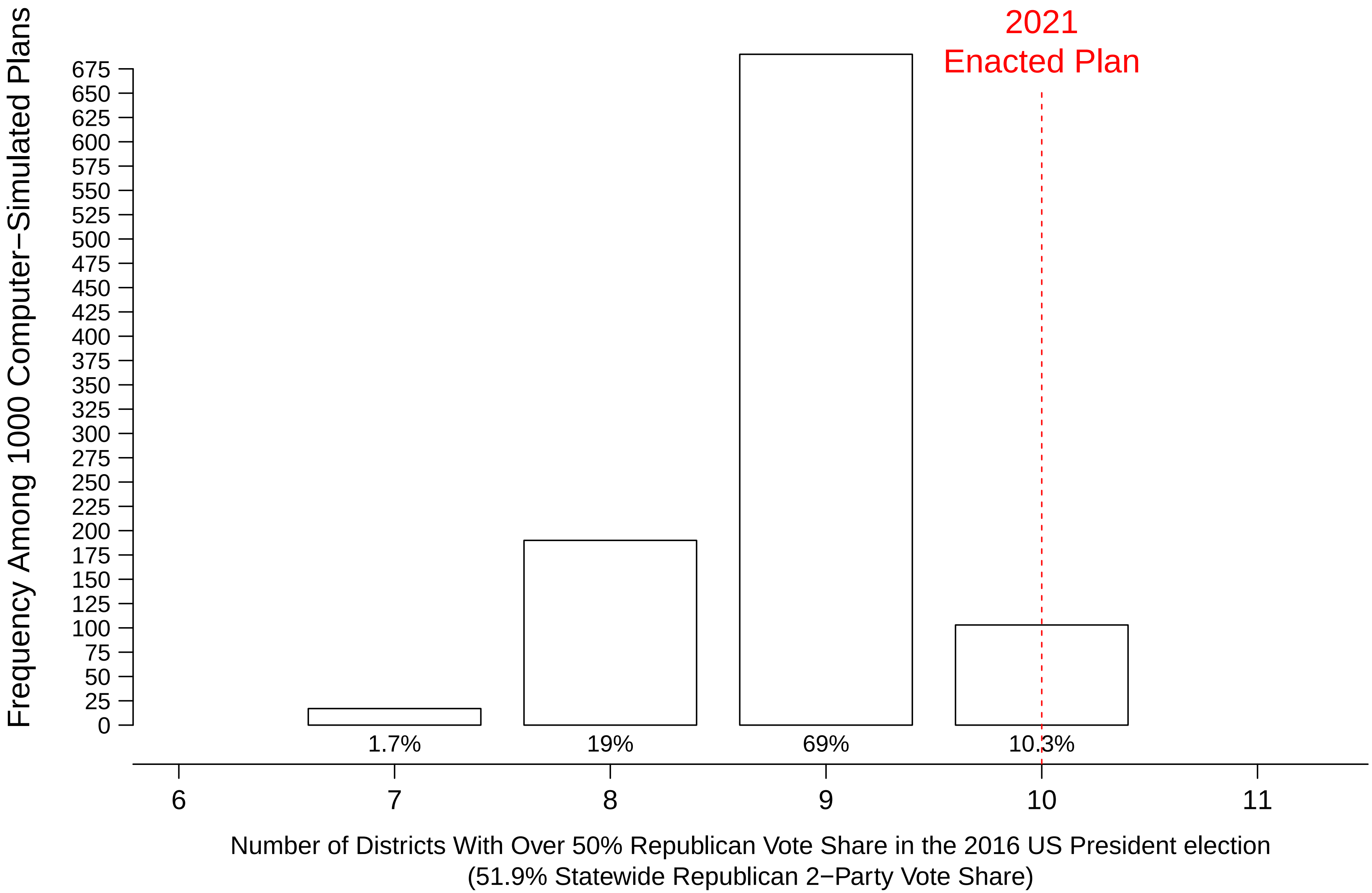


Figure B5: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 US Senator election
(53% Statewide Republican 2−Party Vote Share)

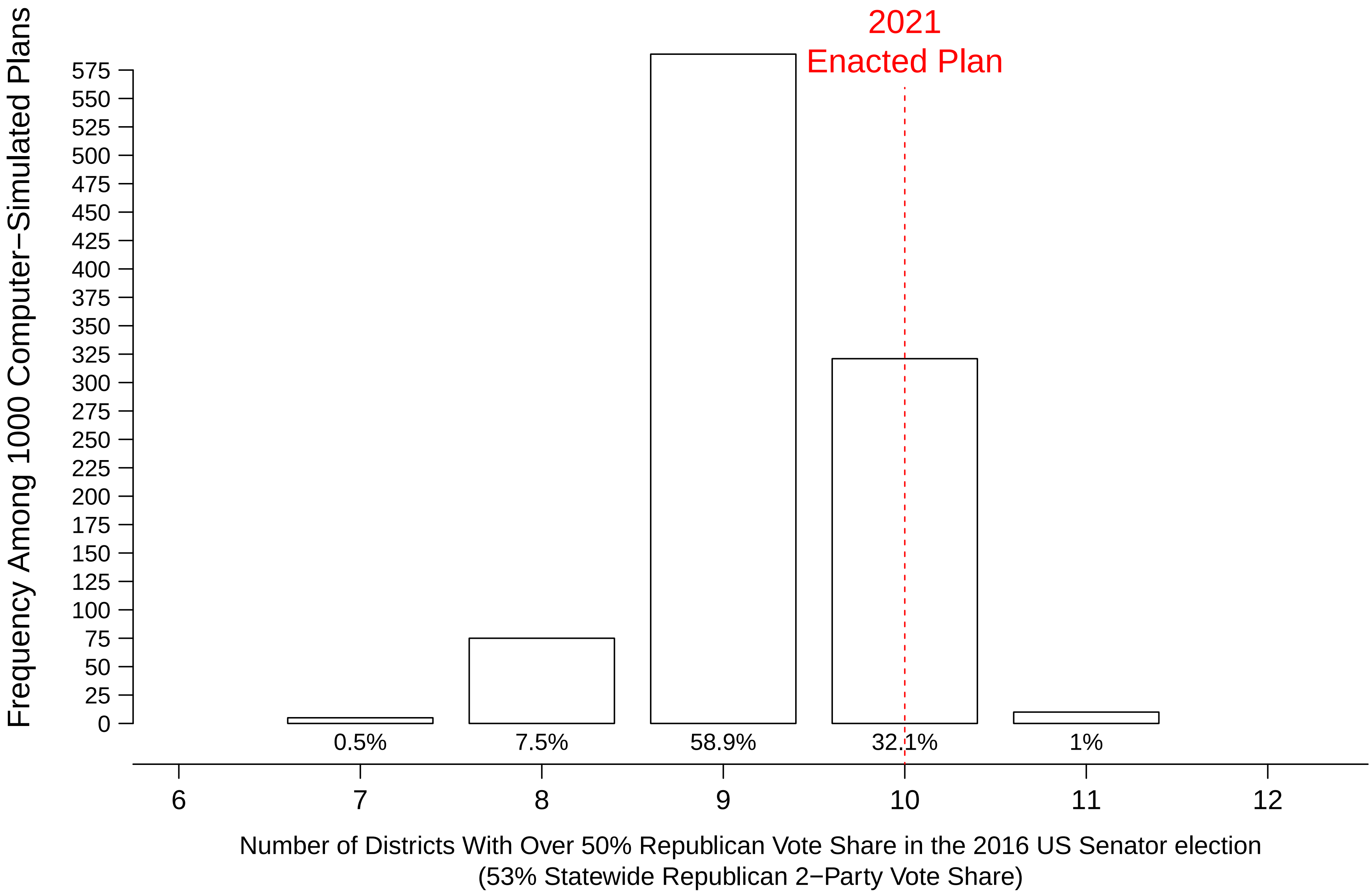


Figure B6: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 Attorney General election
(49.9% Statewide Republican 2−Party Vote Share)

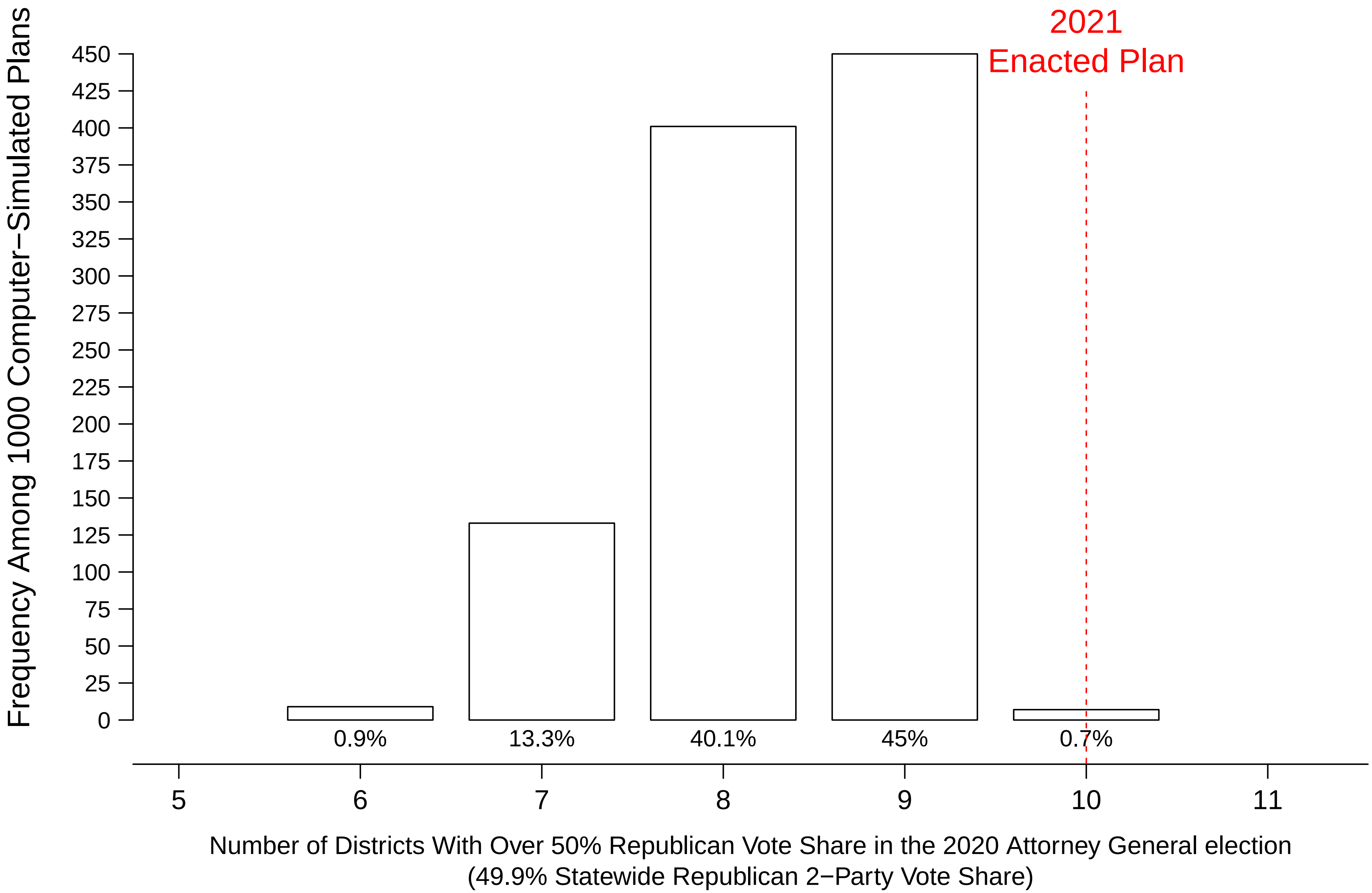


Figure B7: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 Governor election
(47.7% Statewide Republican 2−Party Vote Share)

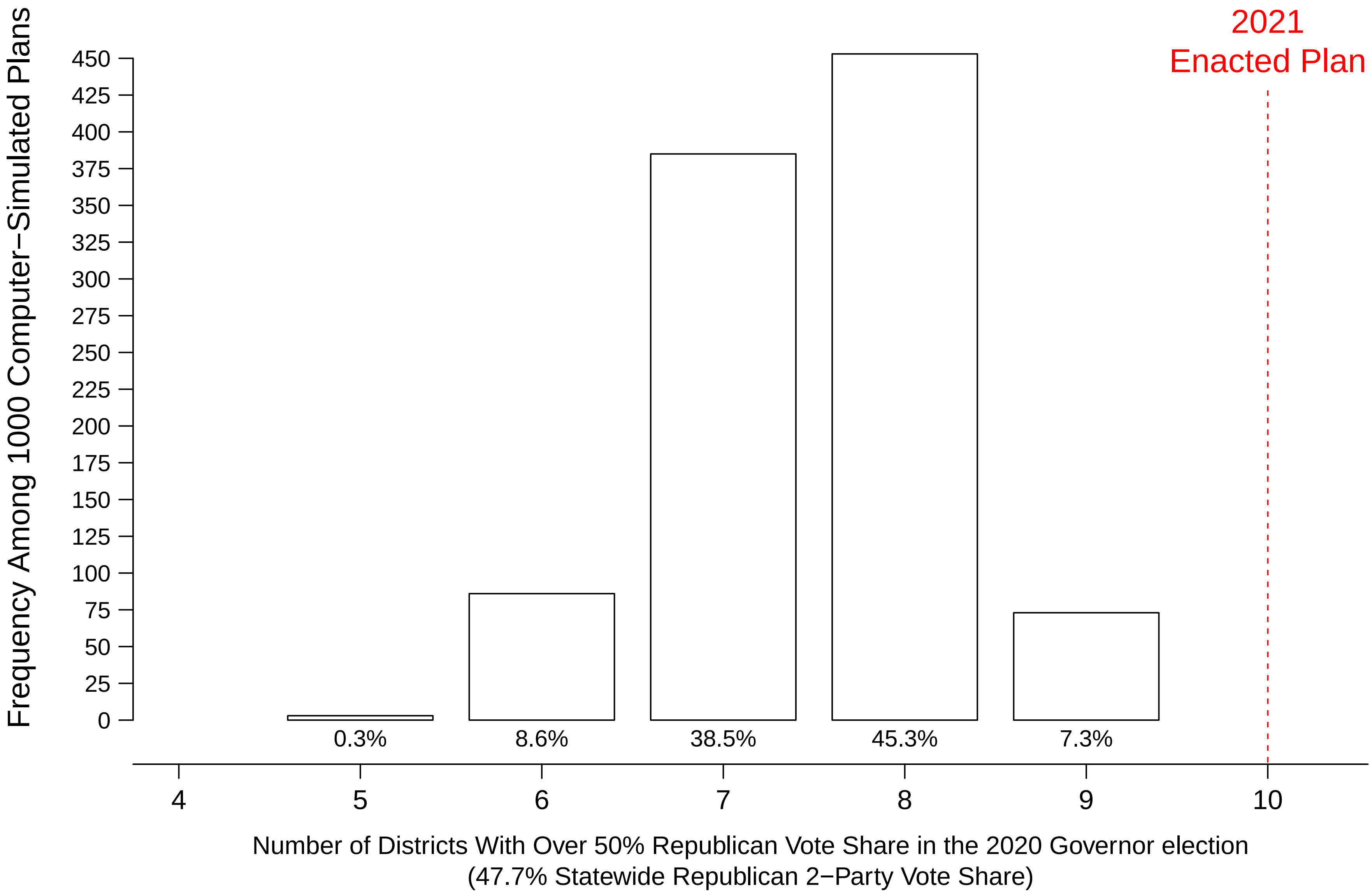


Figure B8: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 Lieutenant Governor election
(51.6% Statewide Republican 2−Party Vote Share)

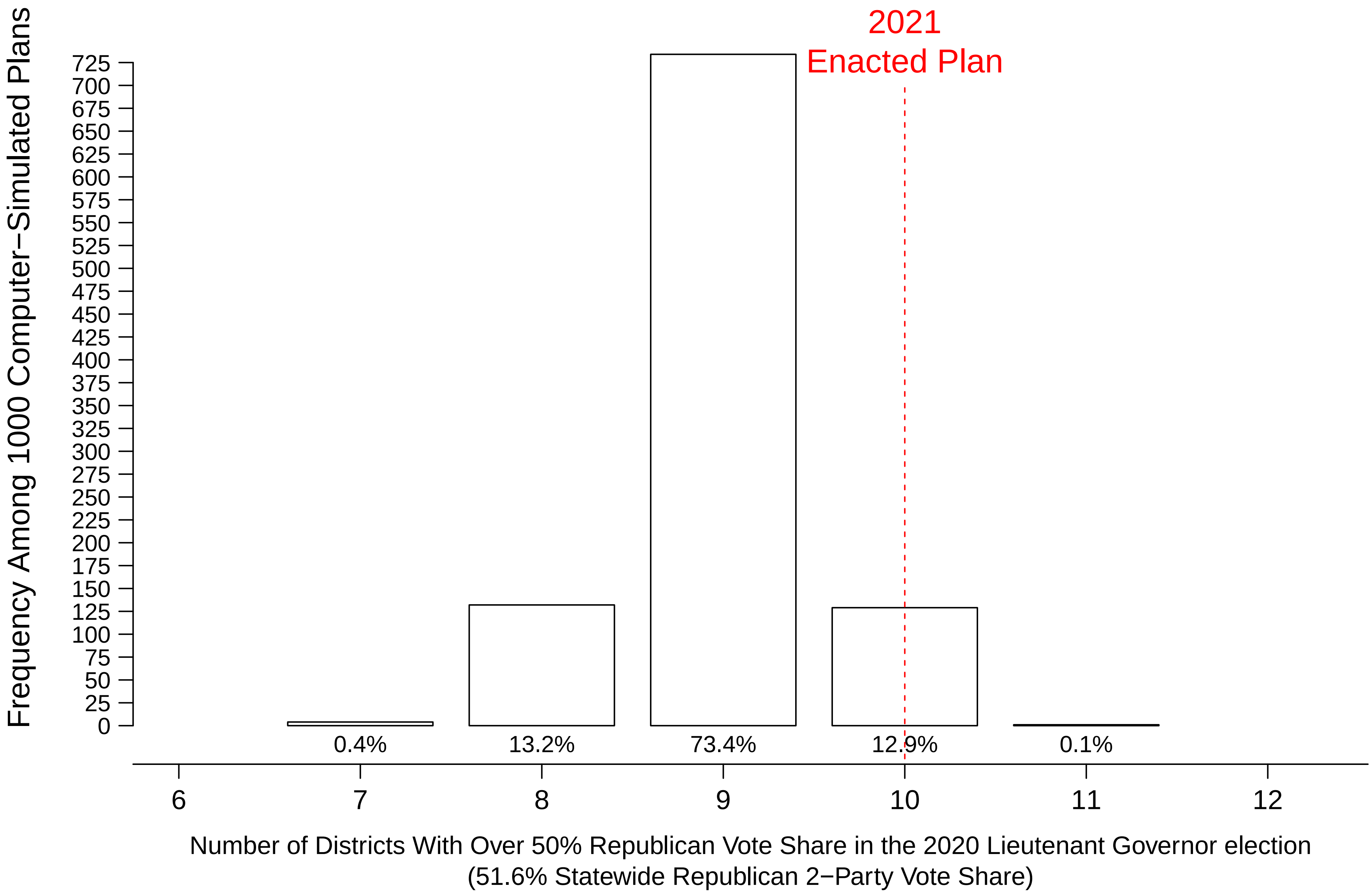


Figure B9: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 US President election
(50.7% Statewide Republican 2−Party Vote Share)

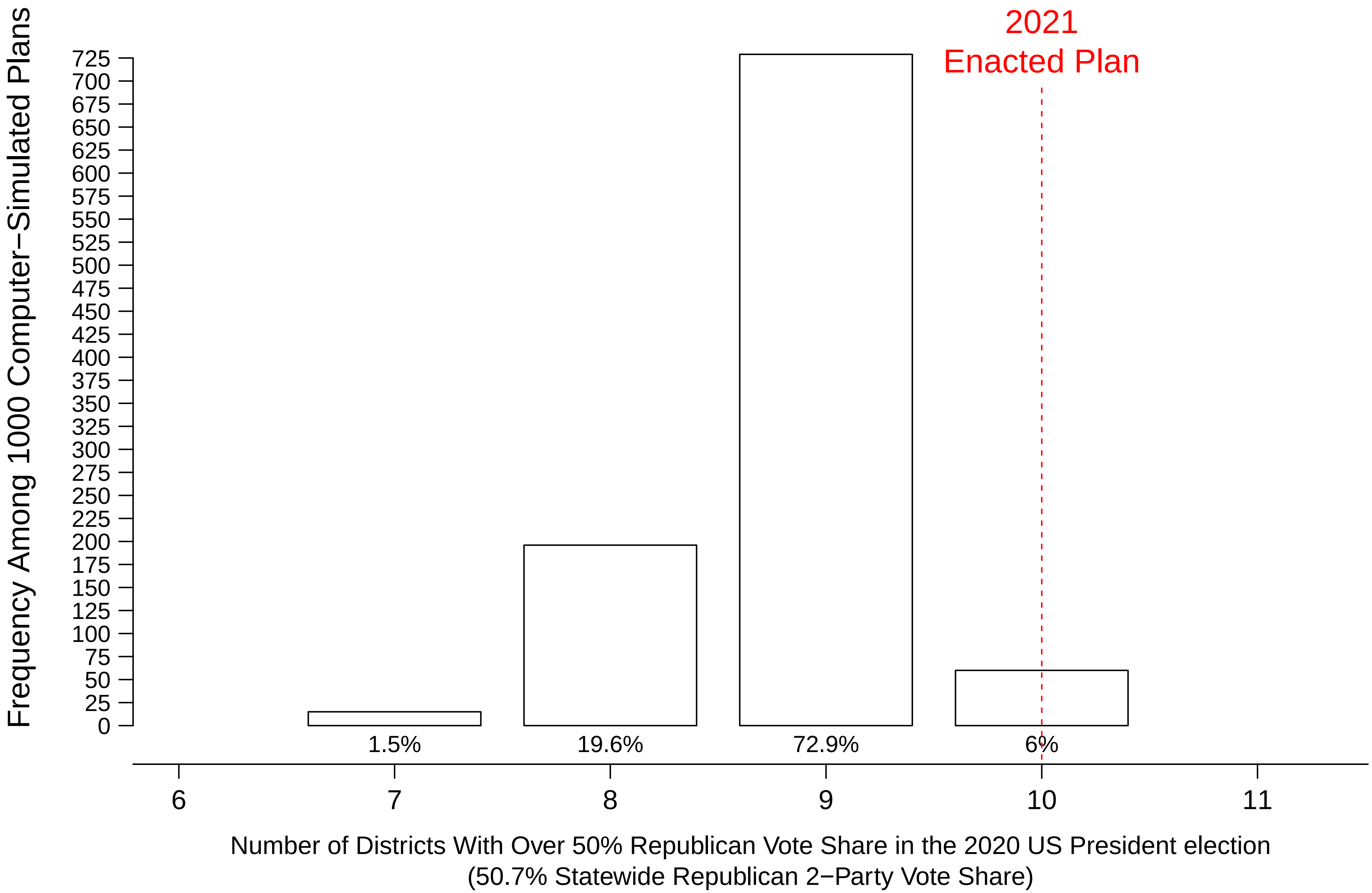
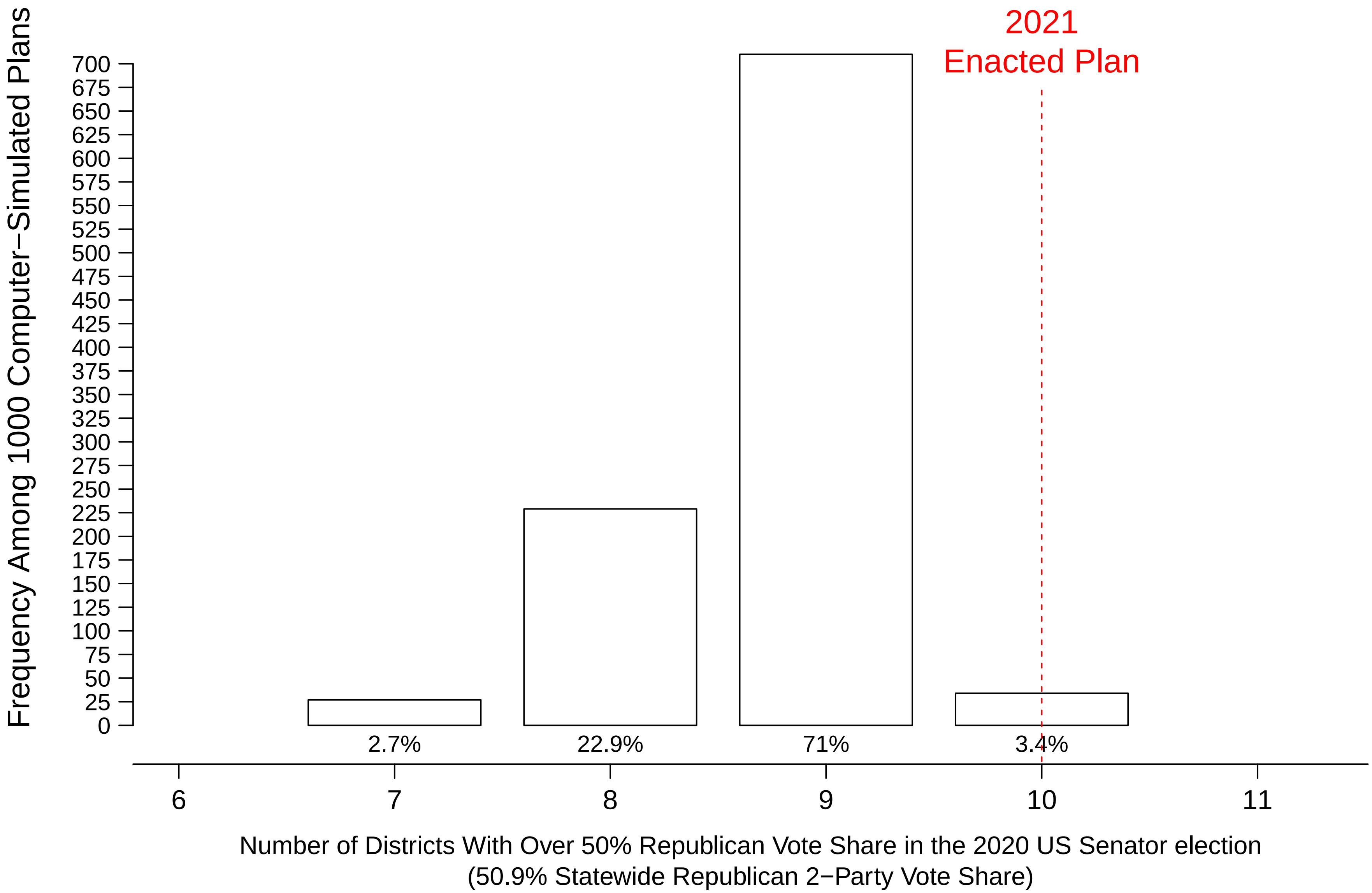


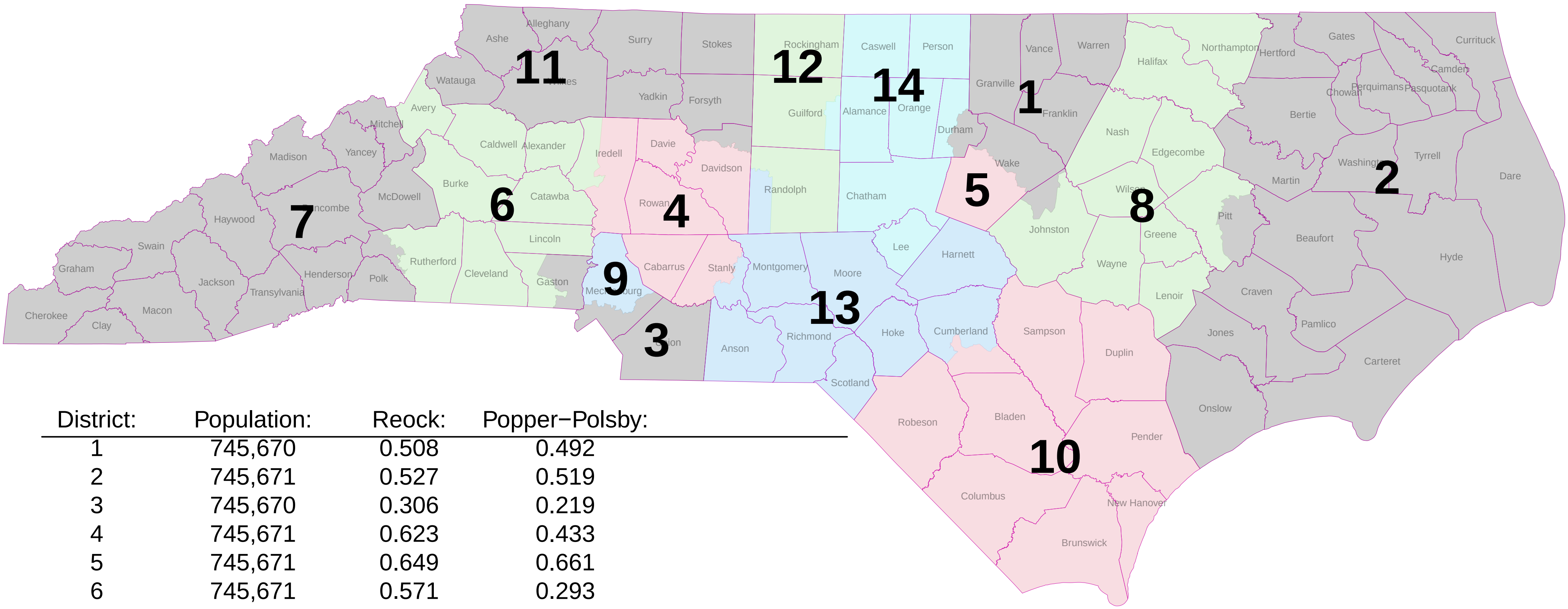
Figure B10: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 US Senator election
(50.9% Statewide Republican 2−Party Vote Share)



– Ex. 4486 –

Map 1:

Example of a Computer−Simulated Congressional Plan Protecting all 11 Incumbents



District:	Population:	Reock:	Popper−Polsby:
1	745,670	0.508	0.492
2	745,671	0.527	0.519
3	745,670	0.306	0.219
4	745,671	0.623	0.433
5	745,671	0.649	0.661
6	745,671	0.571	0.293
7	745,671	0.354	0.303
8	745,670	0.468	0.352
9	745,670	0.576	0.405
10	745,671	0.649	0.534
11	745,670	0.377	0.424
12	745,671	0.4	0.48
13	745,671	0.46	0.301
14	745,670	0.457	0.519
Plan Average:	745,670.6	0.495	0.424

13 Split Counties:
Alamance (Districts 12, 13)
Burke (Districts 10, 3)
Davie (Districts 2, 8)
Granville (Districts 1, 14)
Hoke (Districts 13, 6)
Mecklenburg (Districts 5, 9)
Nash (Districts 1, 11)
Orange (Districts 1, 13)
Pitt (Districts 11, 7)
Rockingham (Districts 12, 2)
Rowan (Districts 10, 8)
Rutherford (Districts 3, 9)
Wake (Districts 14, 4)

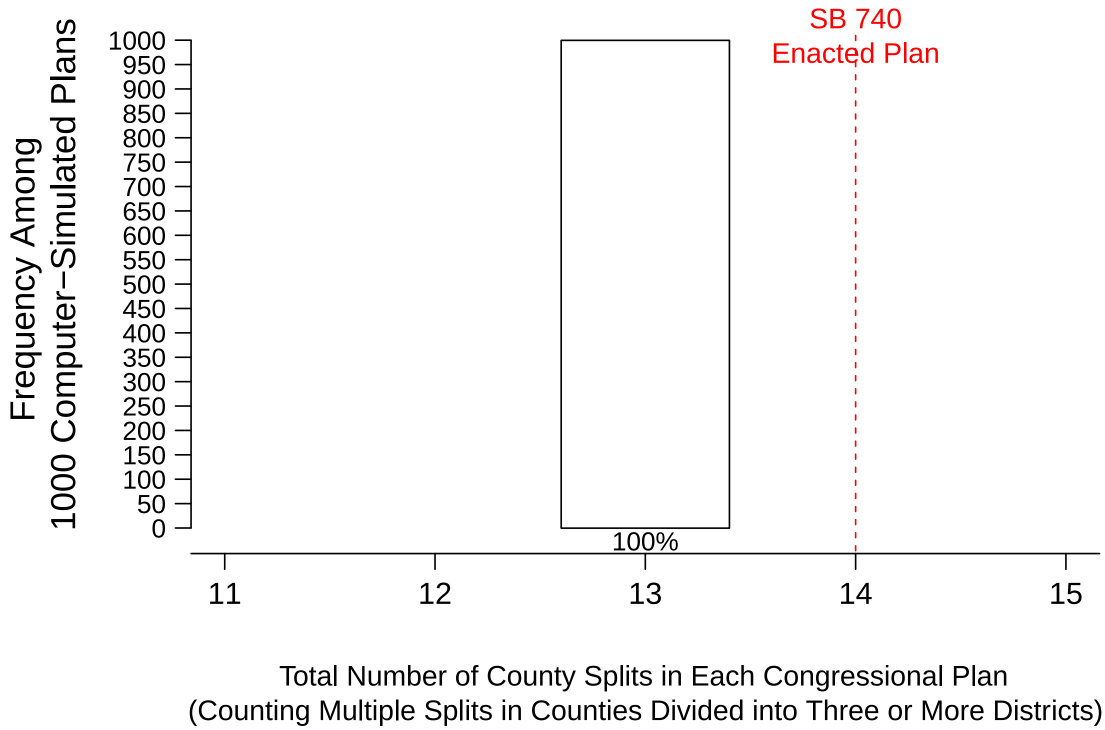
13 Split VTD's:
VTD 00008N in Alamance County (Districts 12 and 13)
VTD 000053 in Burke County (Districts 10 and 3)
VTD 000011 in Davie County (Districts 2 and 8)
VTD 00TYHO in Granville County (Districts 1 and 14)
VTD 000063 in Hoke County (Districts 13 and 6)
VTD 000018 in Mecklenburg County (Districts 5 and 9)
VTD 00P09A in Nash County (Districts 1 and 11)
VTD 0000CX in Orange County (Districts 1 and 13)
VTD 001301 in Pitt County (Districts 11 and 7)
VTD 0000LI in Rockingham County (Districts 12 and 2)
VTD 000033 in Rowan County (Districts 10 and 8)
VTD 000018 in Rutherford County (Districts 3 and 9)
VTD 008−03 in Wake County (Districts 14 and 4)

Table 1: Total Number of County Splits in the 2021 Enacted Plan

	County:	Congressional Districts:	Total County Splits:
1	Davidson	7 and 10	1
2	Guilford	7, 10, and 11	2
3	Harnett	4 and 7	1
4	Iredell	10 and 12	1
5	Mecklenburg	8, 9, and 13	2
6	Onslow	1 and 3	1
7	Pitt	1 and 2	1
8	Robeson	3 and 8	1
9	Wake	5, 6, and 7	2
10	Watauga	11 and 14	1
11	Wayne	2 and 4	1
Total County Splits:			**14**

Figure 1:

Comparison of Total County Splits in Enacted SB 740 Plan and 1,000 Computer−Simulated Plans



Number of Counties Split Multiple Times
in Enacted SB 740 Plan and 1,000 Computer−Simulated Plans

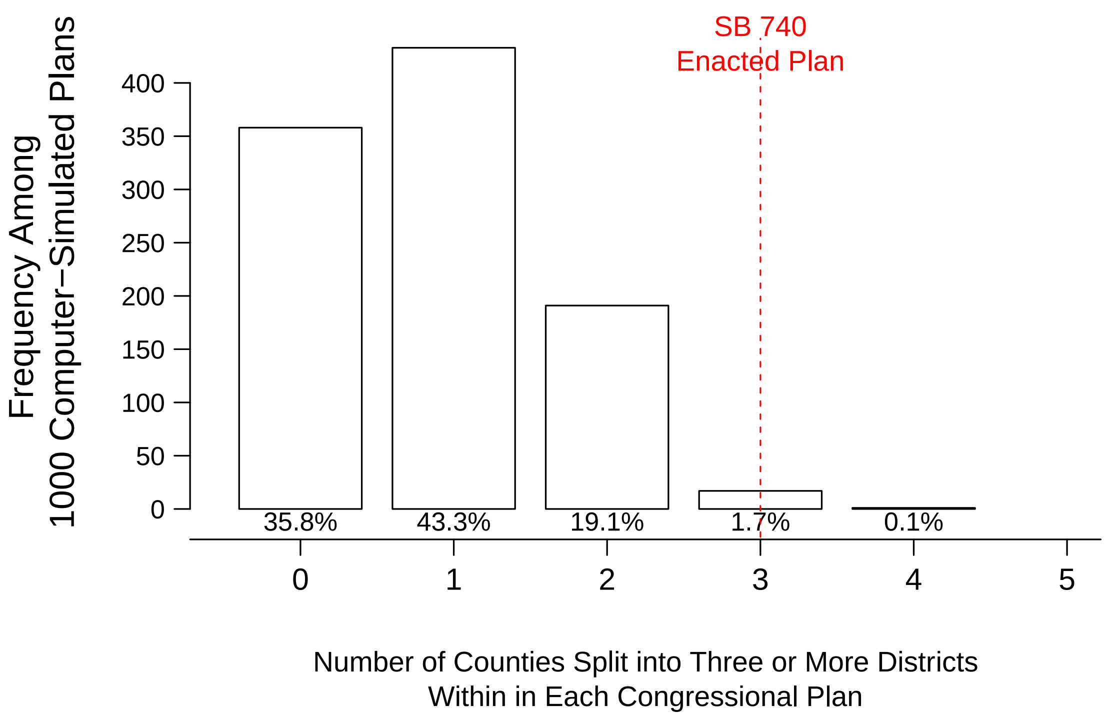


Figure 2:

Comparison of Total VTD Splits in Enacted SB 740 Plan and 1,000 Computer−Simulated Plans

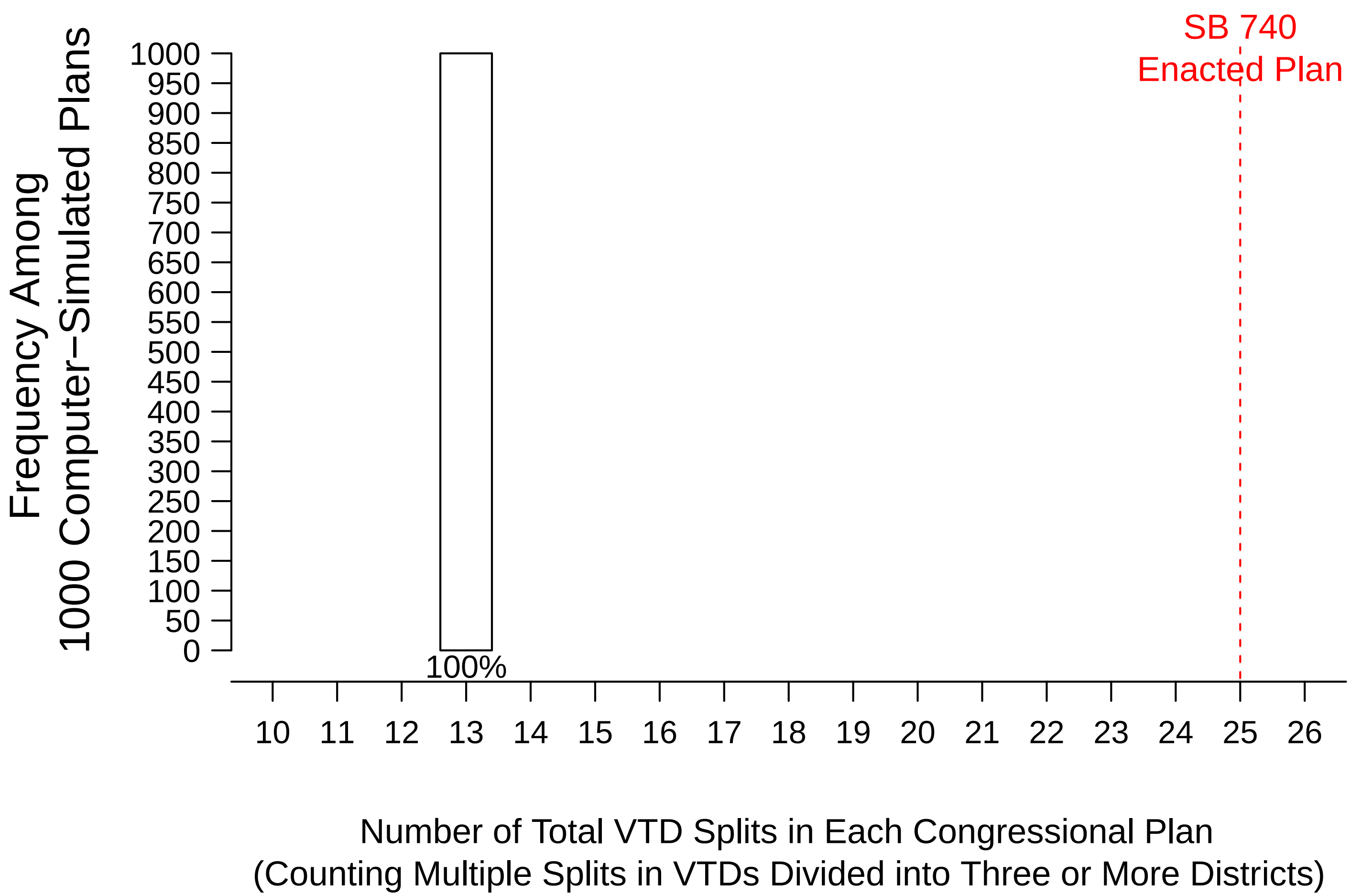


Figure 3:

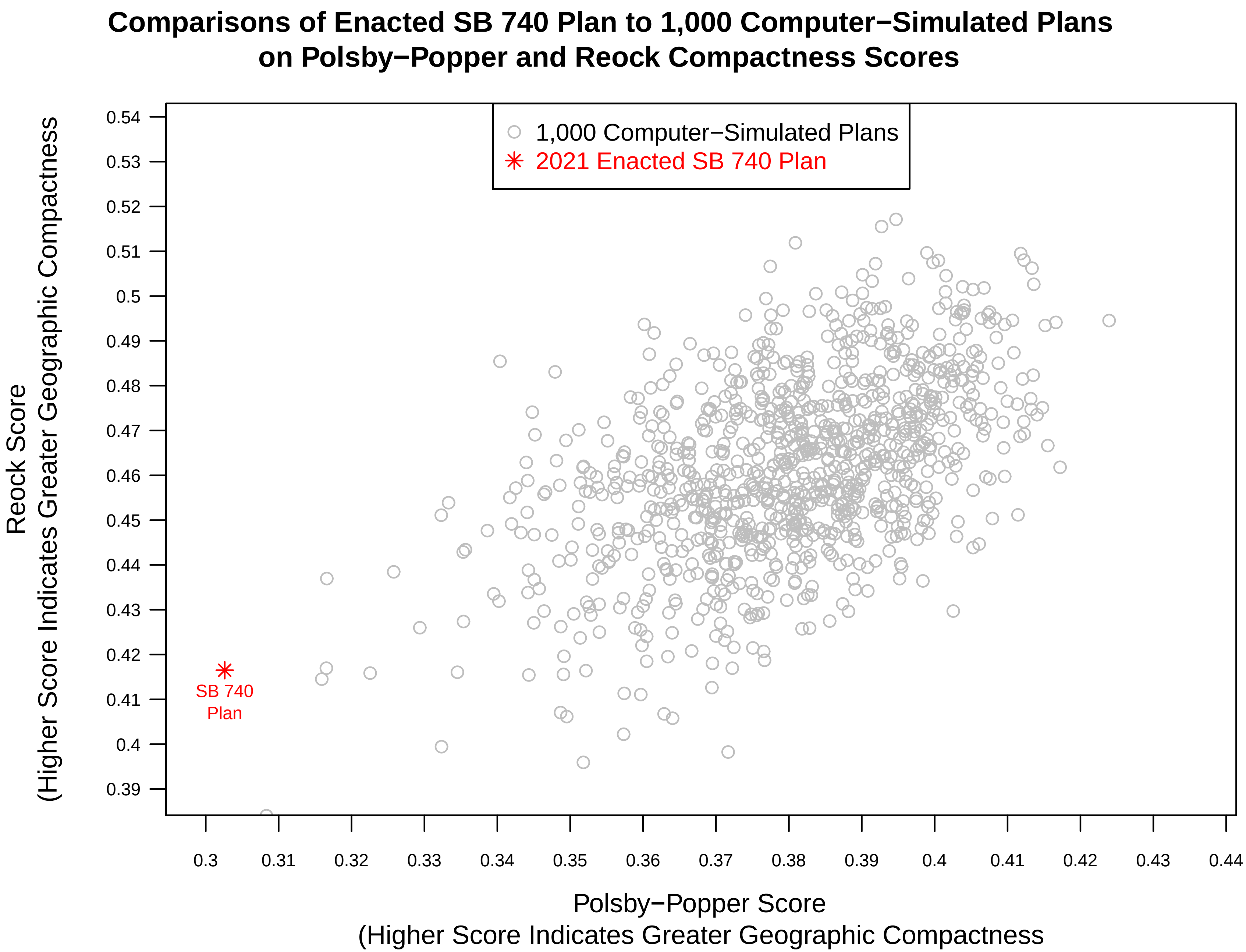

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
on Polsby−Popper and Reock Compactness Scores
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
SB 740
Plan
Reock Score
(Higher Score Indicates Greater Geographic Compactness
Polsby−Popper Score
(Higher Score Indicates Greater Geographic Compactness
0.54
0.53
0.52
0.51
0.5
0.49
0.48
0.47
0.46
0.45
0.44
0.43
0.42
0.41
0.4
0.39
0.3
0.31
0.32
0.33
0.34
0.35
0.36
0.37
0.38
0.39
0.4
0.41
0.42
0.43
0.44

Figure 4:

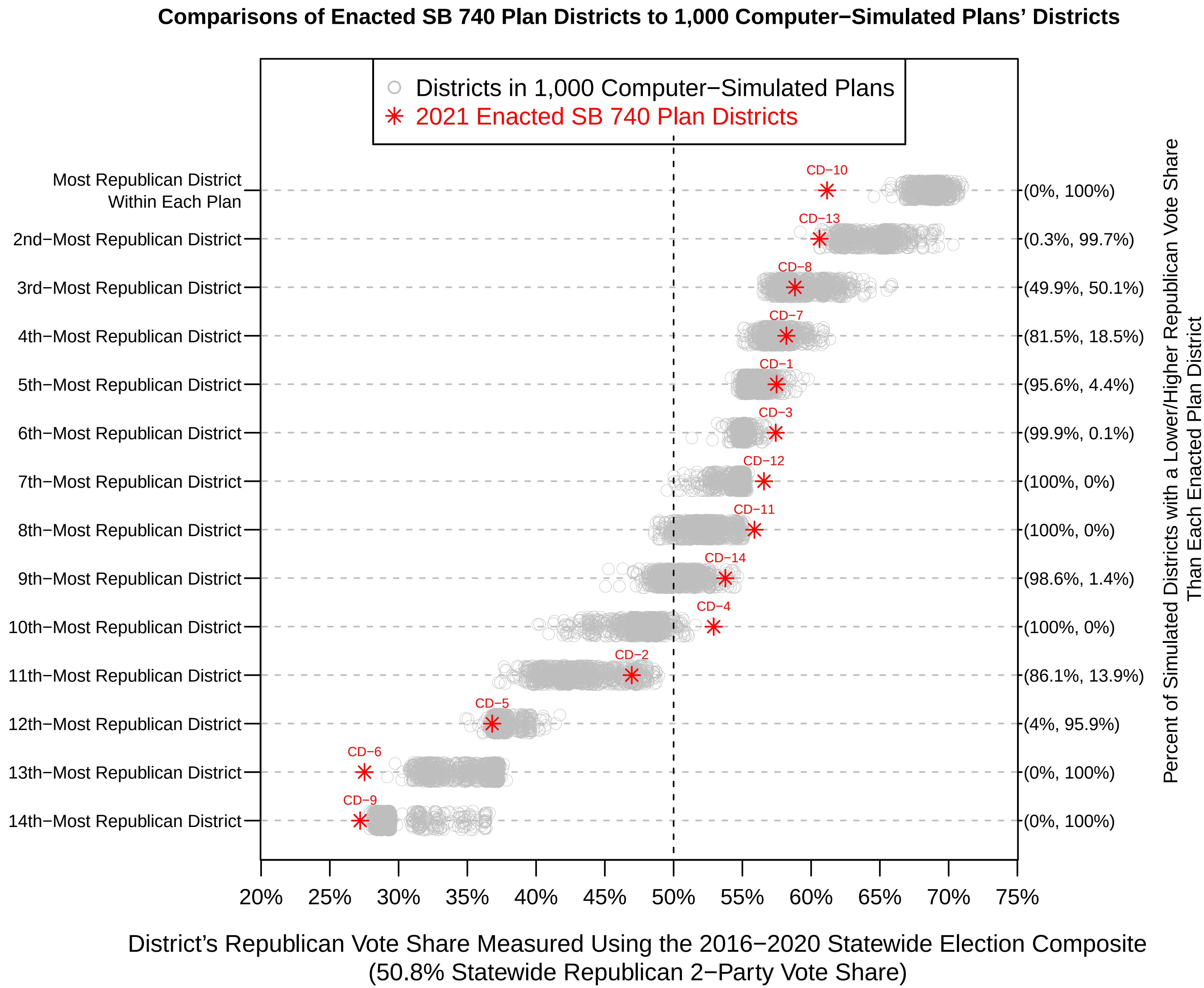

Comparisons of Enacted SB 740 Plan Districts to 1,000 Computer−Simulated Plans' Districts
Districts in 1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan Districts
Most Republican District Within Each Plan
2nd−Most Republican District
3rd−Most Republican District
4th−Most Republican District
5th−Most Republican District
6th−Most Republican District
7th−Most Republican District
8th−Most Republican District
9th−Most Republican District
10th−Most Republican District
11th−Most Republican District
12th−Most Republican District
13th−Most Republican District
14th−Most Republican District
CD−10
CD−13
CD−8
CD−7
CD−1
CD−3
CD−12
CD−11
CD−14
CD−4
CD−2
CD−5
CD−6
CD−9
(0%, 100%)
(0.3%, 99.7%)
(49.9%, 50.1%)
(81.5%, 18.5%)
(95.6%, 4.4%)
(99.9%, 0.1%)
(100%, 0%)
(100%, 0%)
(98.6%, 1.4%)
(100%, 0%)
(86.1%, 13.9%)
(4%, 95.9%)
(0%, 100%)
(0%, 100%)
Percent of Simulated Districts with a Lower/Higher Republican Vote Share Than Each Enacted Plan District
20%
25%
30%
35%
40%
45%
50%
55%
60%
65%
70%
75%
District's Republican Vote Share Measured Using the 2016−2020 Statewide Election Composite
(50.8% Statewide Republican 2−Party Vote Share)

Figure 5:

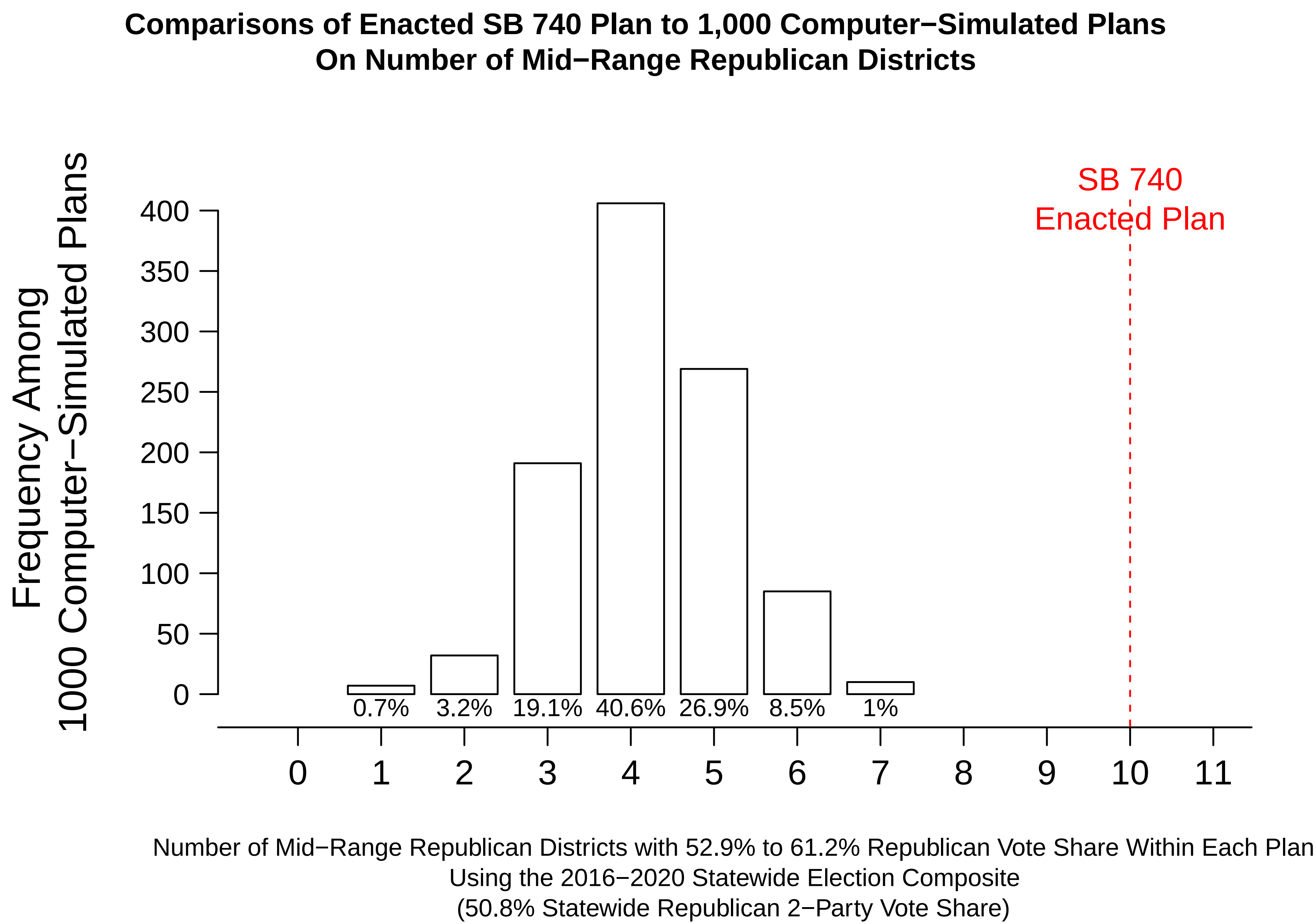

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
On Number of Mid−Range Republican Districts
Frequency Among
1000 Computer−Simulated Plans
400
350
300
250
200
150
100
50
0
0.7%
3.2%
19.1%
40.6%
26.9%
8.5%
1%
SB 740
Enacted Plan
0
1
2
3
4
5
6
7
8
9
10
11
Number of Mid−Range Republican Districts with 52.9% to 61.2% Republican Vote Share Within Each Plan
Using the 2016−2020 Statewide Election Composite
(50.8% Statewide Republican 2−Party Vote Share)

Figure 6:

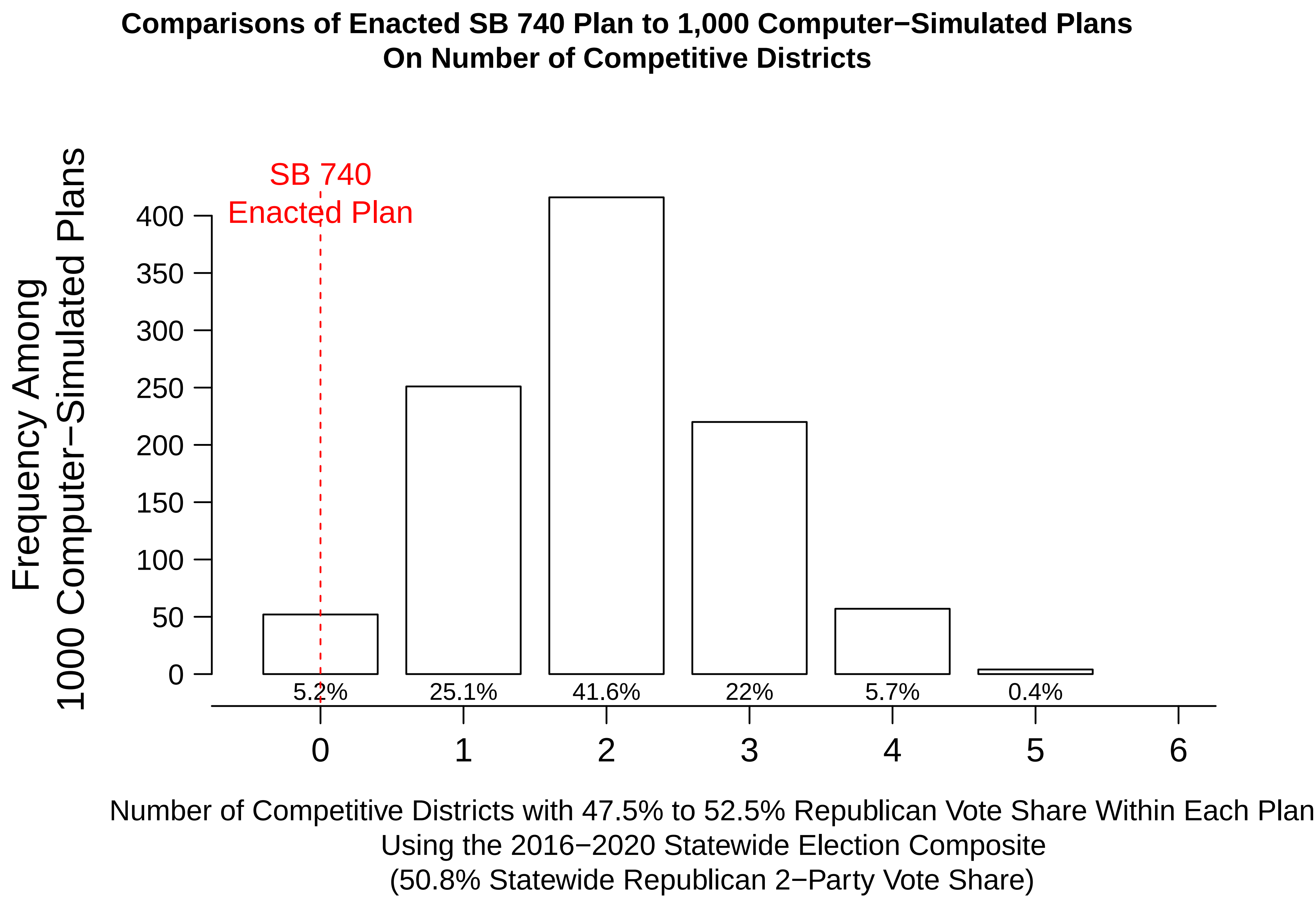

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
On Number of Competitive Districts
SB 740
Enacted Plan
Frequency Among
1000 Computer−Simulated Plans
400
350
300
250
200
150
100
50
0
5.2%
25.1%
41.6%
22%
5.7%
0.4%
0
1
2
3
4
5
6
Number of Competitive Districts with 47.5% to 52.5% Republican Vote Share Within Each Plan
Using the 2016−2020 Statewide Election Composite
(50.8% Statewide Republican 2−Party Vote Share)

Figure 7:

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans

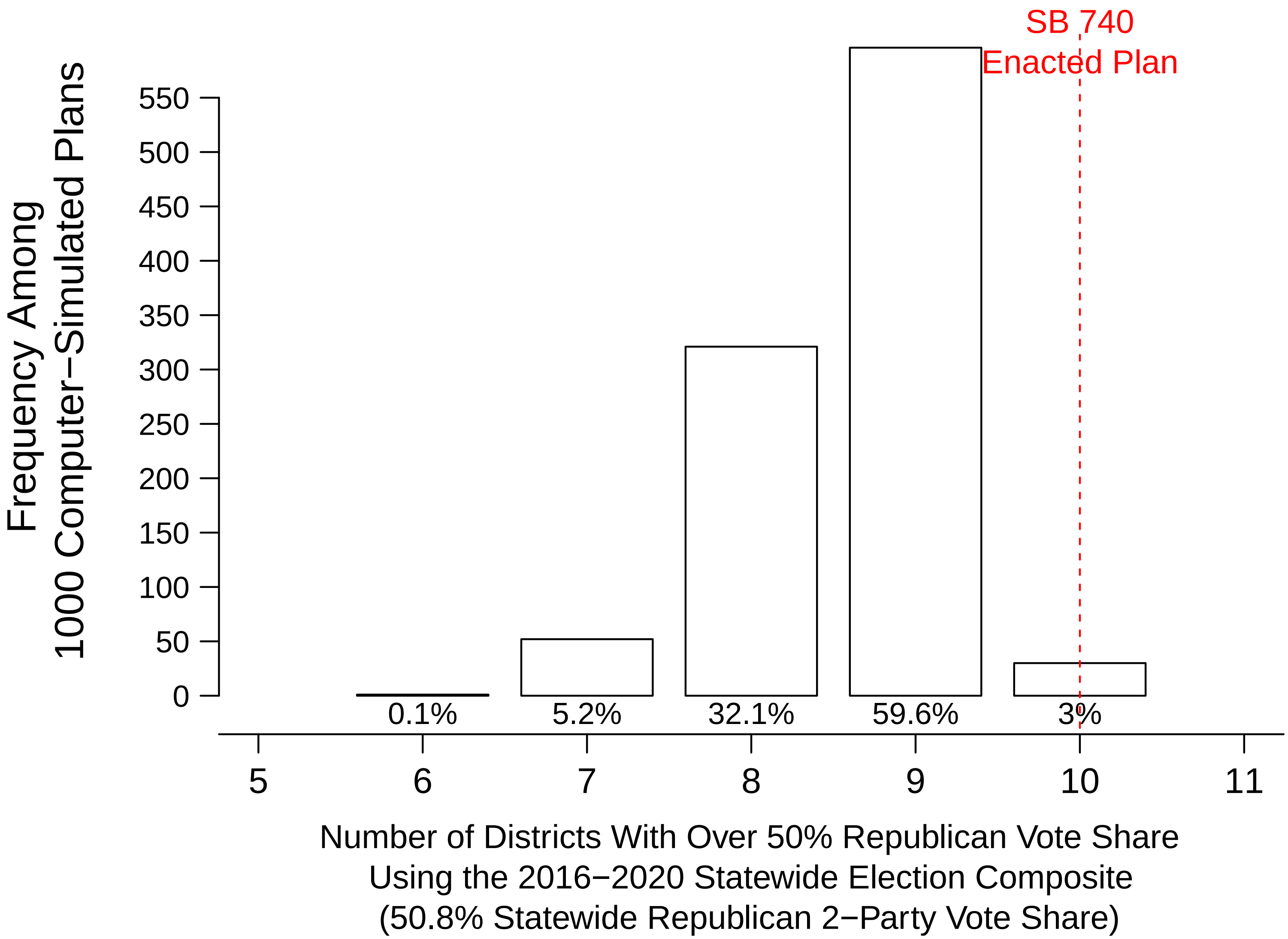


Figure 8:

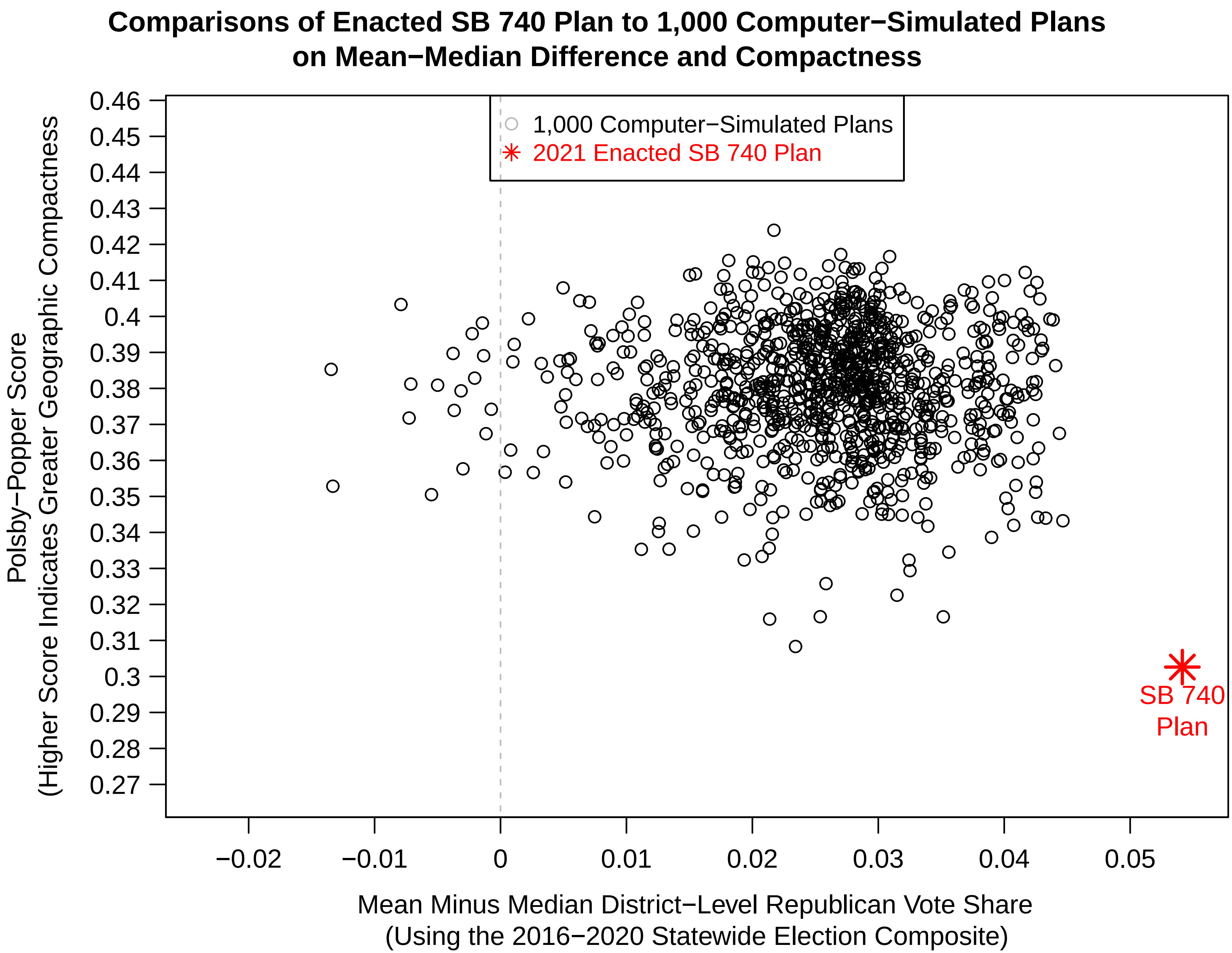

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
on Mean−Median Difference and Compactness
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
SB 740
Plan
Polsby−Popper Score
(Higher Score Indicates Greater Geographic Compactness
0.46
0.45
0.44
0.43
0.42
0.41
0.4
0.39
0.38
0.37
0.36
0.35
0.34
0.33
0.32
0.31
0.3
0.29
0.28
0.27
−0.02
−0.01
0
0.01
0.02
0.03
0.04
0.05
Mean Minus Median District−Level Republican Vote Share
(Using the 2016−2020 Statewide Election Composite)

Figure 9:

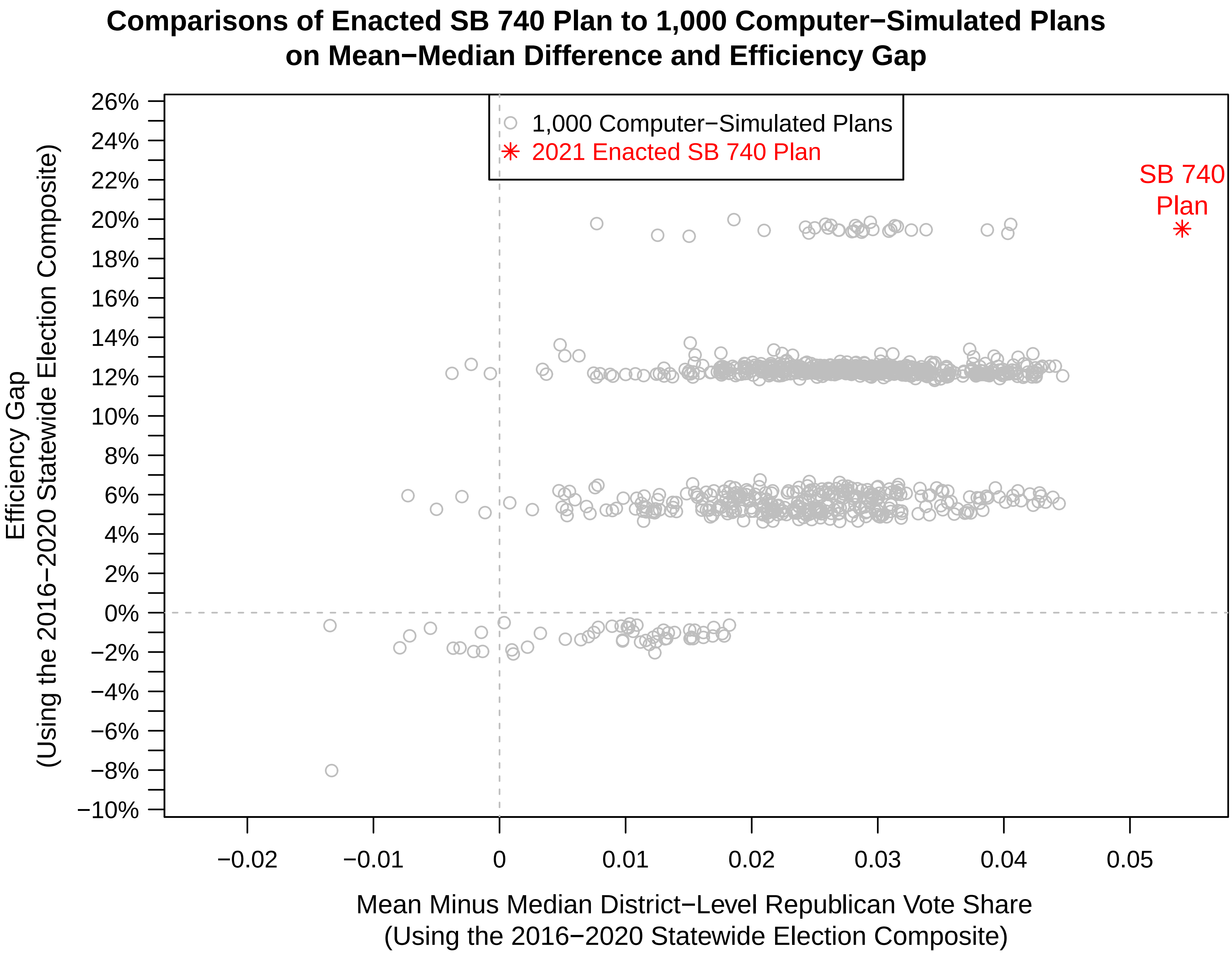

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
on Mean−Median Difference and Efficiency Gap
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
SB 740
Plan
26%
24%
22%
20%
18%
16%
14%
12%
10%
8%
6%
4%
2%
0%
−2%
−4%
−6%
−8%
−10%
Efficiency Gap
(Using the 2016−2020 Statewide Election Composite)
−0.02
−0.01
0
0.01
0.02
0.03
0.04
0.05
Mean Minus Median District−Level Republican Vote Share
(Using the 2016−2020 Statewide Election Composite)

Figure 10:

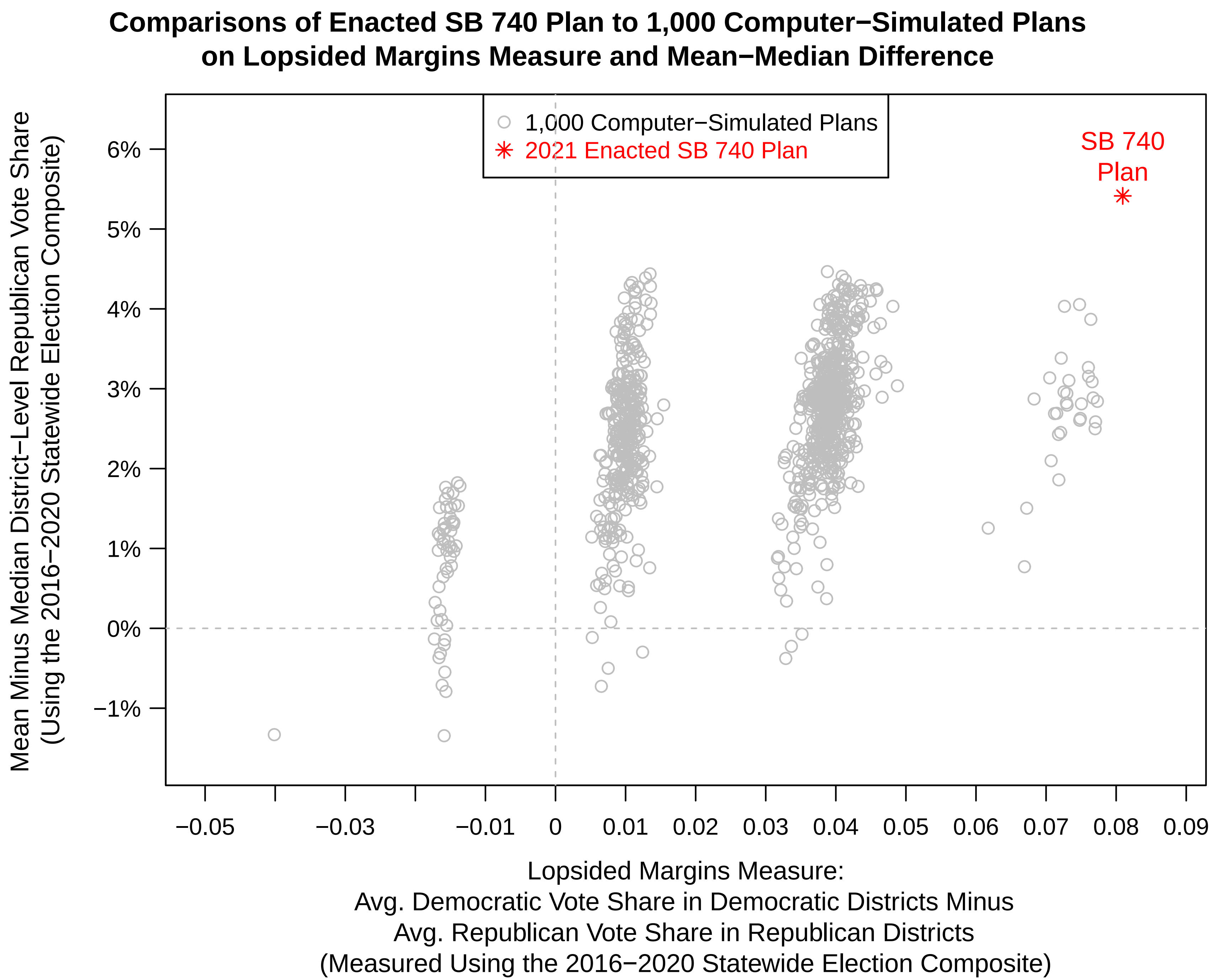

Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
on Lopsided Margins Measure and Mean−Median Difference
1,000 Computer−Simulated Plans
2021 Enacted SB 740 Plan
SB 740
Plan
6%
5%
4%
3%
2%
1%
0%
−1%
Mean Minus Median District−Level Republican Vote Share
(Using the 2016−2020 Statewide Election Composite)
−0.05
−0.03
−0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
Lopsided Margins Measure:
Avg. Democratic Vote Share in Democratic Districts Minus
Avg. Republican Vote Share in Republican Districts
(Measured Using the 2016−2020 Statewide Election Composite)

Figure 11:
Comparisons of SB 740 Enacted Plan to 1,000 Computer−Simulated Plans On Partisan Symmetry Based on Uniform Swing

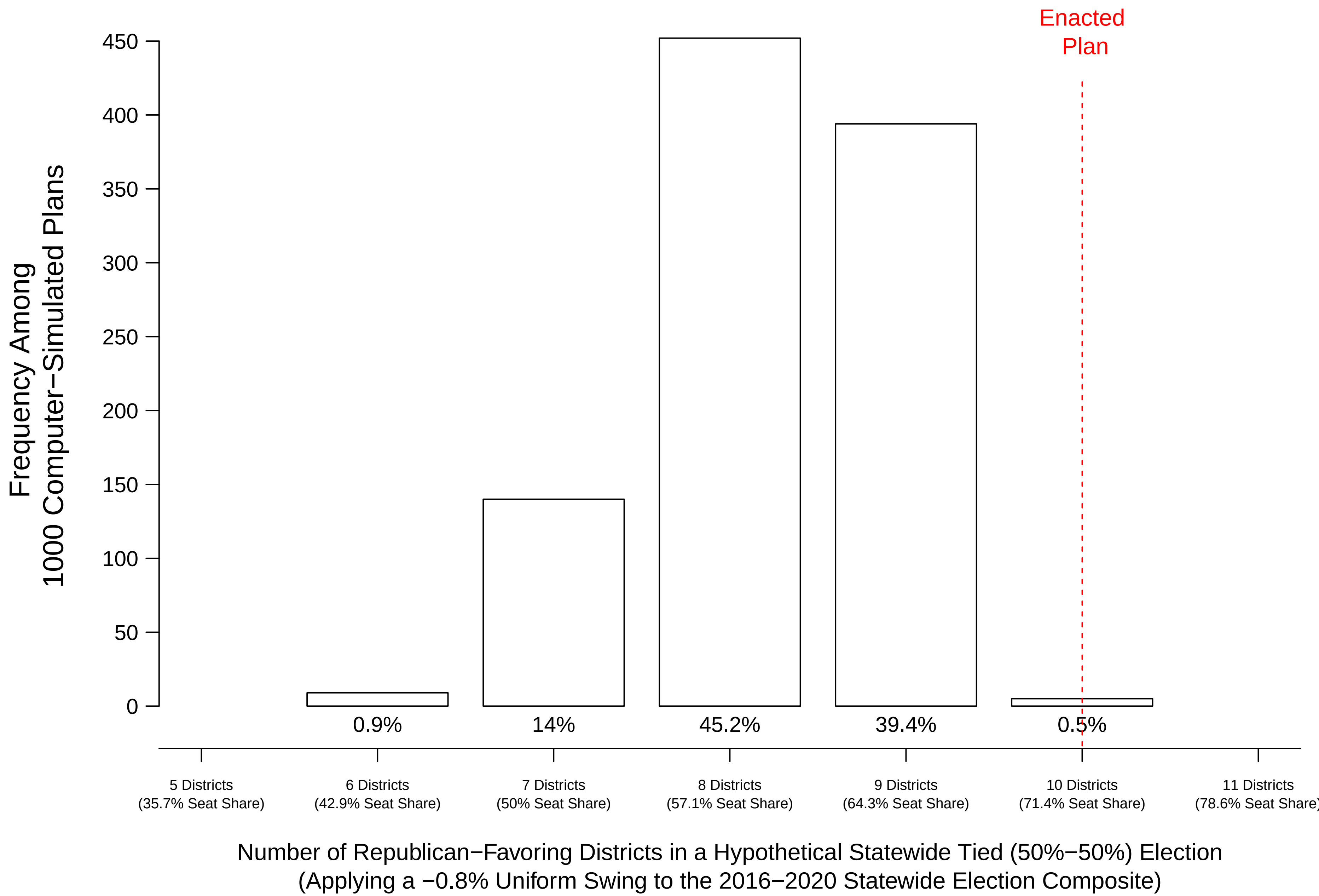


Figure 12: Piedmont Triad Area:
Comparison of Individual Districts' Republican Vote Shares
in the SB 740 Plan and in 1,000 Computer−Simulated Plans

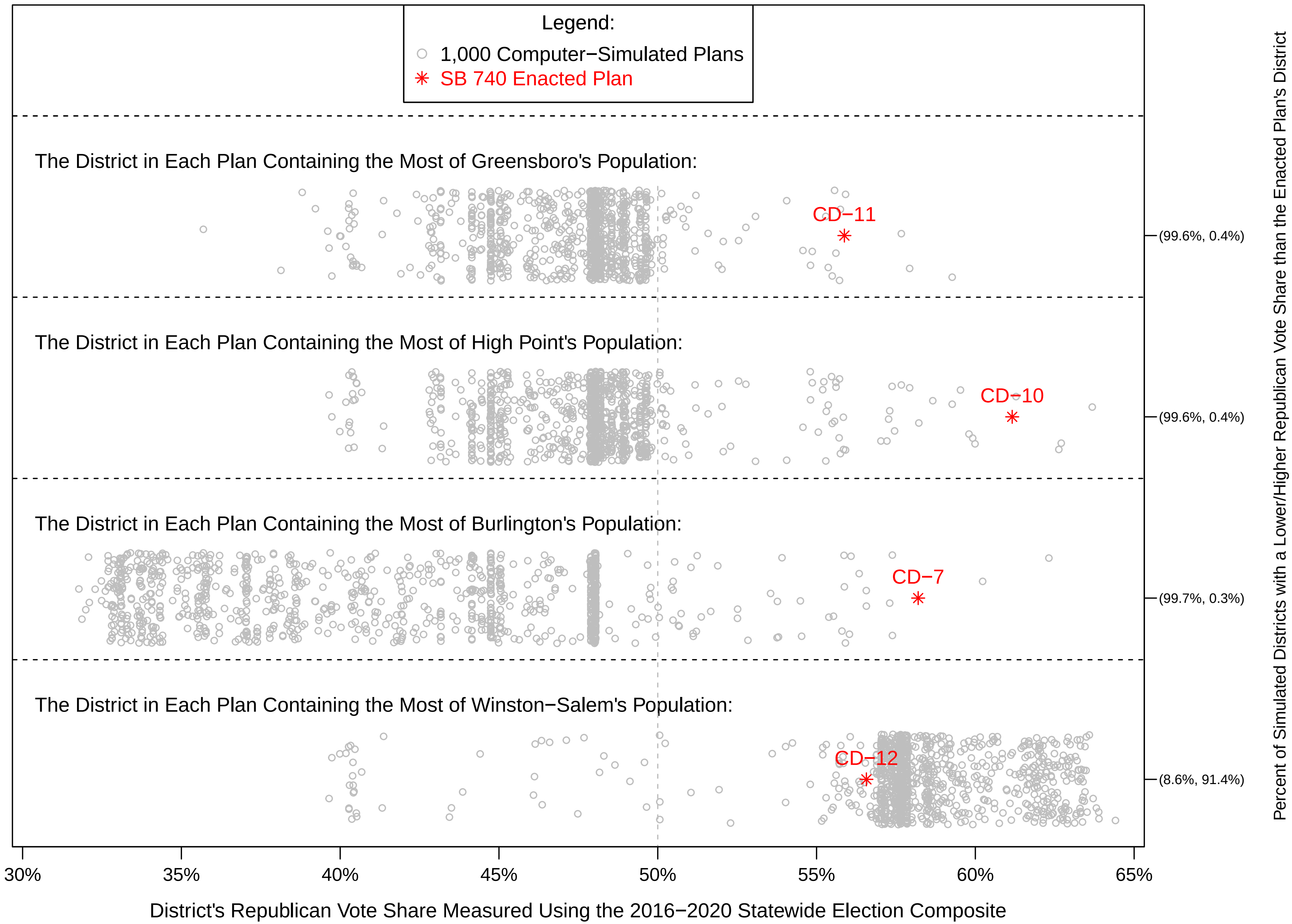


Figure 13: Piedmont Triad Area:
Comparison of Individual Districts' Compactness Scores
in the SB 740 Plan and in 1,000 Computer−Simulated Plans

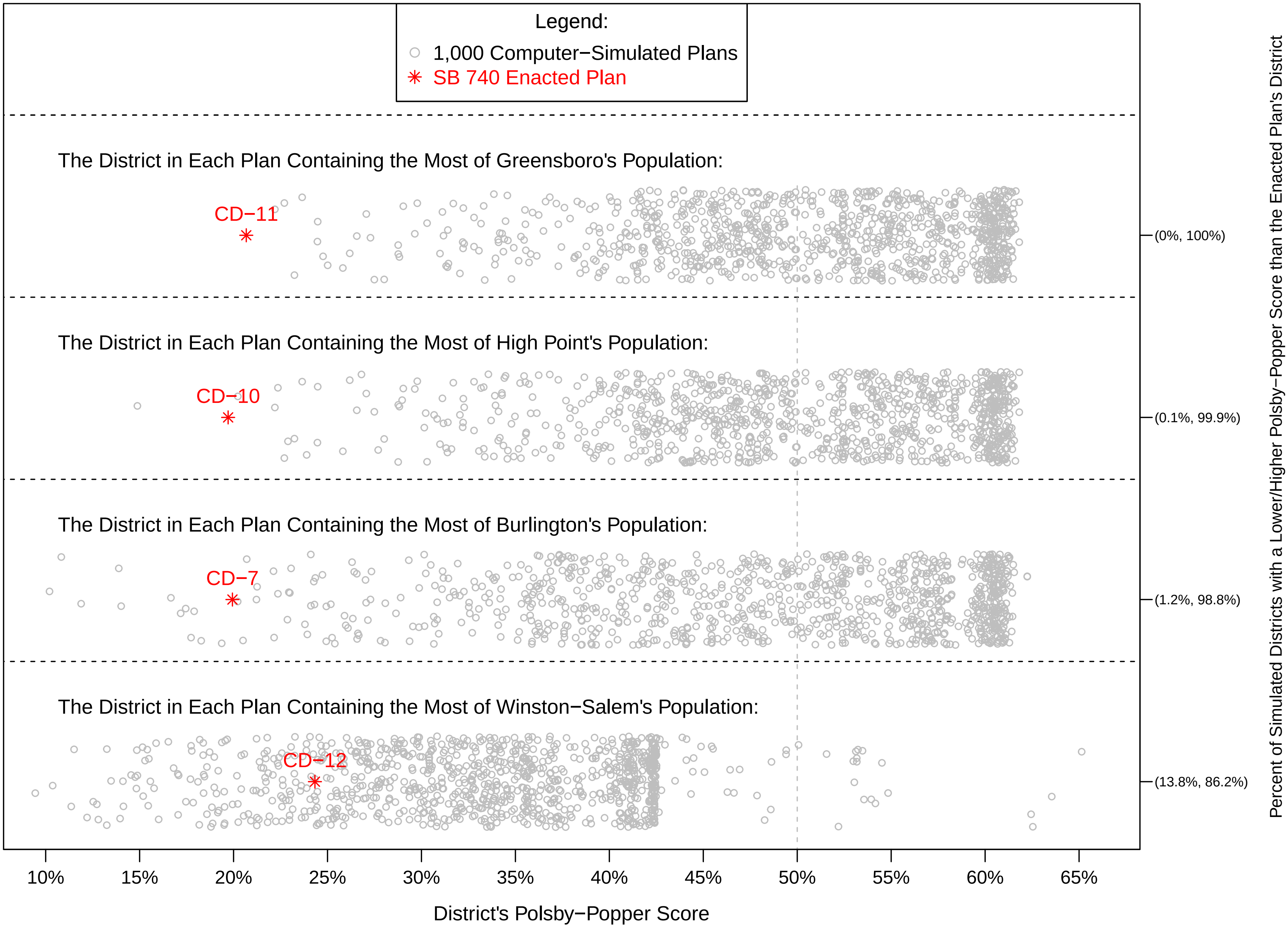
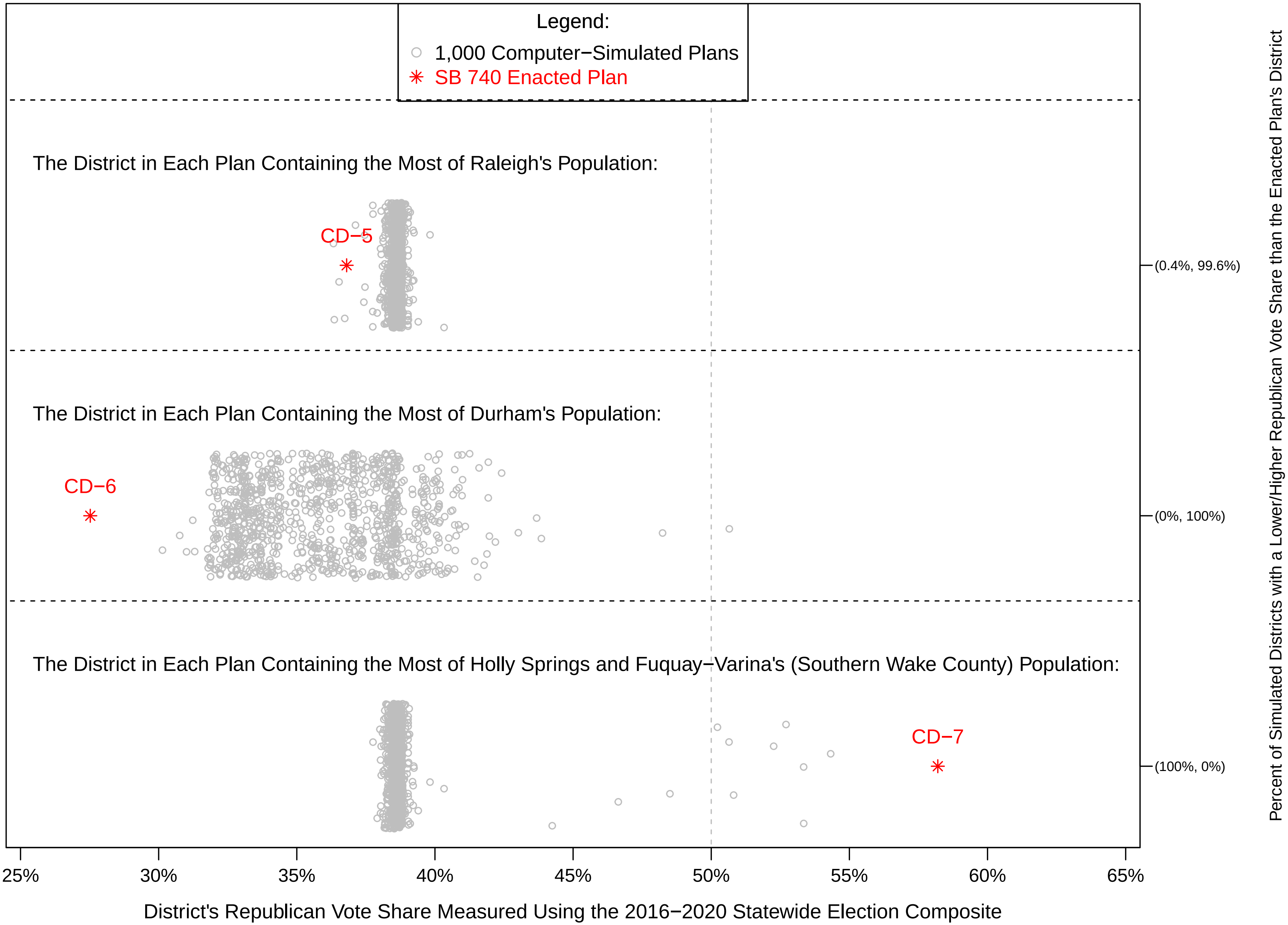


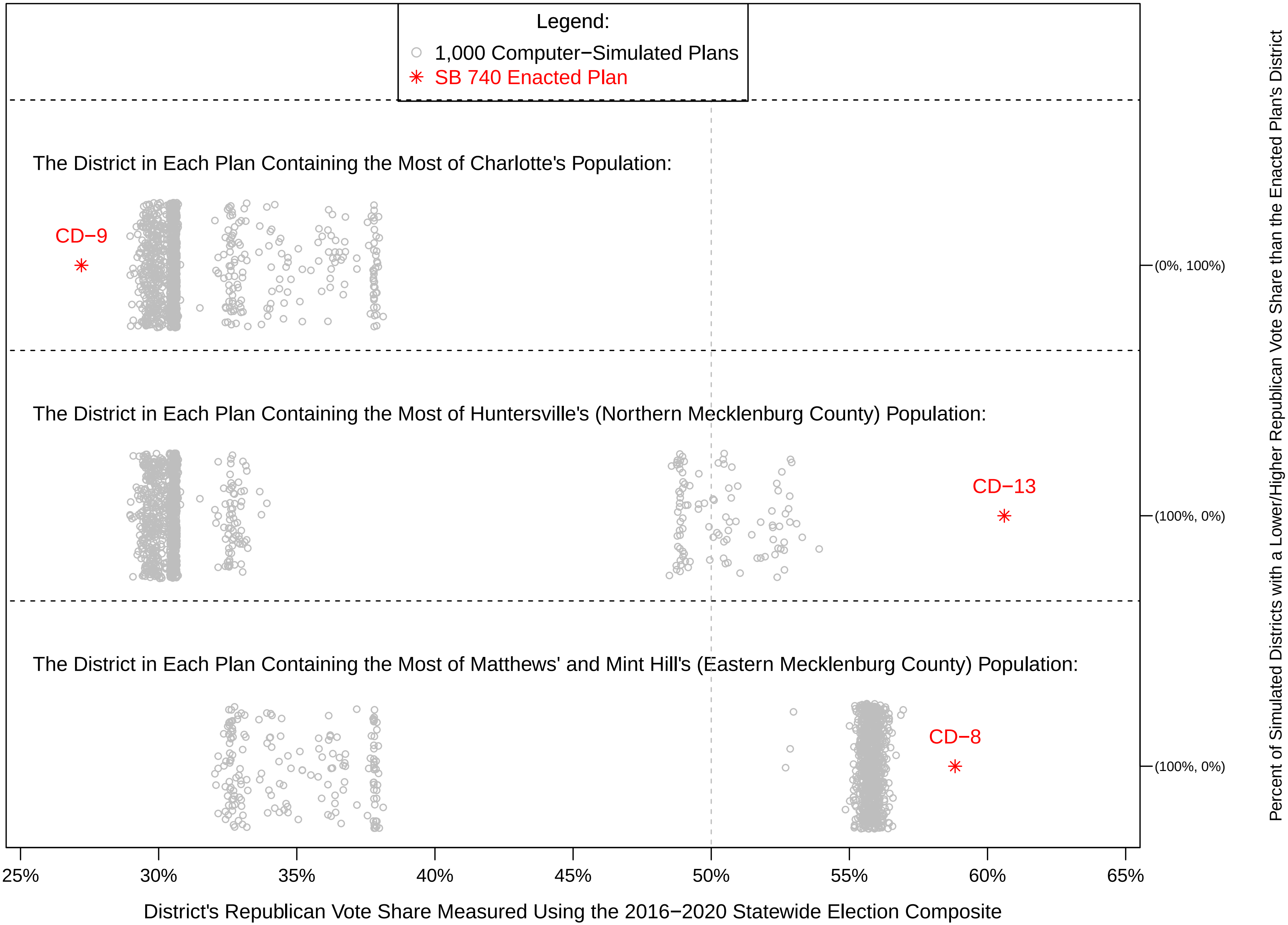
Figure 14: Research Triangle Area: Comparison of Individual Districts' Republican Vote Shares in the SB 740 Plan and in 1,000 Computer−Simulated Plans



– Ex. 4502 –

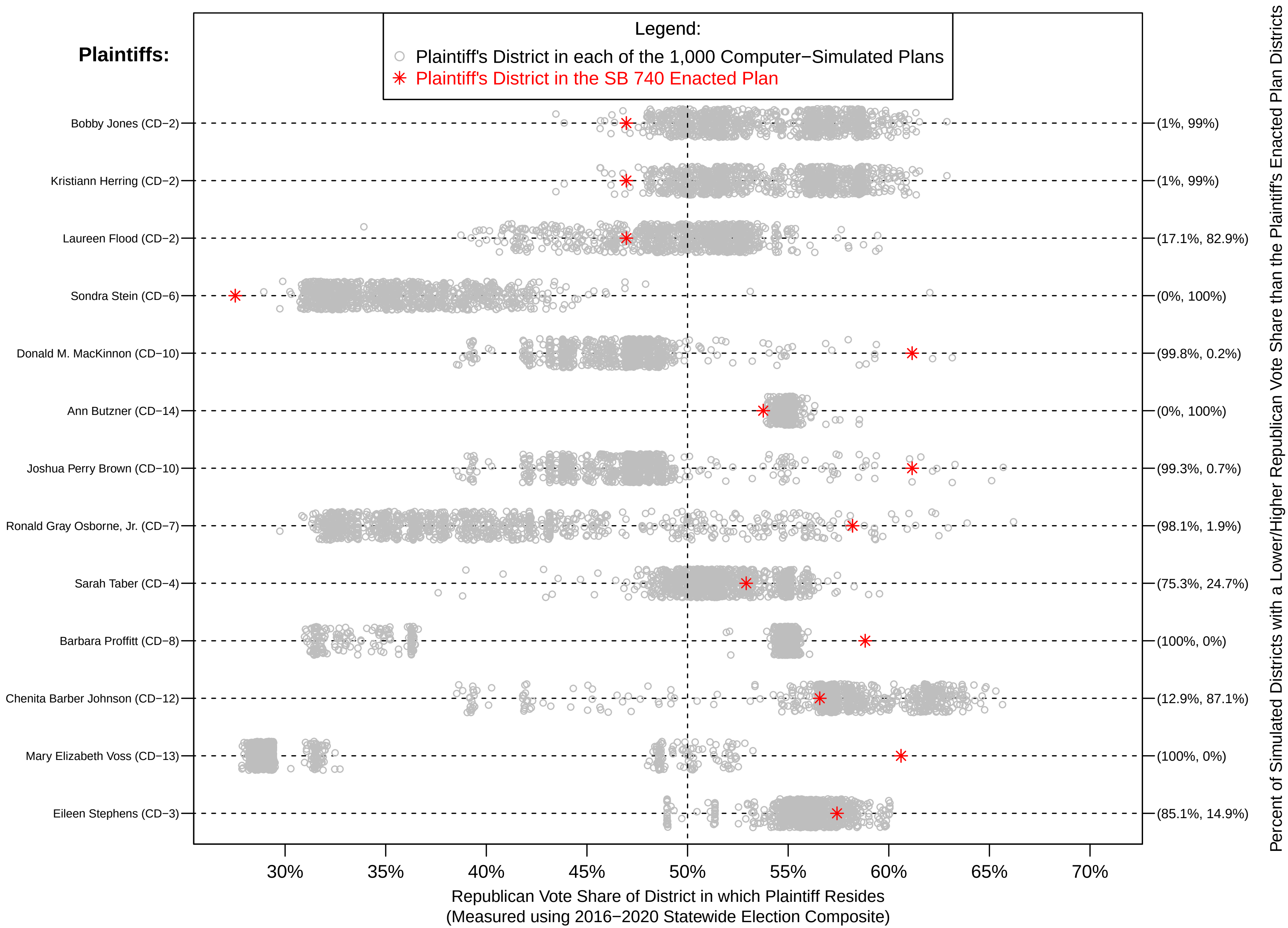
Figure 15: Research Triangle Area: Comparison of Individual Districts' Compactness Scores in the SB 740 Plan and in 1,000 Computer−Simulated Plans

**Figure 16: Mecklenburg County:
Comparison of Individual Districts' Republican Vote Shares
in the SB 740 Plan and in 1,000 Computer−Simulated Plans**



– Ex. 4504 –

Figure 17a:
Plaintiffs' Districts in the SB 740 Plan and in 1,000 Computer−Simulated Plans



– Ex. 4505 –

Figure 17b:
Plaintiffs' Districts in the SB 740 Plan and in 1,000 Computer−Simulated Plans

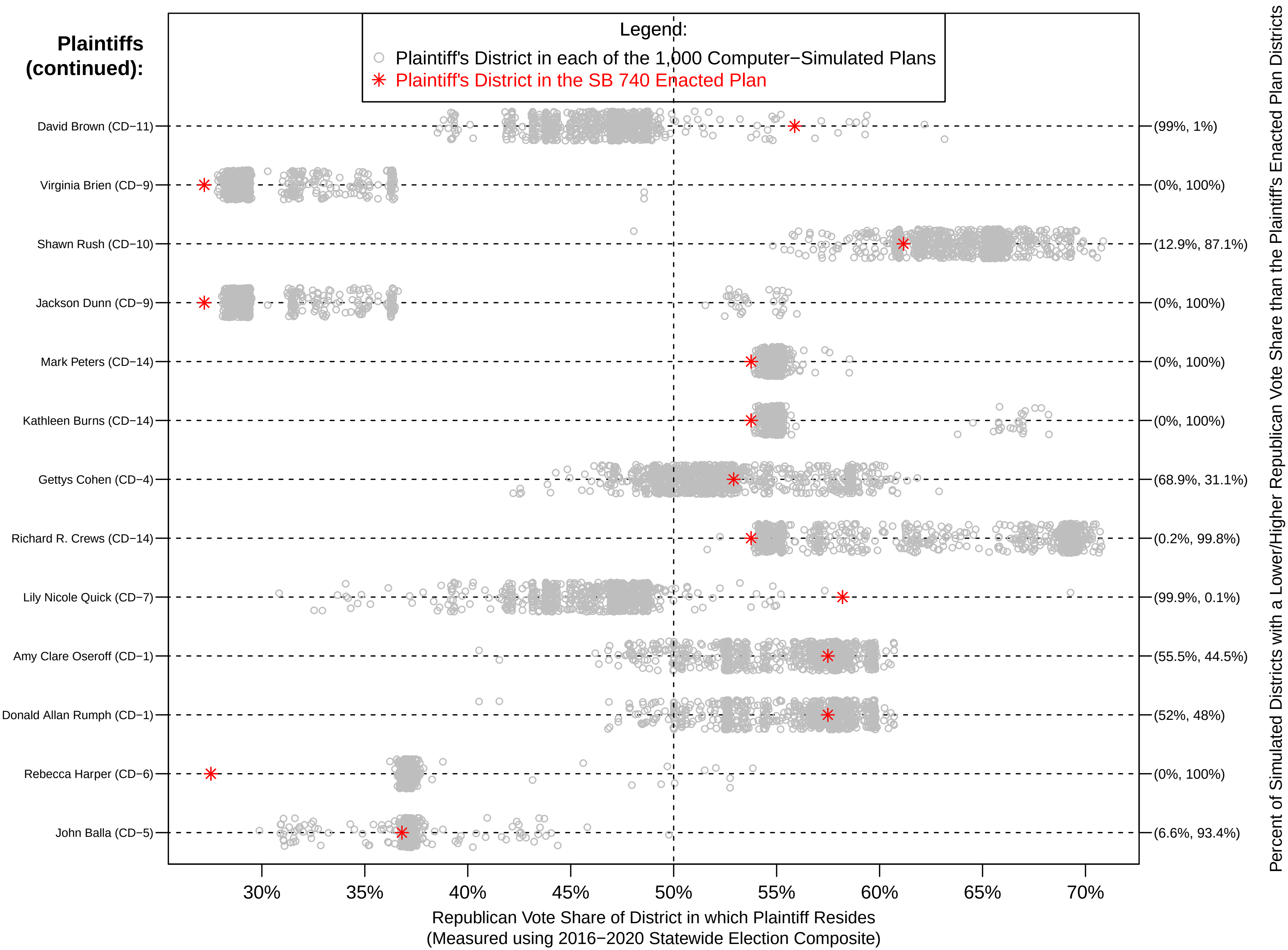
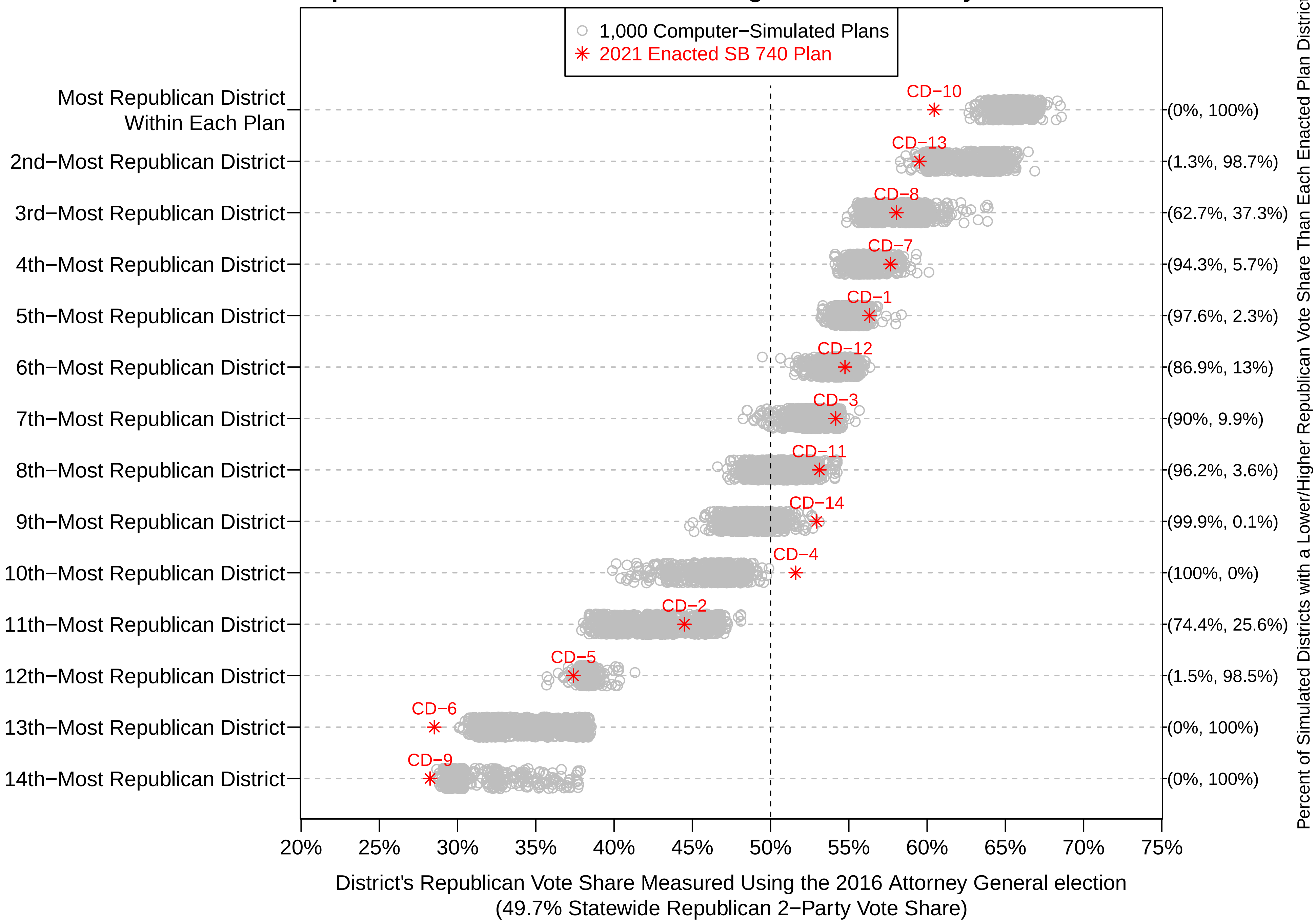


Figure A1: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 Attorney General Election Results



– Ex. 4507 –

Figure A2: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 Governor Election Results

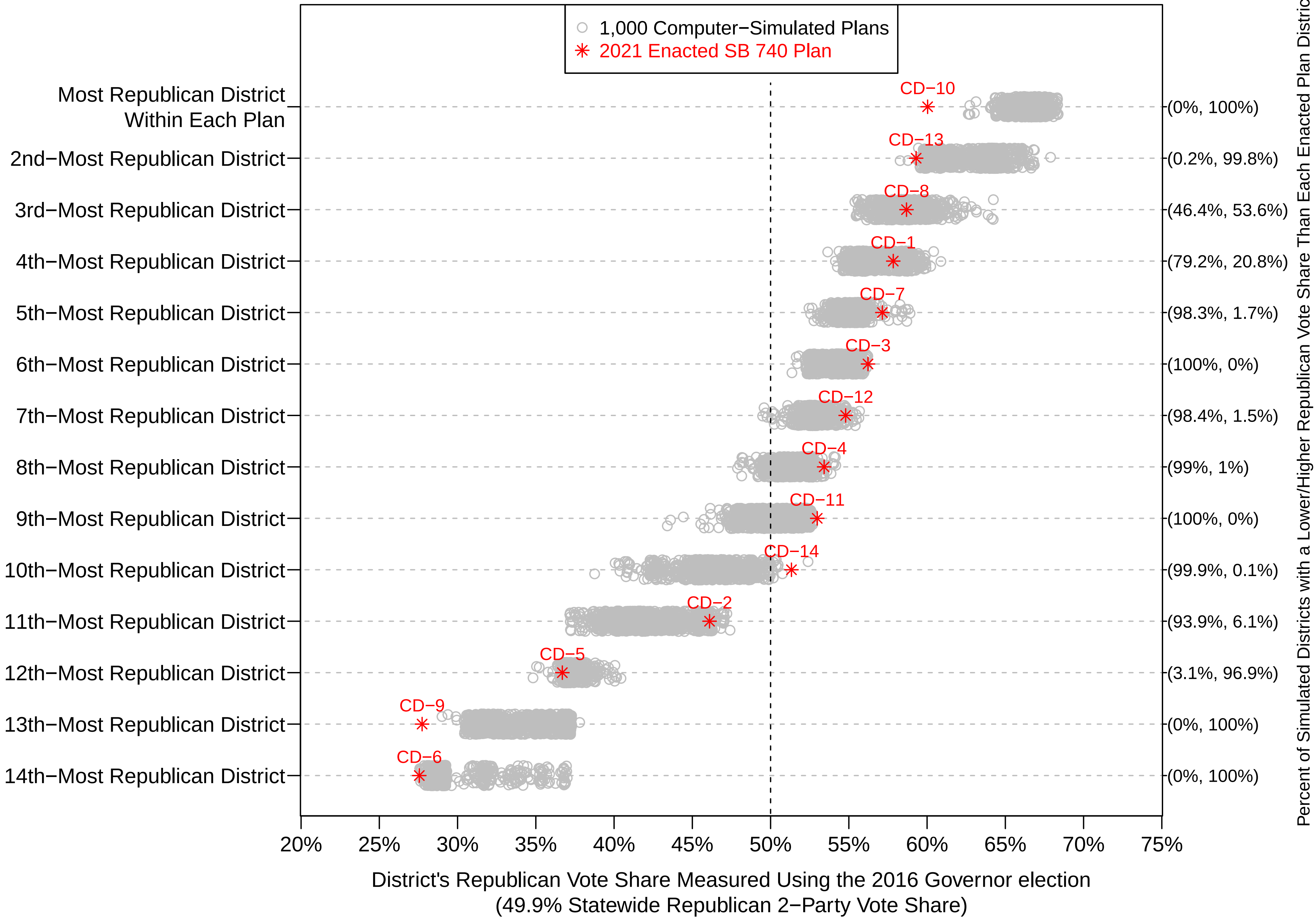
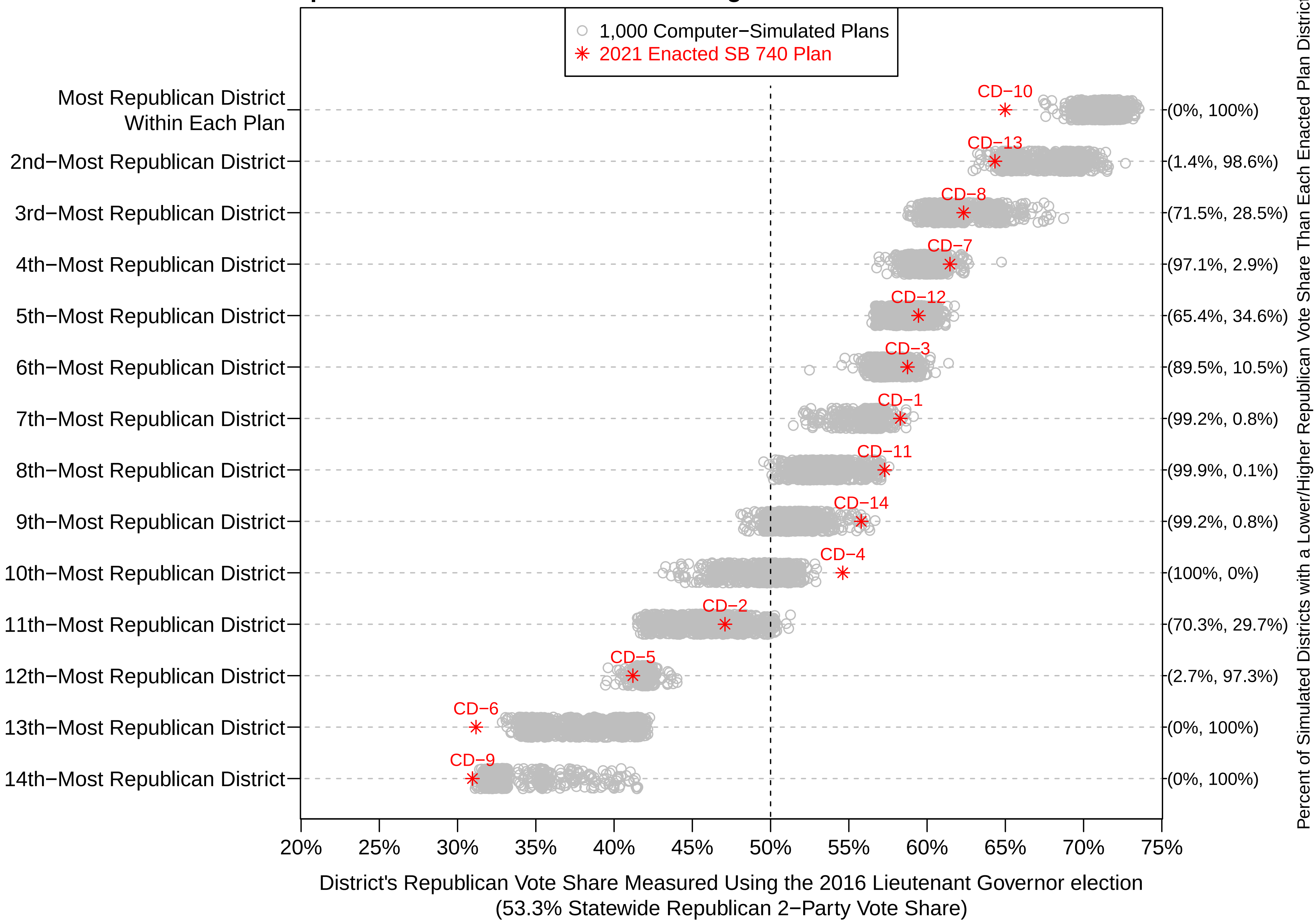


Figure A3: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 Lieutenant Governor Election Results



– Ex. 4509 –

Figure A4: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 US President Election Results

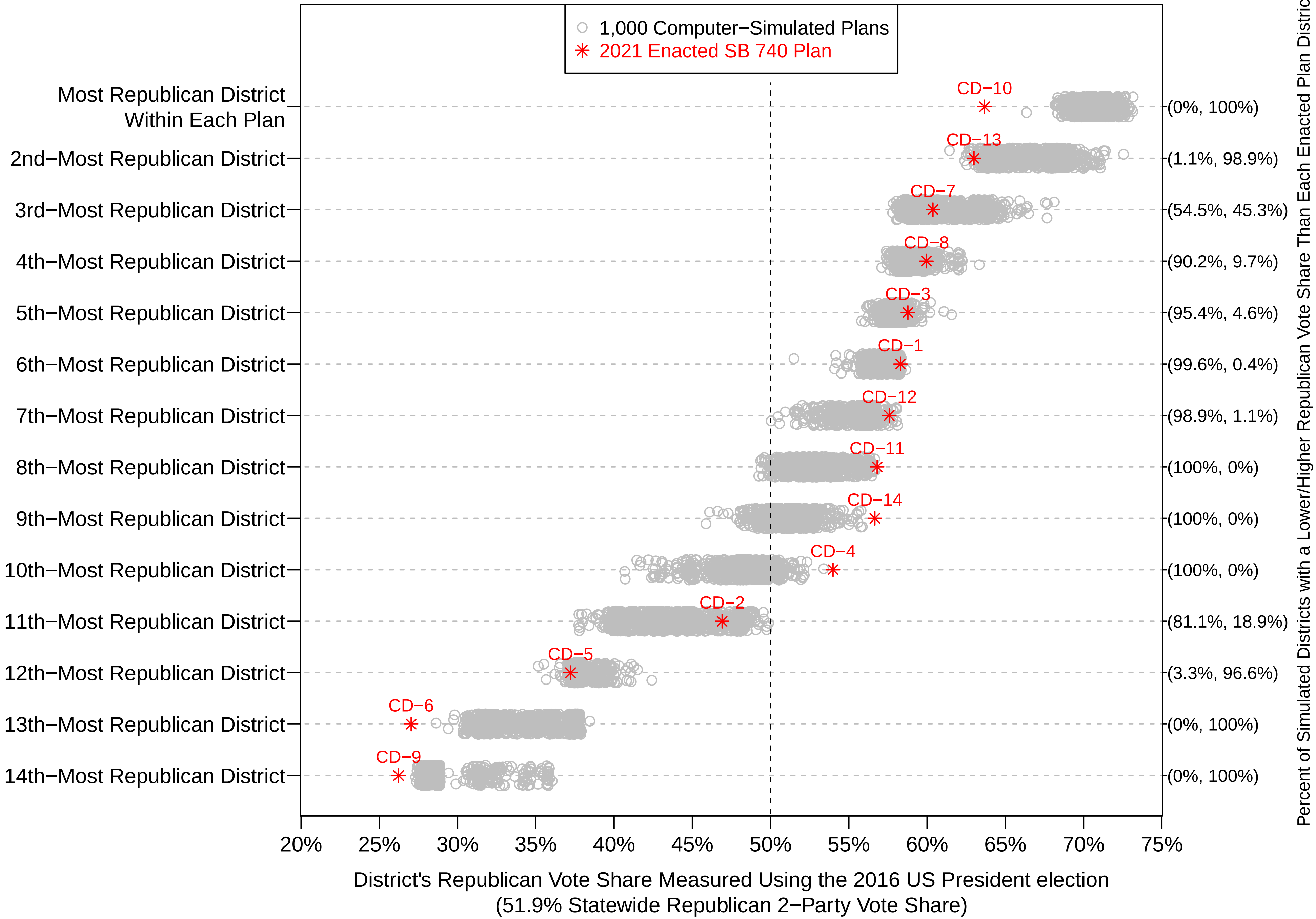
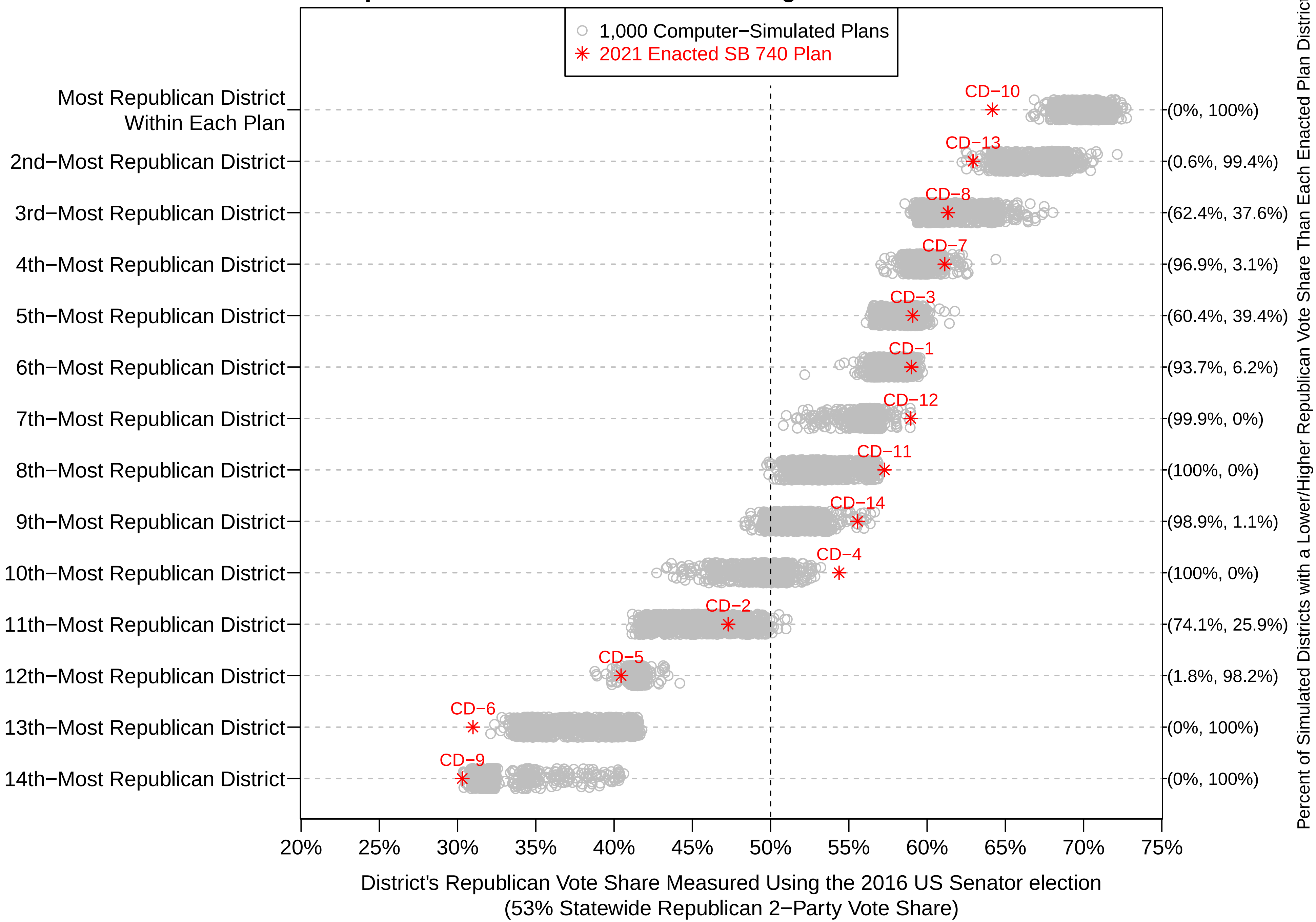


Figure A5: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2016 US Senator Election Results



– Ex. 4511 –

Figure A6: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 Attorney General Election Results

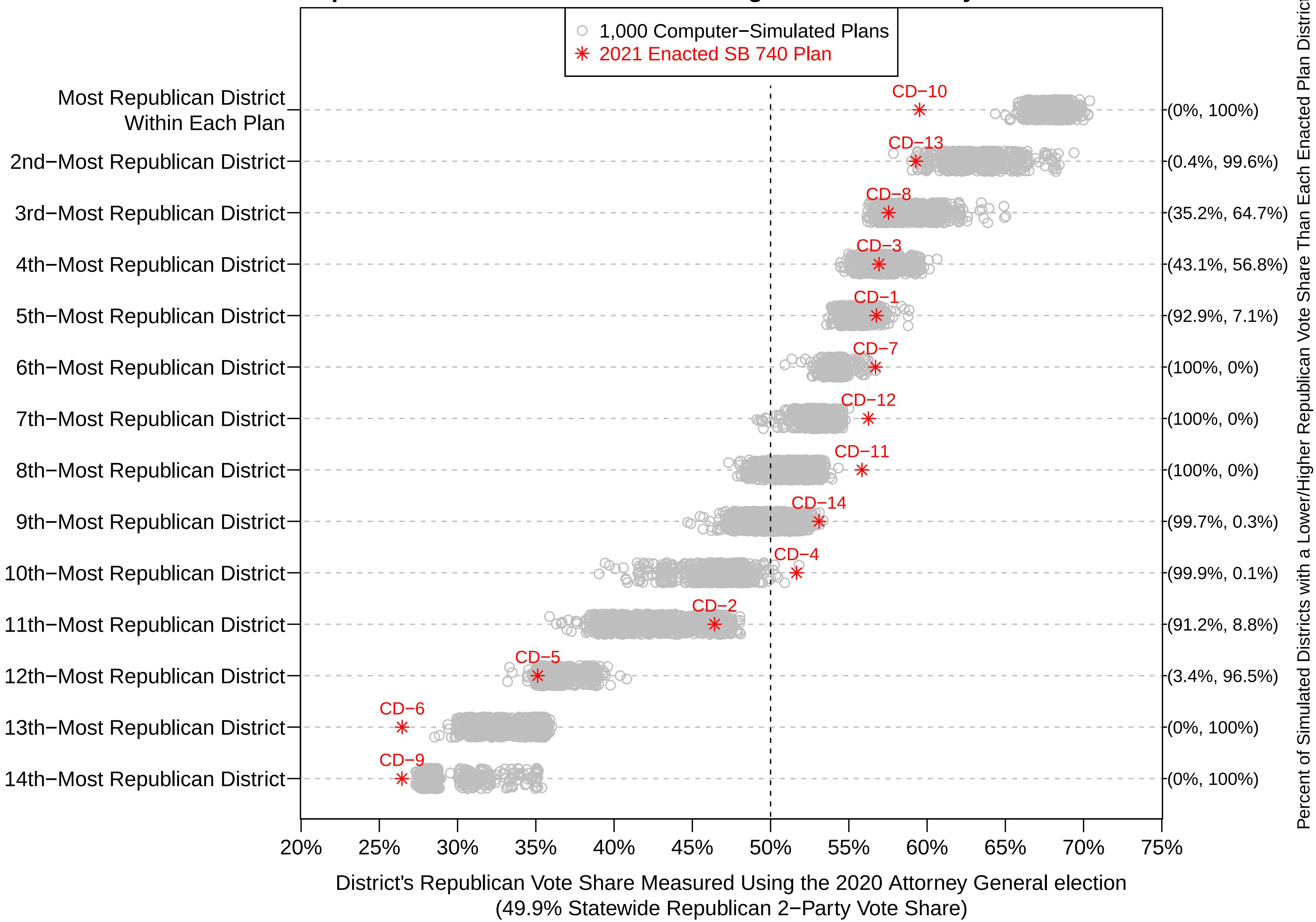


Figure A7: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 Governor Election Results

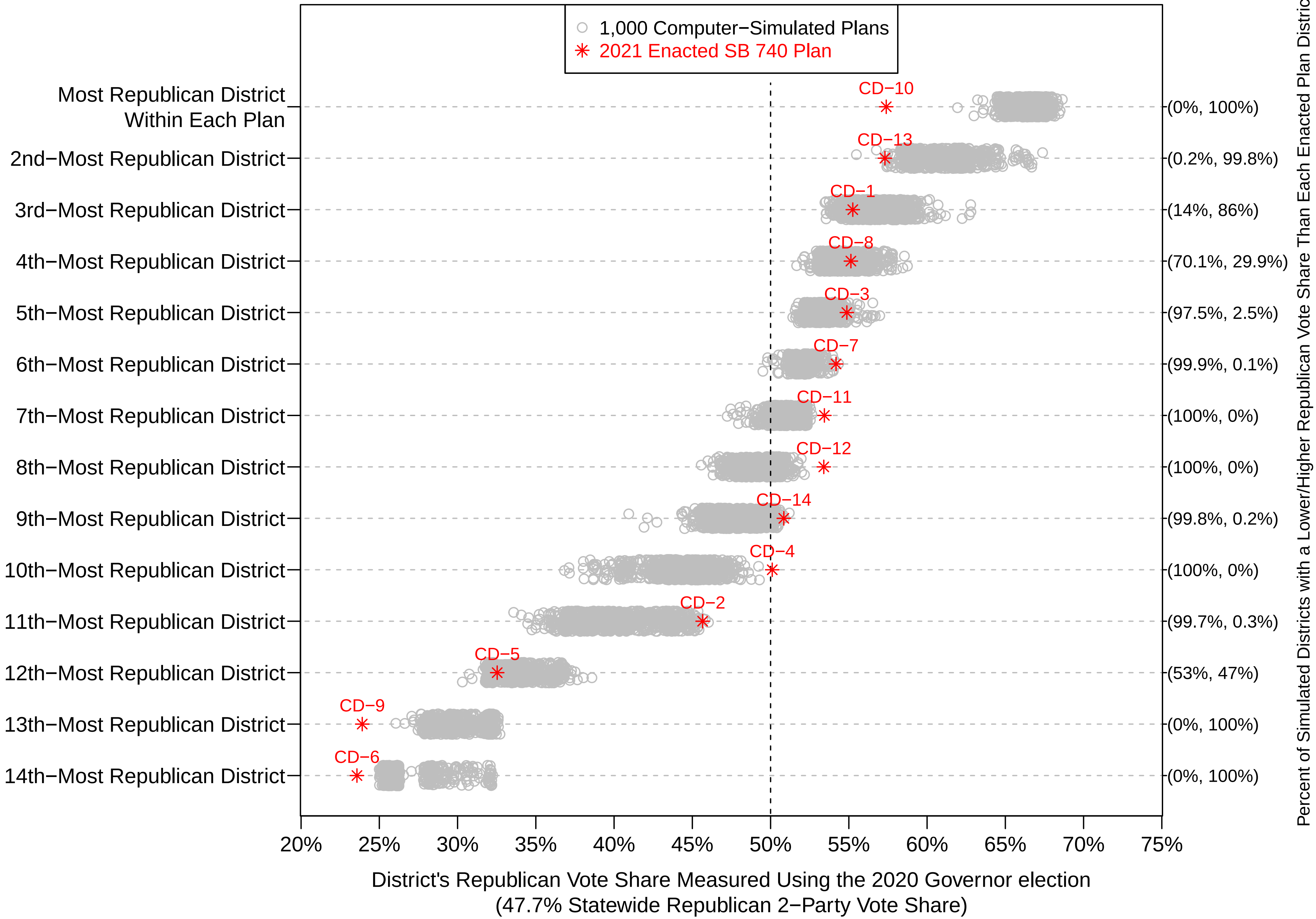


Figure A8: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 Lieutenant Governor Election Results

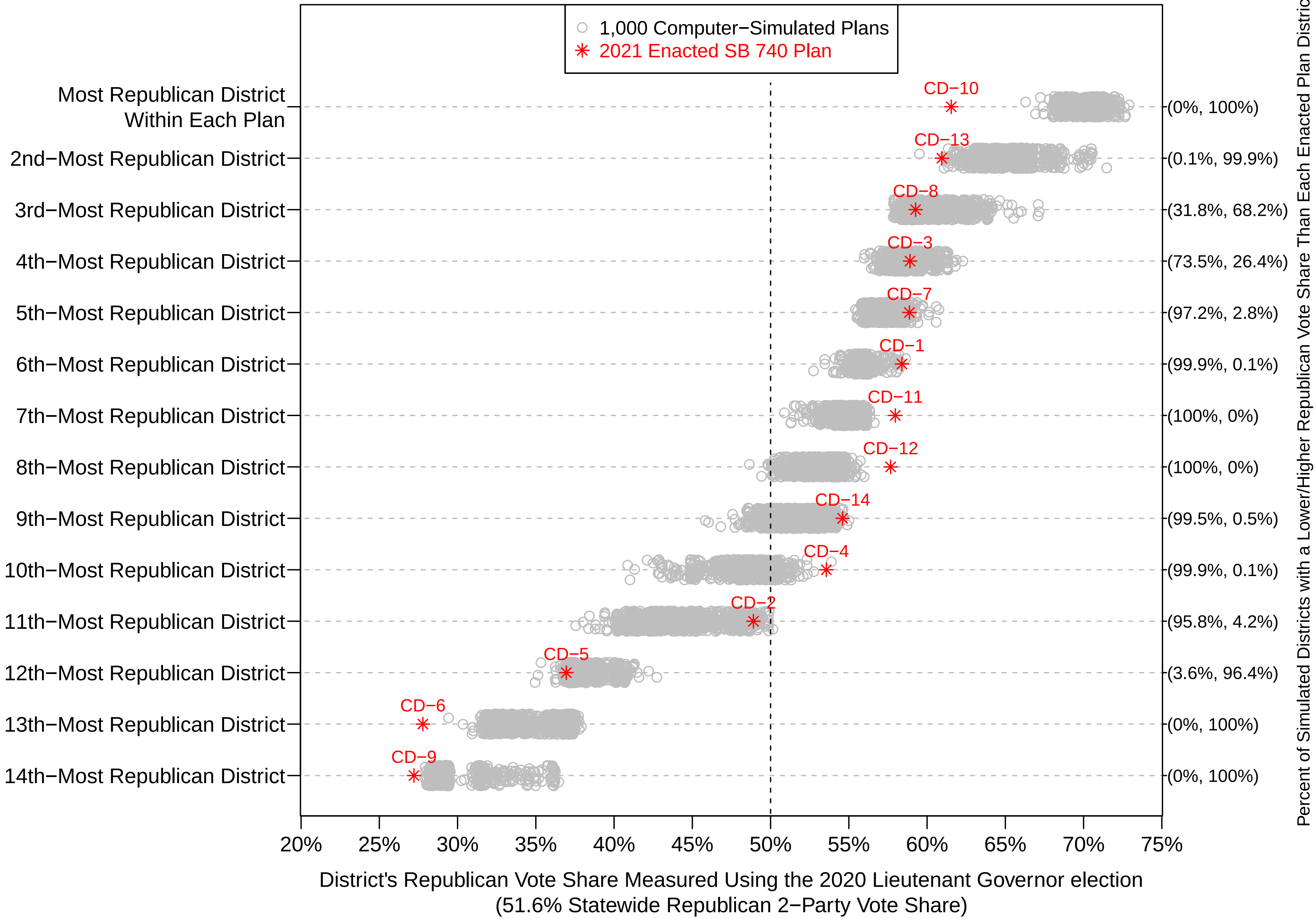


Figure A9: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 US President Election Results

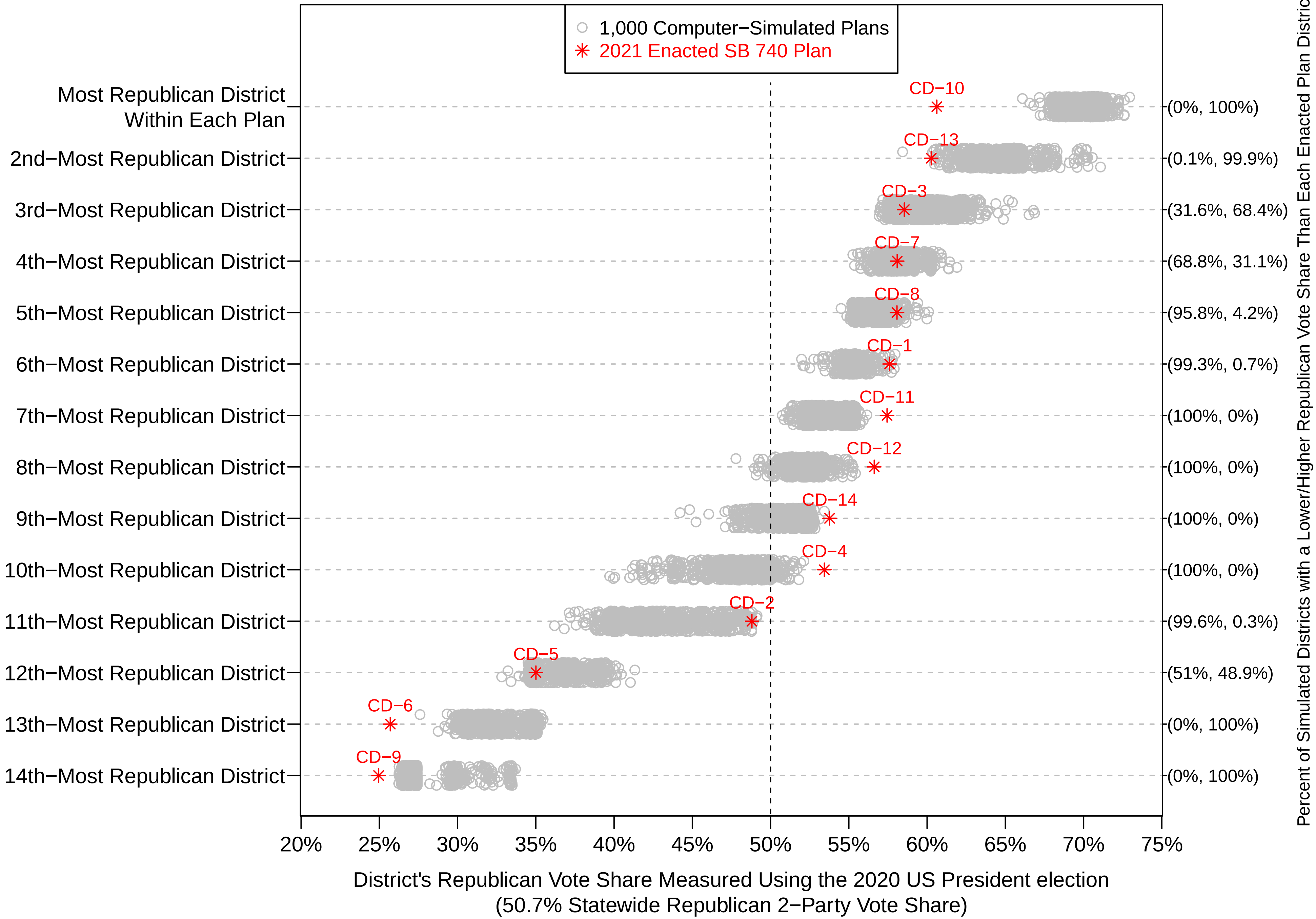


Figure A10: Comparison of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans: Districts' Republican Vote Share Measured Using the 2020 US Senator Election Results

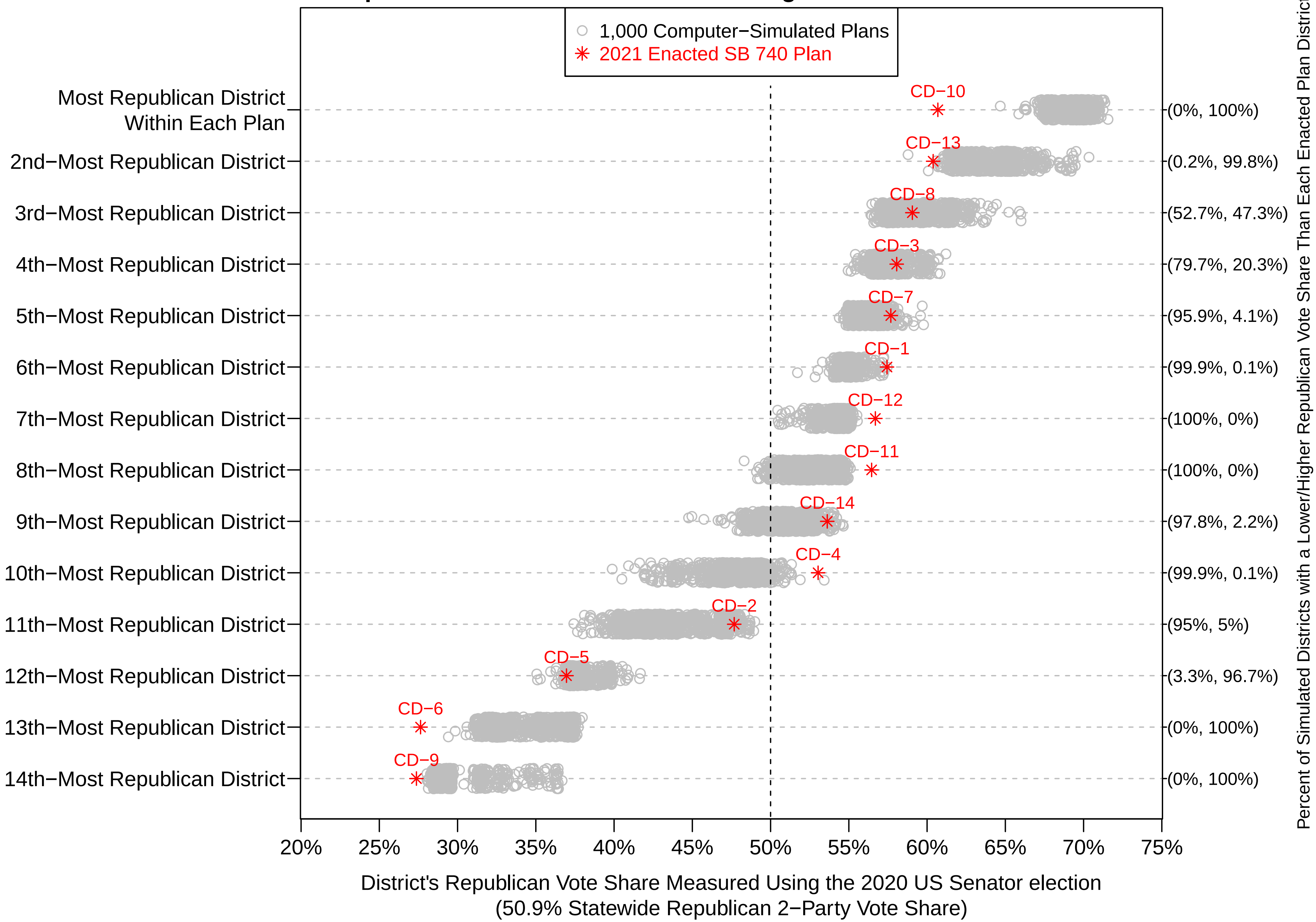


Figure B1: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 Attorney General election
(49.7% Statewide Republican 2−Party Vote Share)

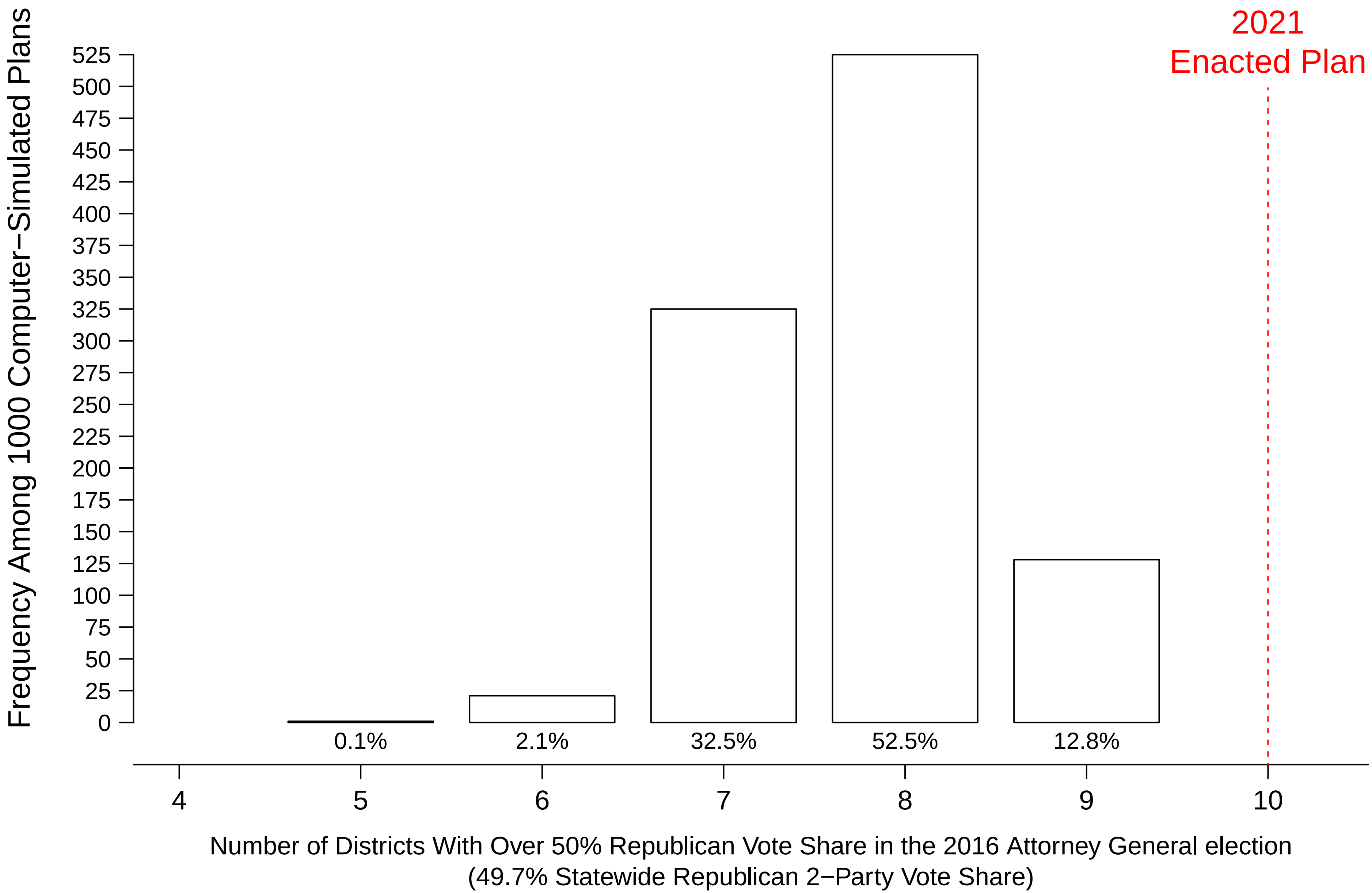


Figure B2: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 Governor election
(49.9% Statewide Republican 2−Party Vote Share)

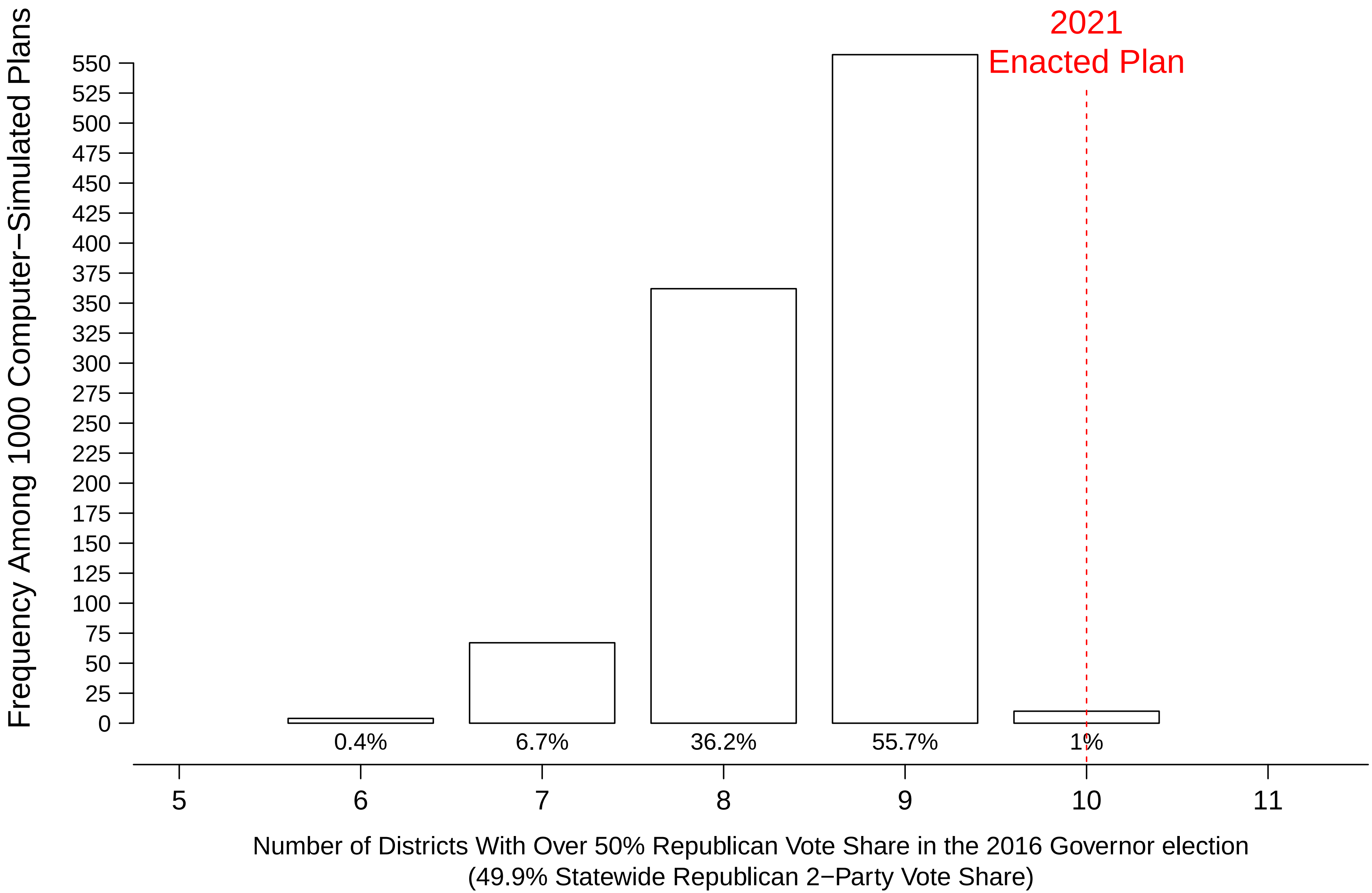


**Figure B3: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 Lieutenant Governor election
(53.3% Statewide Republican 2−Party Vote Share)**

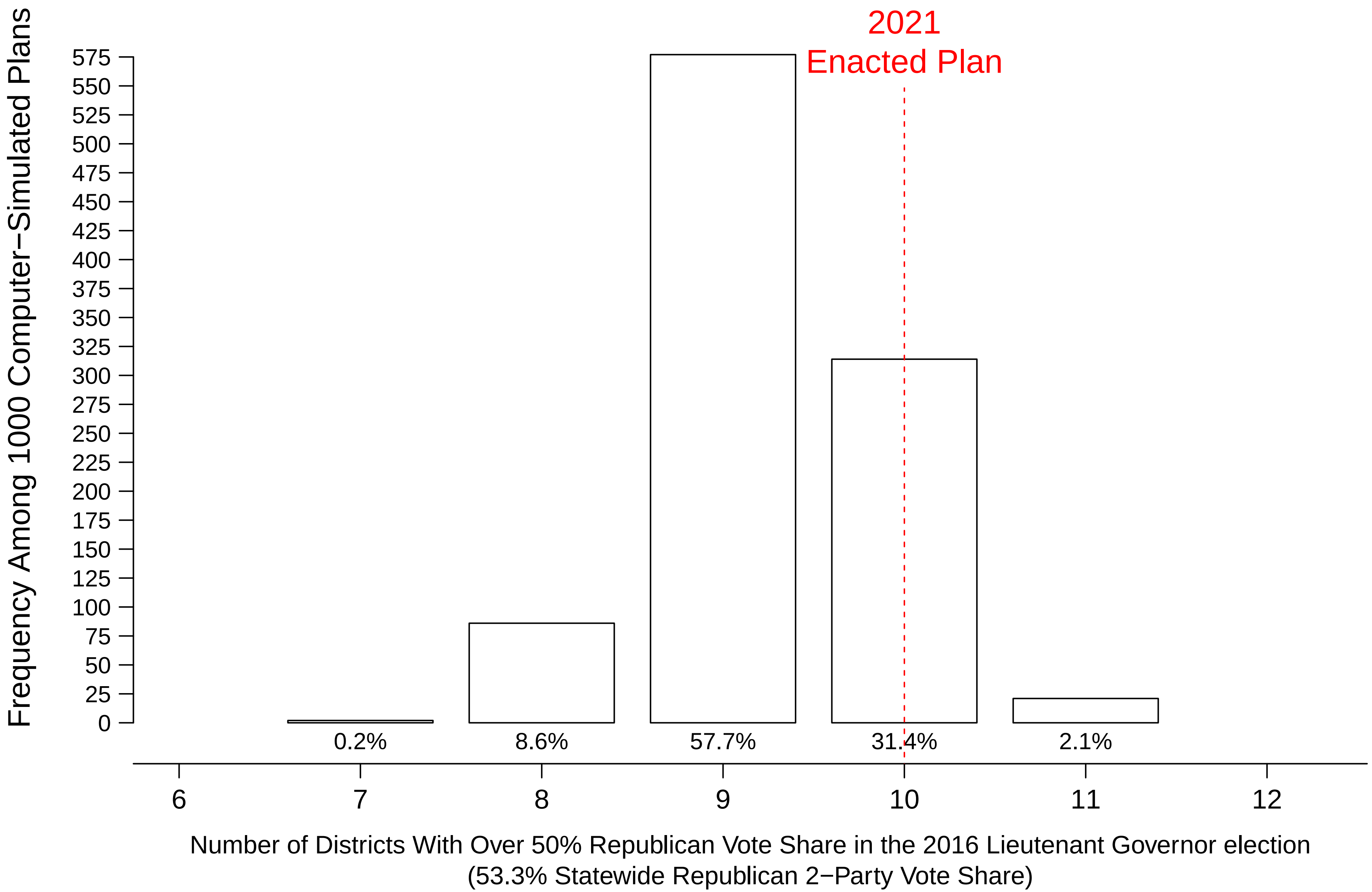


**Figure B4: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 US President election
(51.9% Statewide Republican 2−Party Vote Share)**

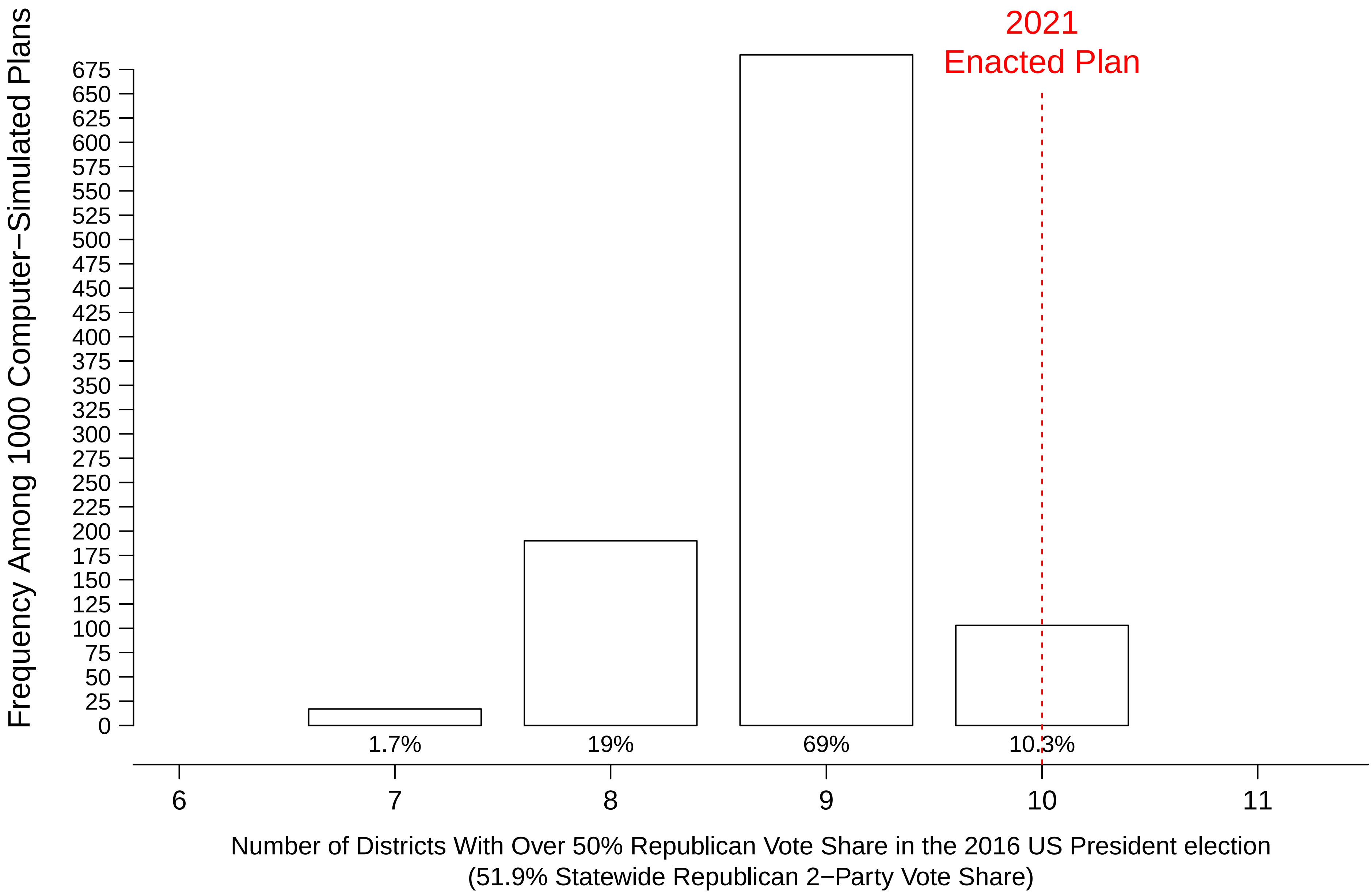


Figure B5: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2016 US Senator election
(53% Statewide Republican 2−Party Vote Share)

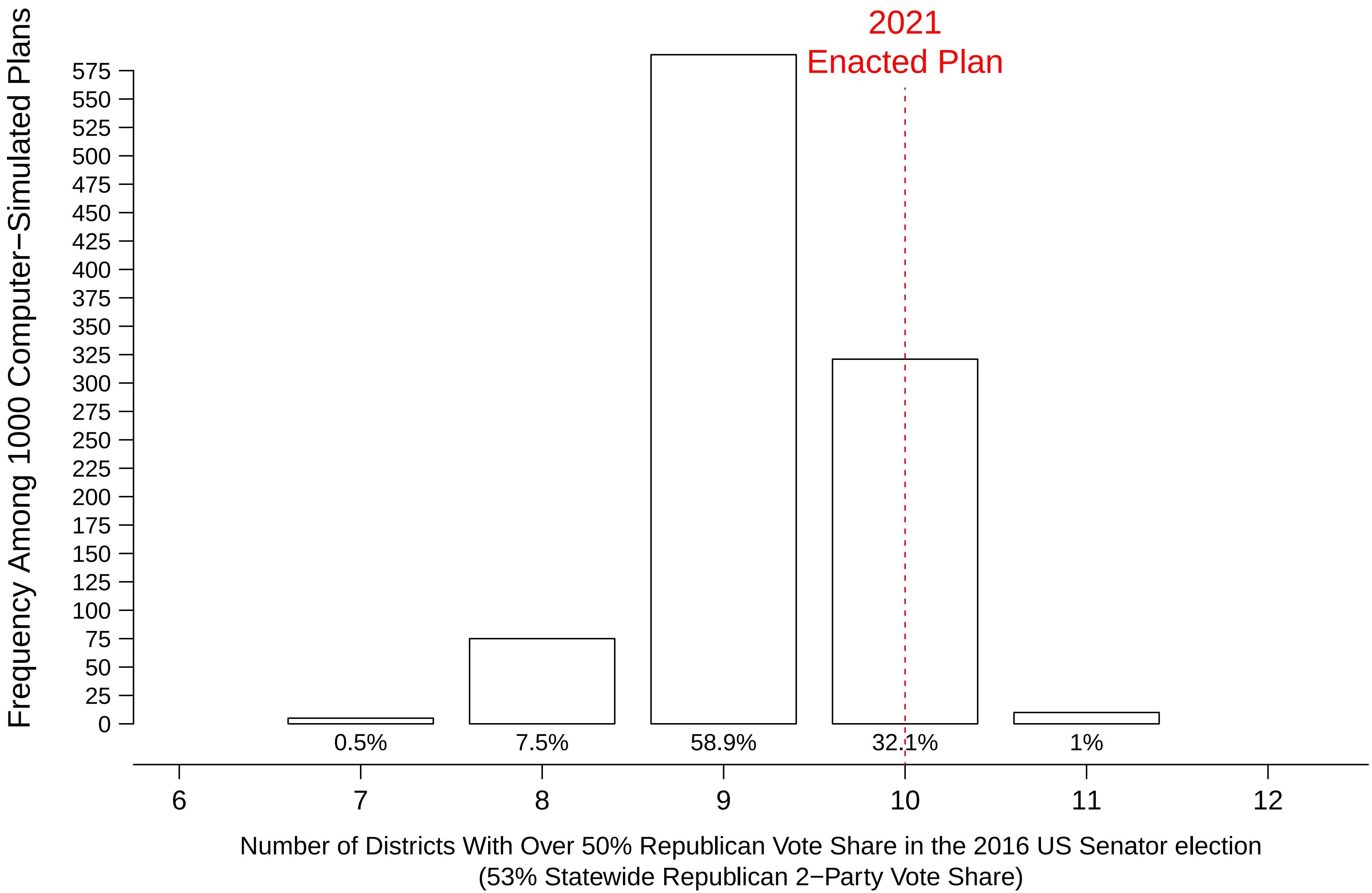


Figure B6: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 Attorney General election
(49.9% Statewide Republican 2−Party Vote Share)

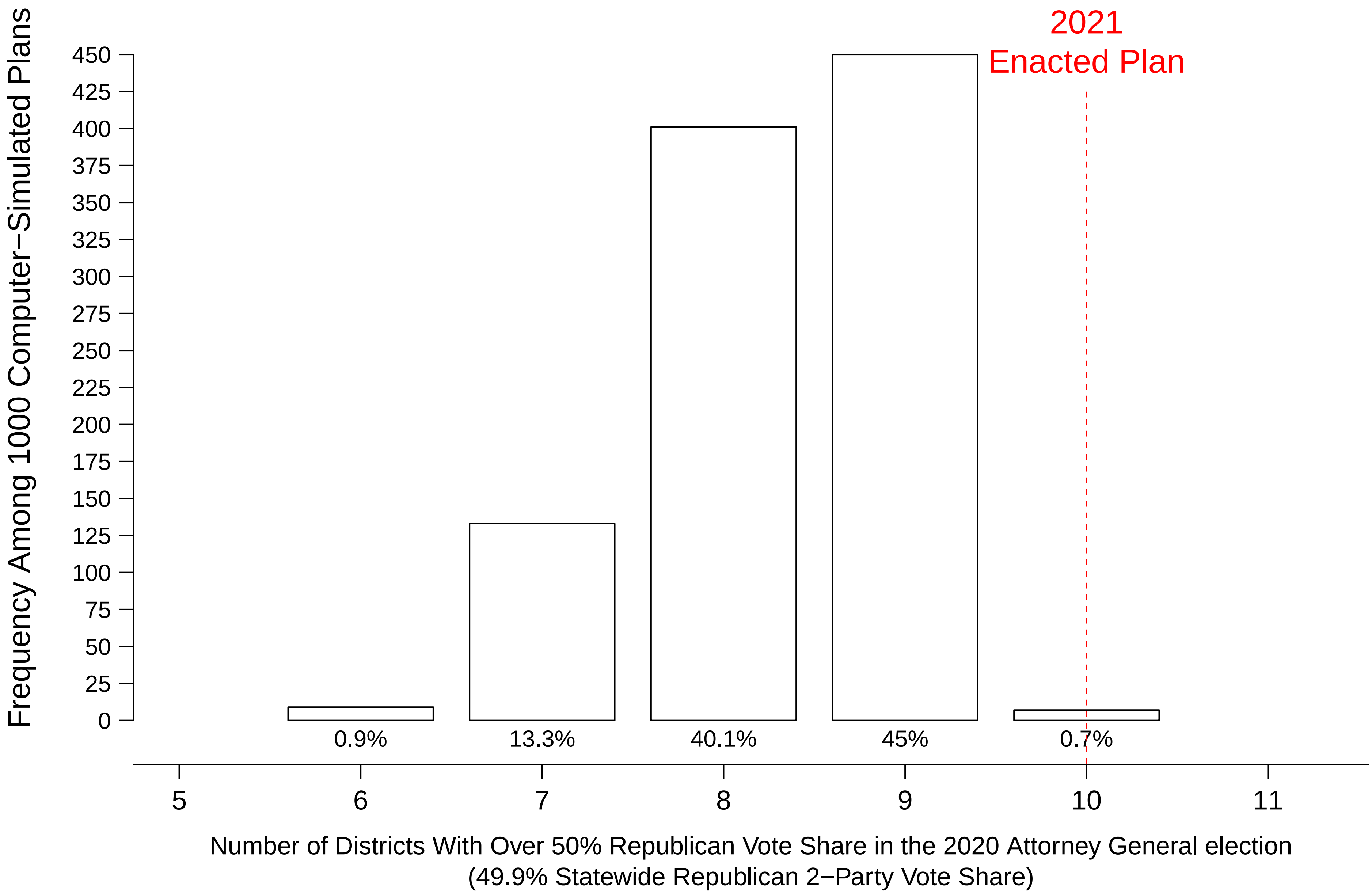


Figure B7: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 Governor election
(47.7% Statewide Republican 2−Party Vote Share)

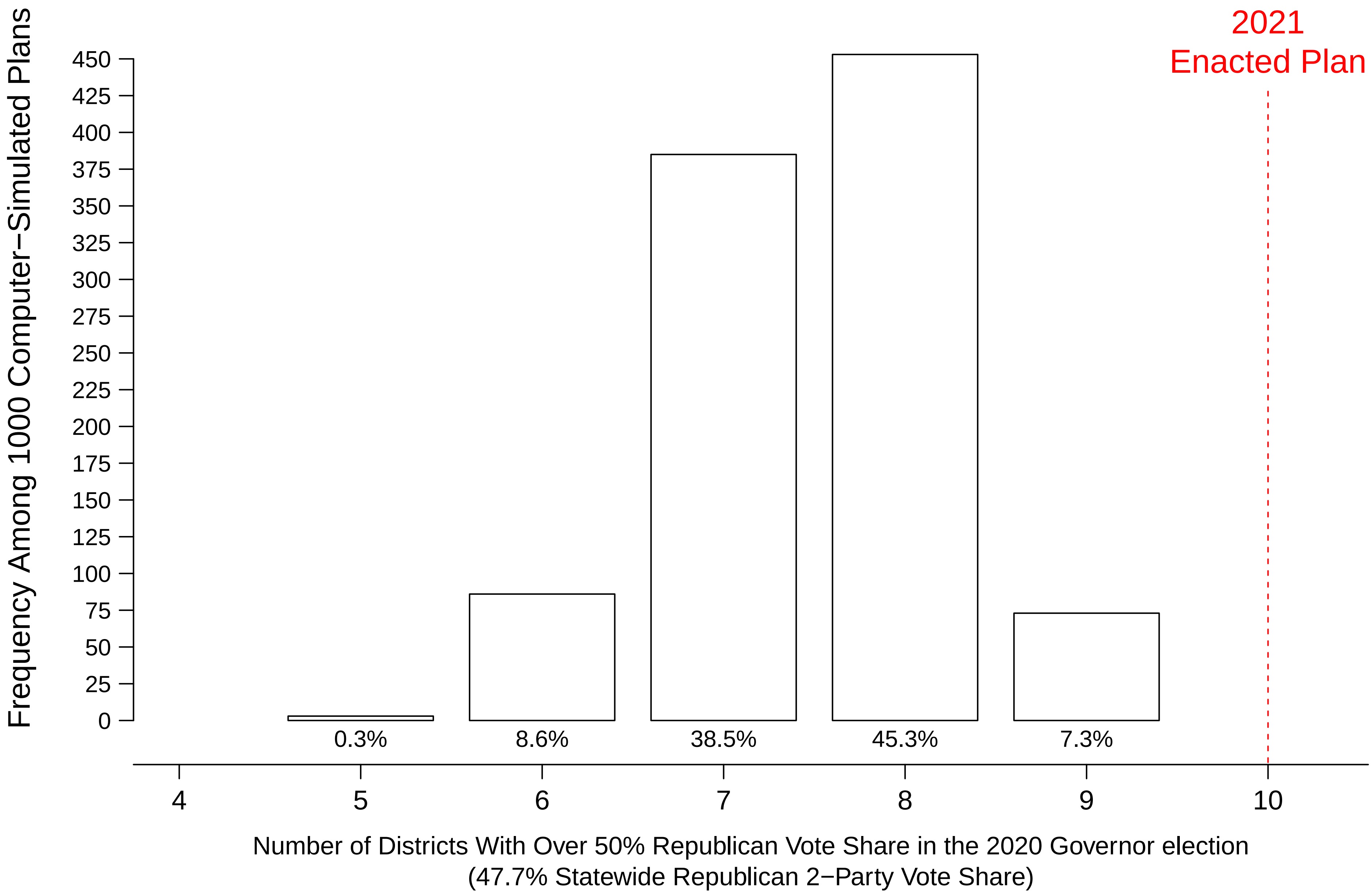


Figure B8: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 Lieutenant Governor election
(51.6% Statewide Republican 2−Party Vote Share)

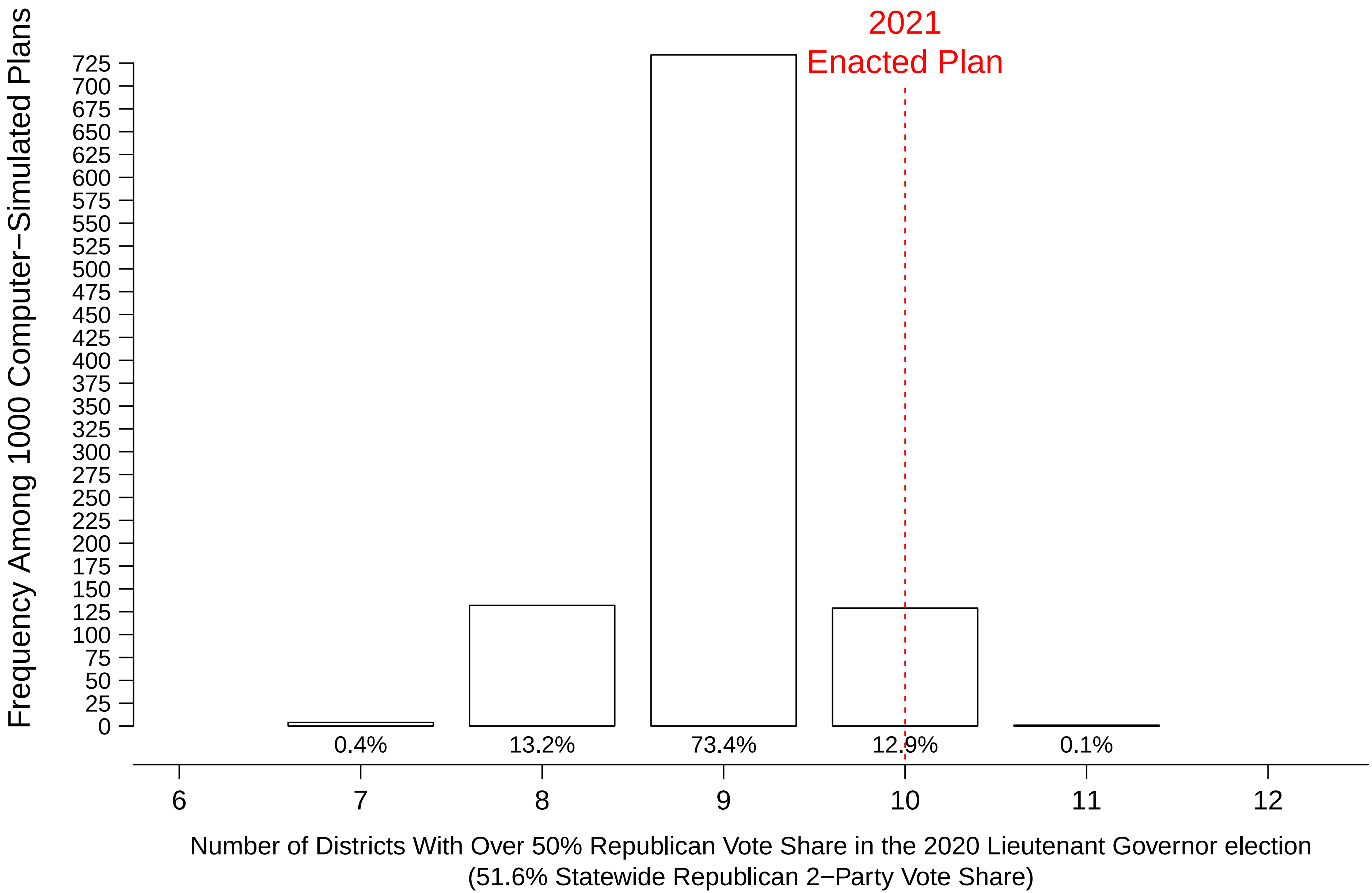


Figure B9: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 US President election
(50.7% Statewide Republican 2−Party Vote Share)

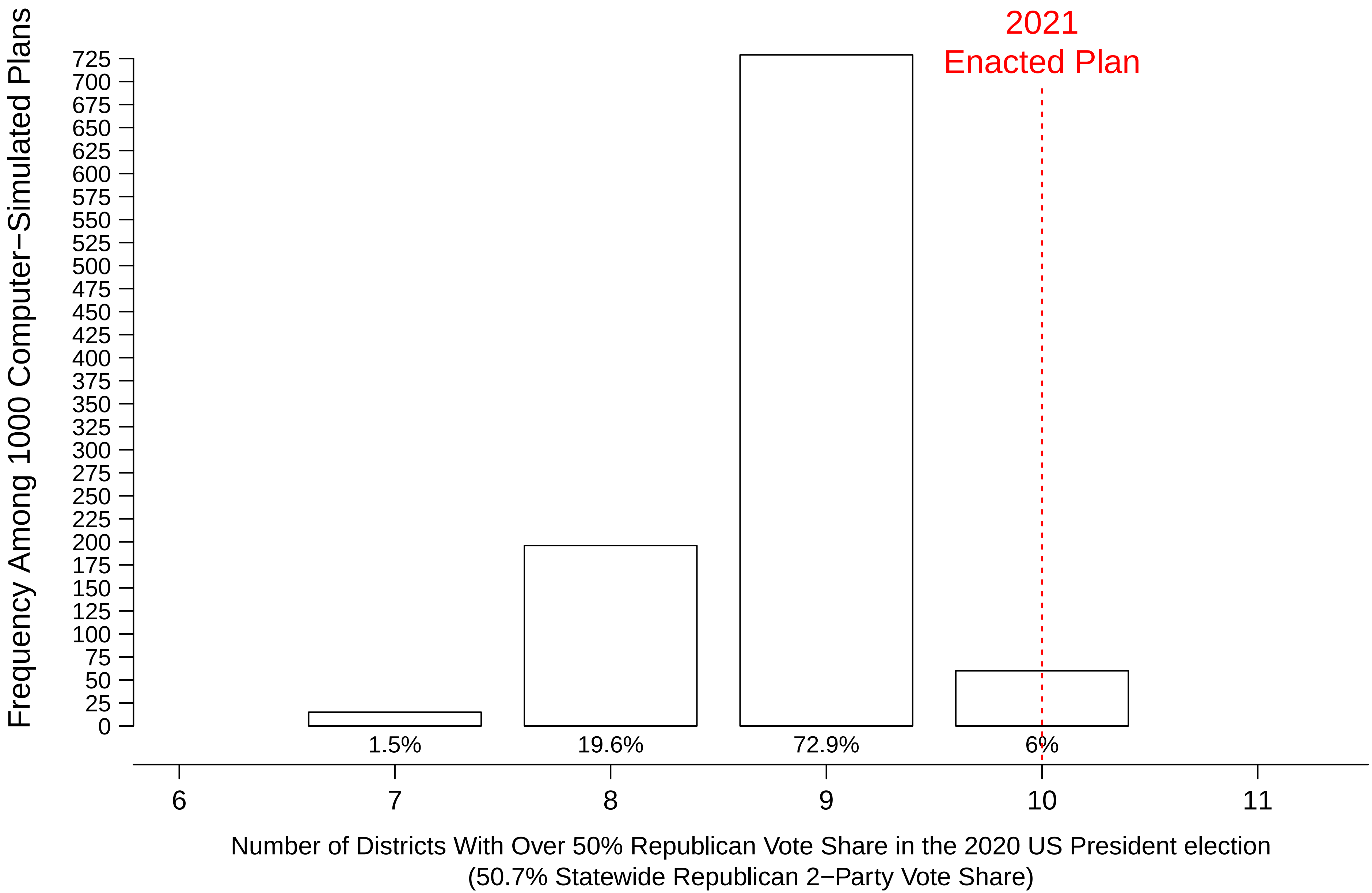
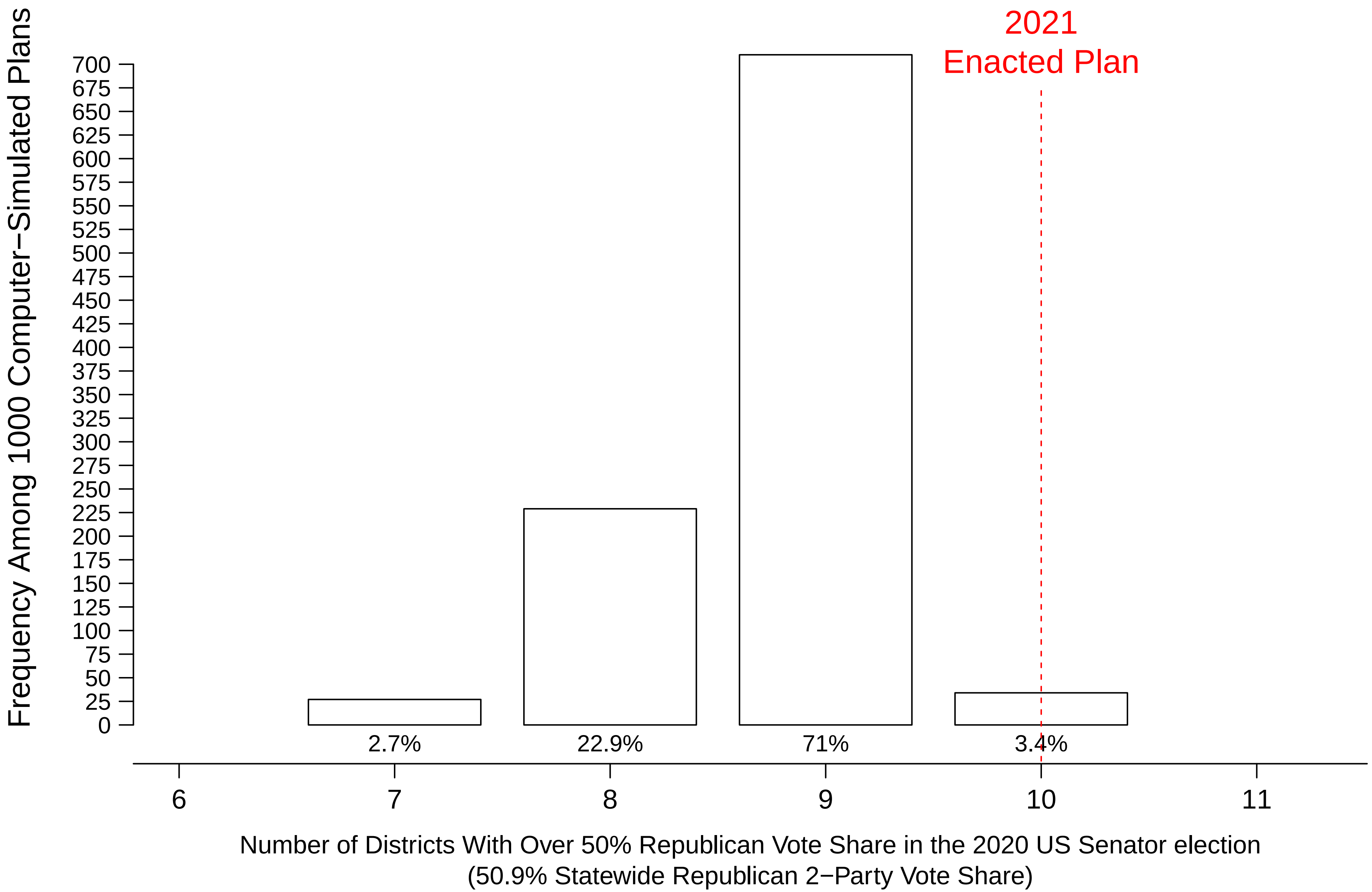


Figure B10: Comparisons of Enacted SB 740 Plan to 1,000 Computer−Simulated Plans
Number of Districts With Over 50% Republican Vote Share in the 2020 US Senator election
(50.9% Statewide Republican 2−Party Vote Share)



An analysis of North Carolina's legislative districtings: Expert Report

Wesley Pegden

December 23, 2021

Contents

1 Qualifications

I am an associate professor in the department of Mathematical Sciences at Carnegie Mellon University, where I have been a member of the faculty since 2013. I received my Ph.D. in Mathematics from Rutgers University in 2010 under the supervision of József Beck, and I am an expert on stochastic processes and discrete probability. My research has been funded by the National Science Foundation and the Sloan Foundation. A current CV with a list of publications is attached as Exhibit A. A list of my publications with links to online manuscripts is also available at my website at `http://math.cmu.edu/~wes`.

I am an expert on the use of Markov Chains for the rigorous analysis of gerrymandering, and have published papers[1] developing techniques for this application in *Proceedings of the National Academy of Sciences* and *Statistics and Public Policy*, hereafter referred to by [CFP] and [CFMP], respectively.

I testified as an expert witness in the *League of Women Voters of Pennsylvania v. Commonwealth of Pennsylvania* case in which the 2011 Congressional districting was found to be an unconstitutional partisan gerrymander, and as well as the *Common Cause v. Lewis* case in North Carolina. I previously served as a member of the bipartisan Pennsylvania Redistricting Reform Commission under appointment by the governor. I am being compensated at a rate of $325 per hour for my work on the current case.

2 Executive Summary

I was asked to analyze whether the proposed Congressional, state House, and state Senate districtings of North Carolina were drawn in a way which made extreme use of partisan considerations.

To conduct my analysis, I take the enacted plan as a starting point and make a sequence of many small random changes to the district boundaries. This methodology is intended to detect whether the district lines were carefully drawn to optimize partisan considerations; in particular, if the plans in question were not intentionally drawn to maximize partisan advantage, then making random changes should not significantly decrease the plan's partisan bias.

Specifically, my method begins with the enacted plan and uses a Markov Chain—a sequence of random changes—to generate trillions of comparison districtings against which I compare the enacted plans. These comparison districtings are generated by making a sequence of small random changes to the enacted plans themselves, and preserve districting criteria such as population deviation, compactness, and splitting of counties, municipalities, and precincts, among other criteria (a complete list is given in Section 4.3.1).

The analysis I conduct of the enacted plan using this data has two levels. The first level of my analysis consists simply of comparing the partisan properties of the enacted plans to the large sets of comparison maps produced by my Markov Chain, and I report how unusual the enacted plans are with respect to their partisan properties, against this comparison set. **Quantitatively, for the enacted Congressional, House, and Senate plans, I find that they have a greater partisan bias than 99.99999%, 99.99999%, and 99.97% of the trillions of districtings produced by my algorithm, respectively.**

The next level of my analysis uses the mathematical theorems I have developed with my co-authors in [CFP] and [CFMP] to translate the results of the above comparison into a statement about how the enacted plans compare against *all* other districtings of North Carolina satisfying the districting criteria I consider in this report. In other words, the theorem that I use in the second level analysis allows me to compare the enacted plan against not only the trillions of plans that my simulations produce through making small random changes, but also against all other possible districtings of North Carolina satisfying the districting criteria I consider.

Consider the following: when I make a sequence of small random changes to an enacted plan as described above, this can be viewed as a test of whether the partisan bias in the current districting is fragile, in the sense that it evaporates when the boundary lines of the district are perturbed. As discussed in Section B, our

[1]

- M. Chikina, A. Frieze, W. Pegden. Assessing significance in a Markov Chain without mixing, in *Proceedings of the National Academy of Sciences* **114** (2017) 2860–2864
- M. Chikina, A. Frieze, J. Mattingly, W. Pegden. Separating effect from significance in Markov chain tests, in *Statistics and Public Policy* **7** (2020) 101–114.

theorems in [CFP] and [CFMP] establish that it is mathematically impossible for the political geography of a state to cause such a result. That is: while political geography might conceivably interact with districting criteria to create a situation where typical districtings of a state are biased in favor of one party, it is mathematically impossible for the political geography of a state to interact with districting criteria to create a situation where typical districtings of a state appear to be *optimized for partisan bias*, in the sense that their bias is fragile and evaporates when small random changes are made. This allows us to rigorously demonstrate that a districting is optimized for partisanship, and is an outlier among *all* districtings of a state satisfying the criteria I consider, with respect to this property.

Quantitatively, my second-level analysis establishes that the enacted plans here are more optimized for partisanship than 99.9999% of all possible Congressional districtings satisfying the districting criteria I account for in my analysis, more than 99.9999% of all possible House districtings satisfying those criteria, and more than 99.9% of all Senate districtings satisfying those criteria. Thus the chance of drawing districtings that are as optimized with respect to their partisan properties as the current House and Senate districtings of North Carolina *without* using partisan considerations is exceedingly small.

In particular, I find that **North Carolina's Congressional, House and Senate districtings were drawn in a way which made extreme use of partisan considerations, a finding which is mathematically impossible to be caused by the interaction of political geography and the districting criteria I consider.**

3 Topic of Expert Report

The question motivating my analysis in this case is: "*How significant a role did partisanship play in the drawing of the enacted Congressional, House and Senate districts of North Carolina?*"

My analysis approaches this question in a rigorous and quantifiable way. In short, I identify how much of an outlier the present districting lines are, with respect to how carefully they are drawn to line up with partisan goals. *A priori*, it is possible that political geography might conceivably interact with districting criteria to bias typical districtings for one party or another. But my analysis provides a rigorous quantifiable answer to the question of the extent to which partisanship was used in the districting process, whose validity *does not depend on the political geography of North Carolina.*

Apart from whole-state analyses of the enacted Congressional, House and Senate plans of North Carolina, I was also asked to conduct separate analyses of the following specific House and Senate clusters:
House:

- Mecklenburg
- Wake
- Forsyth-Stokes
- Guilford
- Buncombe
- Pitt
- Duplin-Wayne
- Alamance
- Durham-Person
- Cumberland
- Cabarrus-Davie-Rowan-Yadkin
- Brunswick-New Hanover

Senate:

- Iredell-Mecklenburg
- Granville-Wake
- Forsyth-Stokes
- Cumberland-Moore
- Guilford-Rockingham

4 Quantifying intentional and excessive use of partisanship

My approach begins with a simple idea: I make small random changes to the boundaries of enacted plans (while maintaining districting criteria) and study the effect this has on the partisan bias of the map. More specifically:

- I begin from the enacted plan I am evaluating, and then repeatedly:
    1. Randomly select a geographical unit (e.g., a voting precinct) on the boundary of two districts, and check: if I change which district this geographic unit belongs to, will the resulting districting still satisfy the districting criteria laid out in Section 4.3.1? If so, I make the change.
    2. Using historical voting data as a proxy for partisan voting patterns, evaluate the partisanship of the districting resulting from the previous step.
- These two steps are repeated many times, resulting in a sequence of districtings, each produced by a small random change to the districting preceding it, with the enacted map I am evaluating as the starting point for the sequence.

This procedure is implemented as a computer algorithm which carries out trillions of the above steps for a districting map.

4.1 First level analysis

The first level of my analysis simply uses the above procedure to generate a large set of comparison districtings against which one can compare the enacted plan. For example, for the Congressional districting, I conducted 32 runs of the above procedure. A “run” in this context consists of a single consecutive sequence of small random changes to the enacted plan, producing a set of comparison districtings. For example, for the Congressional districting, each run consisted of carrying out Steps 1 and 2 in the procedure above $2^{40} \approx 1$ trillion times. As discussed in later sections, these comparison maps adhere to districting criteria in ways that constrain them to be similar in several respects to the enacted map being evaluated. For example, the comparison districtings will preserve the same counties and municipalities preserved by the enacted plan.

In total for this districting, I conducted 32 such runs. I then show the results of these runs in a table, like this:

Congressional districting

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999947%	9	99.9999909%	17	99.9999955%	25	99.999995%
2	99.999968%	10	99.99999966%	18	99.9999973%	26	99.9999961%
3	99.9999988%	11	99.9999943%	19	99.99999972%	27	99.99999977%
4	99.99999931%	12	99.999988%	20	99.9999999981%	28	99.99999979%
5	99.99999999927%	13	99.999988%	21	99.9999999962%	29	99.9999981%
6	99.9999959%	14	99.9999987%	22	99.99999919%	30	99.9999941%
7	99.99999984%	15	99.999996%	23	99.9999908%	31	99.99999901%
8	99.9999999947%	16	99.999985%	24	99.999981%	32	99.9999969%

For example, we see here that in the first run, 99.9999947% of the comparison districtings exhibited less Republican bias than the enacted Congressional districting. Moreover, in *every* run, more than 99.999968% of the comparison districtings exhibited less Republican bias than the enacted plan.

The first level of my analysis simply reports this comparison of the enacted map to the comparison districtings produced in these runs. Even without applying the mathematical theorems we have developed in [CFP] and [CFMP], this gives strong, intuitively clear evidence that the district lines were intentionally drawn to optimize partisan advantage in the enacted plan: if the districting had not been drawn to carefully optimize its partisan bias, we would expect naturally that making small random changes to the districting would not have such a dramatic and consistent partisan effect.

4.2 Second level analysis

In the first level of my analysis, I compare enacted plans to comparison districtings produced by my algorithm (which makes random changes to the existing map while preserving districting criteria).

The next level of my analysis goes further than this, and enables a rigorous comparison to *all* alternative districtings of North Carolina satisfying the districting criteria I consider here. It does this by comparing how "optimized for partisanship" an evaluated plan is to how "optimized for partisanship" alternative plans are.

4.2.1 Defining "optimized for partisanship"

Roughly speaking, when I say that a districting is *optimized for partisanship*, I mean that its partisan characteristics are highly sensitive to small random changes to the boundary lines.

Formally, when I say that a districting is *optimized for partisanship* in this report, I mean that there is a high probability that when I make small random changes to the districting, its partisanship will be an extreme outlier among the comparison maps produced by the small random changes.

The yardstick I use to measure this property of a given map is the *ε-fragility* of a map. Given a small threshold ε—for example, 00.000031%, for the analysis of the Congressional districting given above—I can ask: what is the probability that when I make a sequence of small random changes to the map, the map will be in the most extreme ε fraction of maps encountered in the sequence of random changes? The probability of this occurrence is the ε-fragility of the map, and it is this probability that I use to quantify how optimized for partisanship a map appears to be.

In other words, **one districting is considered more optimized for partisanship than another if it is more likely to have its partisan bias consistently reduced when making a random sequence of small changes to its boundary lines.**

4.2.2 Comparing an enacted plan to the set of *all* alternatives

My analysis enables a rigorous comparison of an enacted plan to **all possible districting plans of the state** satisfying the districting criteria I consider, with respect to how optimized for partisanship the districtings are. I can report the maximum fraction of all such possible redistricting plans which could appear as optimized for partisanship as the enacted plan, in the sense of the test described above. For example, I report that the enacted Congressional districting of North Carolina is among the most optimized-for-partisanship 00.000031% of **all possible House districtings of North Carolina** satisfying the districting criteria I consider here, as measured by it's ε-fragility.

My method produces a rigorous p-value (statistical significance level) which precisely captures the confidence one can have in the findings of my "second level" analyses. In particular, for my statewide analyses, my second-level claims are all valid at a statistical significance of $p = .002$. This means that the probability that I would report an incorrect number (for example, claiming that a districting is among the most optimized-for-partisanship 00.01% of all districtings, when in fact it is merely among the most 00.015% optimized-for-partisanship) is at most 00.2%. To put this in context, clinical trials seeking regulatory approval for new medications frequently target a significance level of $p = .05$ (5%), a looser standard of statistical significance than I hold myself to in this report.

4.2.3 Some intuition for why this is possible

It may seem remarkable that I can make a rigorous quantifiable comparison to *all* possible districtings, without actually generating all such districtings; this is the role of our theorems from [CFP] and [CFMP], which have simple proofs which have been verified by the mathematical community.

To give some nontechnical intuition for why this kind of analysis is possible, these results roughly work by showing that in a very general sense, it is not possible for an appreciable fraction of districtings of a state to appear optimized for partisanship in the sense defined in Section 4.2.1. In other words, it is *mathematically impossible* for any state, with any political geography of voting preferences and any choice of districting criteria, to have the property that a significant fraction of the possible districtings of the state satisfying the chosen districting criteria appear optimized for partisanship (as measured by their ε-fragility).

4.3 Implementation details

Here I specify the particulars of the random changes my algorithm makes to a map, my implementation of districting criteria, and my method of comparing the partisanship of a districting to that of districtings encountered on the sequence of random changes.

4.3.1 Districting criteria

All comparison maps produced by my algorithm are required to satisfy the following districting criteria:

(a) **Contiguity**: I require comparison districtings to contain only contiguous districts.

(b) **Compact districts**: I require comparison districtings to be at least as compact as the enacted plan being evaluated, up to an error of 5%. Districting compactness is quantified by taking the average, over each district, of the ratio of the perimeter squared to the area (Polsby-Popper reciprocal).

(c) **County clusters**: For the House and Senate plans, I require comparison maps to respect the same county clustering as used by the enacted House and Senate plans.

(d) **Country traversals**: I require comparison districts to not contain more county traversals than the enacted plan. Additionally, I constrain the total length of all district boundary which is not also county boundary to be at most that of the enacted map, up to an error of 5%.

(e) **Municipality preservation:** There are at most as many municipal splits as in the enacted plan.

(f) **VTD preservation:** The total number of VTD splits in comparison districtings must not exceed the total number of VTD splits in the enacted plan.

(g) **Incumbency protection:** Any incumbent who, in the enacted plan, is not paired with any other incumbent must remain unpaired in the comparison districtings.

(h) **Population deviation**: For House and Senate districtings, I require comparison districtings to have district populations within 5% of the ideal district population. For the Congressional districting, I use a 2% threshold in my main analysis. I discuss robustness of my Congressional analysis to differences in population criteria in Section 5.0.2. Population is measured by the 2020 decennial Census.

4.3.2 A conservative application of the criteria

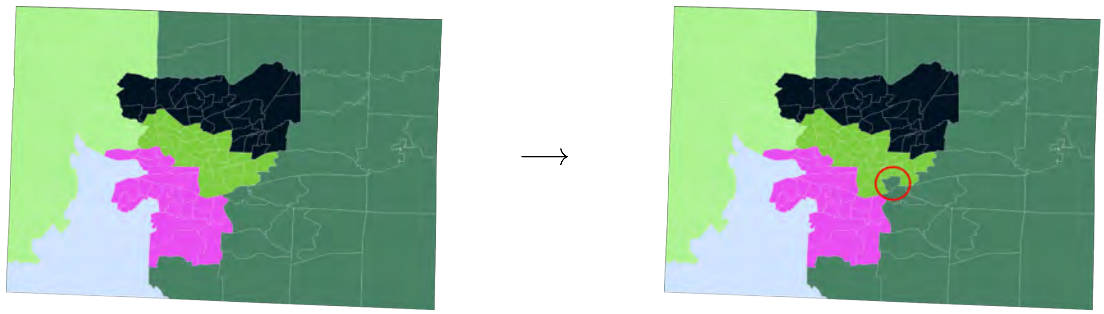
It is important to note that my analysis is designed to avoid second-guessing the mapmakers' choices in how they implemented the districting criteria. In particular, while it is reasonable to ask whether the mapmakers could have drawn districtings which adhered better to nonpartisan criteria (more compact, preserving more municipalities, *etc*), my approach is different, and much more conservative.

In particular, my analysis asks the question: even if we accept that the mapmakers have made appropriate choices with respect to nonpartisan criteria such as compactness, population deviation, municipality preservation, incumbency protection, and so on, does their plan nevertheless stand out with respect to its *partisan* qualities?

Note that, for example, I choose my compactness threshold within 5% of value of the enacted map. And with respect to incumbents, I do not try to protect as many incumbents as are protected in the enacted map, but exactly the same incumbents as protected by the mapmakers. With respect to municipality preservation, I am not trying to answer the question: "if the mapmakers had tried to preserve more municipalities, would this have resulted in a more favorable districting for Democrats?" Instead, I am asking, among all alternative districtings of North Carolina with the same *nonpartisan* characteristics as the enacted map—their compactness, how many municipalities they preserve, etc.—whether the enacted plan is an extreme outlier with respect to the extent to which it is optimized for partisanship.

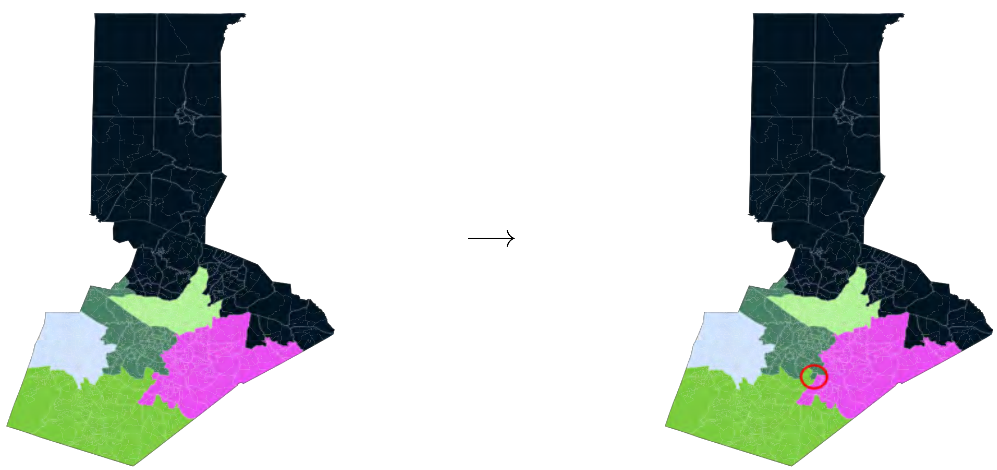
5 Random Changes

As described earlier, my method involves making small random changes to a map. For example, depicted here is a small random change made to the enacted House districting within the Guilford county cluster:



The geographical units used for these small random changes in this district are voting tabulation districts—VTDs. In particular, at each step of the sequence of random changes for the house districting within Guilford county, I move a randomly VTD that is at the boundary of two districts from one of those districts to the other (unless it would violate the constraints laid out in Section 4.3.1.

For House and Senate clusters that split VTDs, my analysis operates below the VTD level. In particular, my procedure in these case manipulates sub-VTD units (referred to hereafter as geounits). These are compact combinations of Census Blocks which respect VTD and district lines and contain on average approximately 1000 people. In particular, there are an average of around 4 geounits per VTD. In the following example from the Granville-Wake senate districting, we see an example of a random change at the geounit level:



The thick white lines here indicate current VTD boundaries. A geounit within an already broken VTD has changed district membership. When analyzing any districting at the below-VTD level, my algorithm constrains comparison maps to split at most as many VTDs as the enacted map.

For my whole-state analyses, my algorithm operates at the VTD level. This means that the algorithm is prohibited from splitting any VTD's not split in the enacted map. In Section C, I include runs where the Congressional districting is analyzed at the geounit level.

In each run, my chain generates comparison maps from a given enacted plan by making billions or trillions of these small changes to the enacted plan, while preserving districting criteria in specific ways chosen by the mapmakers, as discussed in Section 4.3.2.

These random changes can be either be made one-at-a-time or with several steps made simultaneously; the latter allows comparison maps to be generated when any single move would lead to a violation of the constraints laid out in Section 4.3.1 (e.g., because population would become too imbalanced), but combinations of moves can be found which would preserve all these criteria. My mathematical analysis applies equally well when using these "multi-move swaps" and I could analyze all clusters in this way if I wanted to, but the algorithm is slower in this mode. In general, in the interest of efficiency, I conduct all state-level analysis with single-move swaps, cluster-level VTD-level runs with multi-move swaps, and cluster-level geounit runs with single-move swaps, but additionally use multi-move swaps any time it enables the algorithm to generate more comparison maps.

Technical details of my implementation of these multi-moves are found in Appendix A. A related implementation detail for VTD splitting is also discussed there.

5.0.1 The *seats expected* metric for comparing districtings

As described in Section 4.2.1, my definition of optimized for partisanship involves comparing the partisanship of an enacted plan to the partisanship of comparison districtings produced from it by a sequence of random changes. Here I describe the *seats expected* metric of partisanship I use for this comparison throughout this report. In short, the seats expected metric for the districting is **the average number of seats Democrats would expect to win in the districting**, based on a uniform swing model with the historical voting data I use.

The *uniform swing* is a simple model frequently used to make predictions about the number of seats a party might win in an election, based on partisan voting data. Suppose, for example, that given data from the last North Carolina House election, we would like to predict how many seats Democrats will win in an upcoming House election (with the same districting), assuming that at a statewide level, we expect them to outperform by 1.5 percentage points their results from the last election.

A uniform swing would simply add 1.5 percentage points to Democrat performance in every district in data from the last election, and then evaluate how many seats would be won with these shifted voting outcomes.

When I am evaluating the partisanship of a comparison districting (to compare it to the enacted plan), I am interested in the number of seats we expect Democrats might win in the districting, given unknown shifts in partisan support. In particular, the metric I use is:

> How many seats, on average, would Democrats win in the given districting, if a random uniform swing is applied to the historical voting data being used?

As an example, let us consider the enacted Congressional plan, using the 2020 Attorney General election as a proxy for partisan voting patterns. Using these results as a direct proxy for future voting patterns, the enacted map would produce a 4:10 split of Democrat:Republican seats. If the Democrat vote share was increased by 1.68% in every district, the split would change to 5:9, and if it was increased by 3.05%, the split would rise to 6:8.

The random choice of my uniform swing is made from a normal distribution whose standard deviation is 4 percentage points, which is roughly the standard deviation of the swing in the past five North Carolina gubernatorial elections. Figure 1 visualizes the probabilities that this distribution assigns to the various seat splits which would arise from the enacted Congressional map under uniform swings of the 2020 Attorney General election:

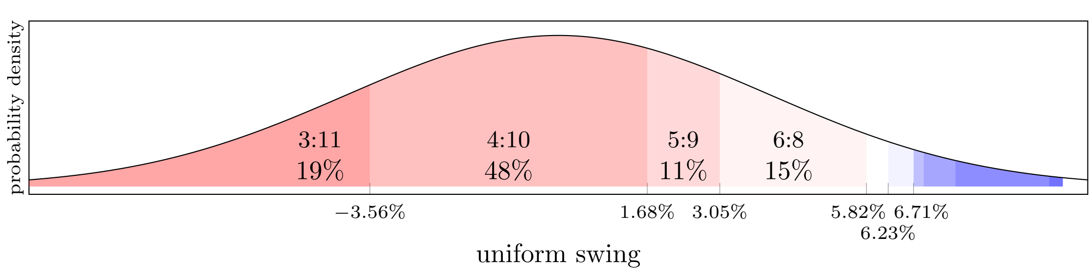


Figure 1: A normally distributed uniform swing applied to the enacted Congressional districting.

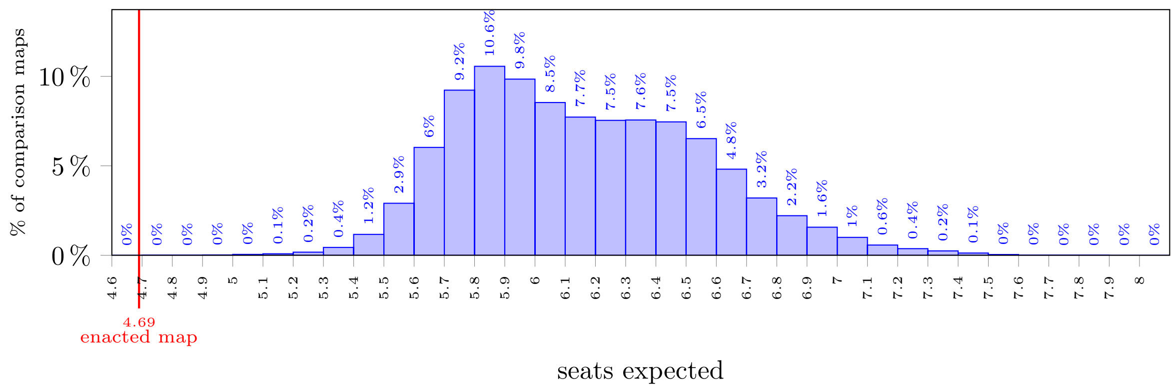
In particular, we can list the probability of any number of Democratic seats for the enacted Congressional plan according to this uniform swing model using the 2020 Attorney General race:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0.0%	0.0%	0.0%	19%	48%	11%	15%	1.3%	1.3%	0.1%	0.5%	1.2%	2.0%	0.1%	0.9%

The weighted average of these seat outcomes is computed as

$$.19 \times 3 + .48 \times 4 + .11 \times 5 + .15 \times 6 + .013 \times +.013 \times 8 + .001 \times 9 + .005 \times 10 + .012 \times 11 + .02 \times 12 + .001 \times 13 + .009 \times 14 = 4.69. \quad (1)$$

This "seats expected" number for the Congressional plan shows up in our analysis page for the Congressional districting (page 13), in a histogram we reproduce here for the purpose of illustration:



It is important to note that **my method does not evaluate the fairness of a districting by whether it produces a "small" or "large" number of seats for one party, or whether the uniform swing score calculated in this way is lower or higher than would be expected in a system of proportional representation**. Instead, this score is merely a metric used to **compare** one map to another. The only way these scores are used in my method is to evaluate which of two maps may be more advantageous to a particular political party, and when I find that a districting made extreme use of partisan consideration, it means that the enacted map is extreme outlier with respect to how optimized for partisanship it is **compared to the set of alternative comparison districtings of North Carolina** satisfying the districting criteria I impose.

5.0.2 Note on Population Deviation

My method does not simulate the results of hypothetical elections at the per-person level, and I do not enforce 1-person population deviation on Congressional districts. Instead, I use a cutoff 2%, as described above. I verify that the distinction between 1-person and 2% population deviation do not drive the results of my analysis in two ways.

First, in Section C, I show a run of my whole Congressional analysis exactly the same way but with a 1% population deviation constraint and obtain similar results. I also show a geounit-level analysis which operates at just 0.5% population deviation and still finds the enacted plan to be an extreme outlier.

Second, I analyze a coarse VTD-level version of the enacted map (itself with nearly 2% population deviation), and show that even this coarse version of the enacted map is an extreme outlier with respect to partisan bias, before small changes are made to it to produce the enacted 1-person-deviation map. This demonstrates that the coarse VTD-level "blueprint" for the map is an extreme outlier, optimized for partisan considerations, among alternative VTD-level maps with similar population deviation, even before the small changes used to achieve 1-person deviation are accounted for.

Finally, I note that by design, the seats-expected metric I use is not sensitive to the kinds of small changes that need to be made to districts to equalize population. This can already be seen by comparing the seats-expected metric for the enacted Congressional plan to the "VTD-level blueprint" version we analyze in Section C.8. As calculated above, the enacted map, with 1-person popluation deviation, scores 4.69 on the seats expected metric. The whole-VTD level blueprint, which has 1.8% population deviation, scores 4.70 by the same metric, as seen in the plot in C.8. This difference of 0.01 is much smaller than the sizes of differences in the seats-expected metric that are driving the results in my report.

5.1 A note on comparing results

For my cluster-by-cluster analysis of the House and Senate districtings, we will see that even among clusters for which we find that the enacted plan is an extreme outlier, there is quite a bit of variation from cluster to cluster for how extreme an outlier we find the enacted plan to be.

For example, in our second-level analysis of the Guilford county house districting, we find that it is among the most optimized-for-partisanship 00.000089% of all alternative districtings of the county satisfying our districting criteria, while for the Mecklenburg county districting, we find that it is among the most optimized-for-partisanship 5% of districtings.

Because it is tempting to compare results from cluster to cluster, it is important to emphasize that the mathematical results we employ in these findings are one-directional. In particular, while they imply that the Mecklenburg cluster is among the most optimized-for-partisanship 5% of districtings, they do *not* imply that it could not also be among the most optimized-for-partisanship 00.000089%.

What we know from my analysis is that we have extreme statistical certainty that the Guilford cluster districting is among the most optimized-for-partisanship 00.000089% of all districtings satisfying the criteria I consider, and we have extreme statistical certainty that the Mecklenburg cluster is among the most optimized-for-partisanship 5% of all districtings satisfying the criteria. The Mecklenburg cluster may be even more of an outlier, but my analysis does not address this latter question in either direction.

It should also be noted that it is natural to expect that my very conservative application of the districting criteria (discussed in Section 4.3.2) will affect some clusters more than others. In some clusters (e.g., Duplin/Wayne), it even prevents any comparison districtings from being generated by my algorithm at all. Of course, this should not seen as settling in either direction the question of whether the enacted map of the Duplin/Wayne cluster is gerrymandered.

6 Results of Analysis

The following pages show the results of my analysis for the enacted Congressional, state House, and state Senate districting plans.

Each page has the following components:

Comparison map examples

I show four maps in each case. The first map is the enacted map. The other three are examples of comparison maps used by by method. In each case, these maps are either the final map from runs 1, 2 and 3, or, from just the first run, the last map, the map from the halfway point of the run, and the run from the 25% point of the run.

Results

Under results I show a **table**, with an entry for each run conducted for the districting. The table shows the fraction of maps in that run that exhibited less partisan bias in favor of Republicans than the enacted map under evaluation. In particular, this is the fraction of maps for which the "seats expected" metric was higher than for the enacted map. For example, on the next page, we will see that in the first run, 99.9999947% of comparisons exhibited less partisan bias in favor of Republicans than the enacted plan.

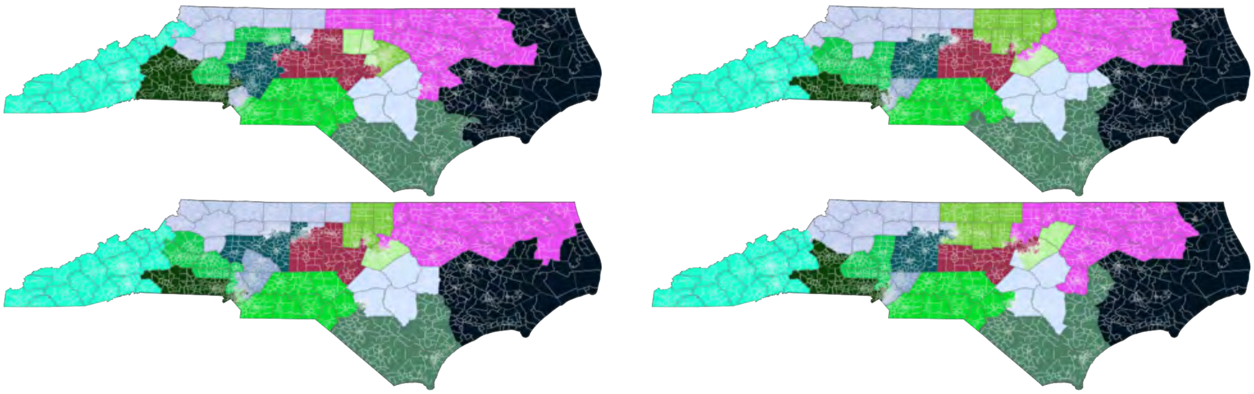
Below this table I show a **histogram** which plots the number of comparison maps whose "seats expected" value fell in various ranges. For example, on the next page, we see that 10.6% of comparison maps had a seats-expected value between 5.8 and 5.9. The histogram also shows the seats-expected value for the enacted map, which for the Congressional districting is 4.69. Note that the computation of this value 4.69 was illustrated earlier in Section 5.0.1. The same computation can be applied to every comparison map to build the histogram of resulting seats-expected values.

I present in each case a **First-level analysis**, which is simply a summary of the how the enacted map compares to the set of comparison districtings generated by my algorithm. For example, for the Congressional map, we will see that in every one of the 32 runs I conducted, 99.999968% of maps produced exhibited less partisan bias than the enacted map itself.

After this I present the **Second-level analysis**, which is a rigorous evaluation of how the enacted map compares to *all* alternative districtings of North Carolina satisfying the districting criteria I consider here. For example, for the Congressional districting as evaluated on the next page, we see that it is more optimized-for-partisanship than 99.999905% of *all* alternative districtings of North Carolina satisfying the criteria I impose as outlined in Section 4.3.1.

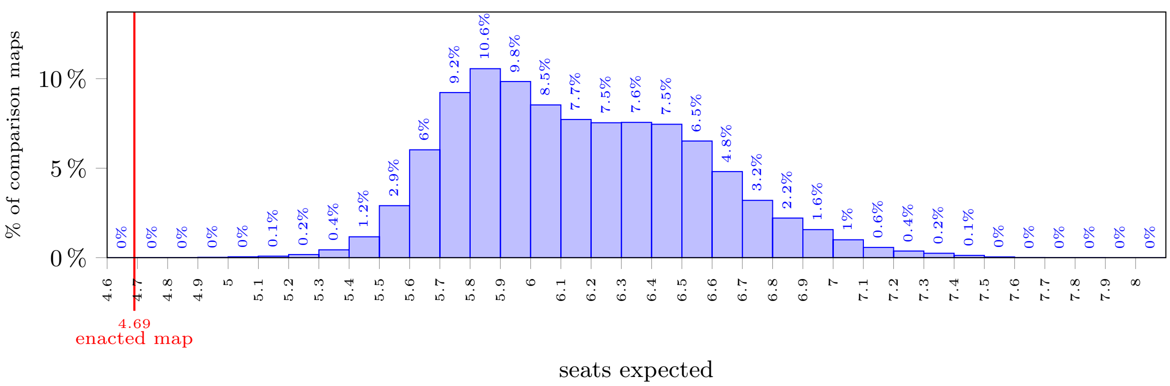
6.1 Congressional districting

6.1.1 Comparison map examples



6.1.2 Results

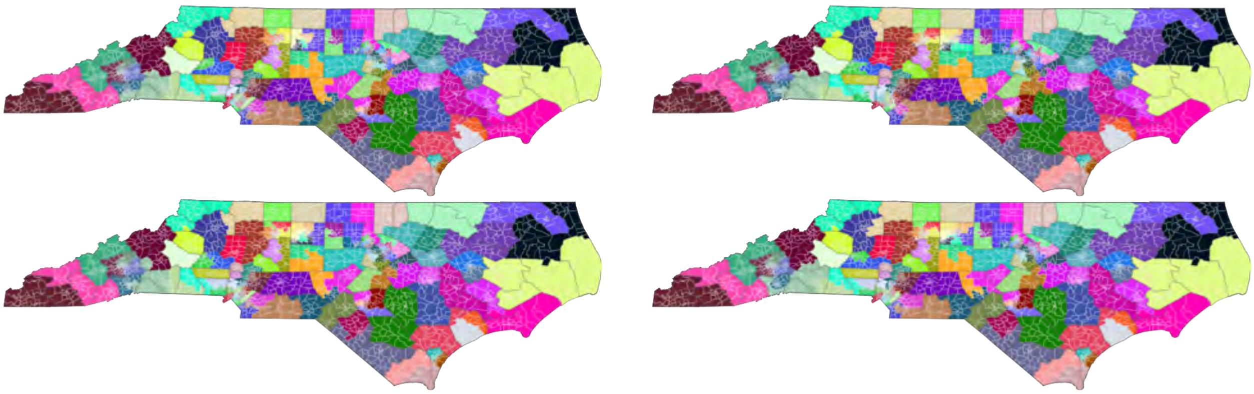
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999947%	9	99.9999909%	17	99.9999955%	25	99.999995%
2	99.999968%	10	99.99999966%	18	99.9999973%	26	99.9999961%
3	99.9999988%	11	99.9999943%	19	99.99999972%	27	99.99999977%
4	99.99999931%	12	99.999988%	20	99.9999999981%	28	99.99999979%
5	99.99999999927%	13	99.999988%	21	99.9999999962%	29	99.9999981%
6	99.9999959%	14	99.9999987%	22	99.99999919%	30	99.9999941%
7	99.99999984%	15	99.999996%	23	99.9999908%	31	99.99999901%
8	99.9999999947%	16	99.999985%	24	99.999981%	32	99.9999969%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.000031% of districtings (in other words, 99.999968% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted House districting is among the most optimized-for-partisanship 0.000094% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999905% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000031\%$.

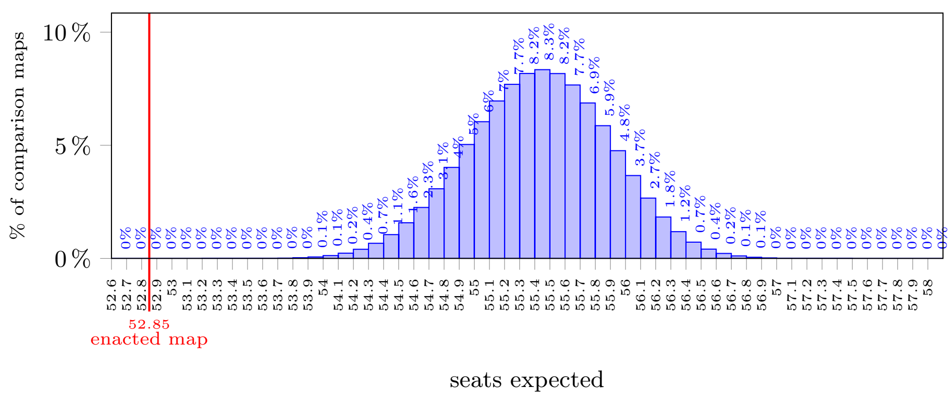
6.2 House districting

6.2.1 Comparison map examples



6.2.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999999985%	9	99.99999957%	17	99.9999989%	25	99.9999989%
2	99.99999942%	10	99.99999904%	18	99.99999966%	26	99.9999918%
3	99.99999997%	11	99.9999984%	19	99.99999982%	27	99.99999984%
4	99.9999969%	12	99.9999986%	20	99.9999986%	28	99.9999988%
5	99.9999975%	13	99.99999989%	21	99.9999935%	29	99.99999987%
6	99.9999999959%	14	99.99999996%	22	99.9999999967%	30	99.99999908%
7	99.999999985%	15	99.9999984%	23	99.9999975%	31	99.9999966%
8	99.999999951%	16	99.99999954%	24	99.999999939%	32	99.999999939%

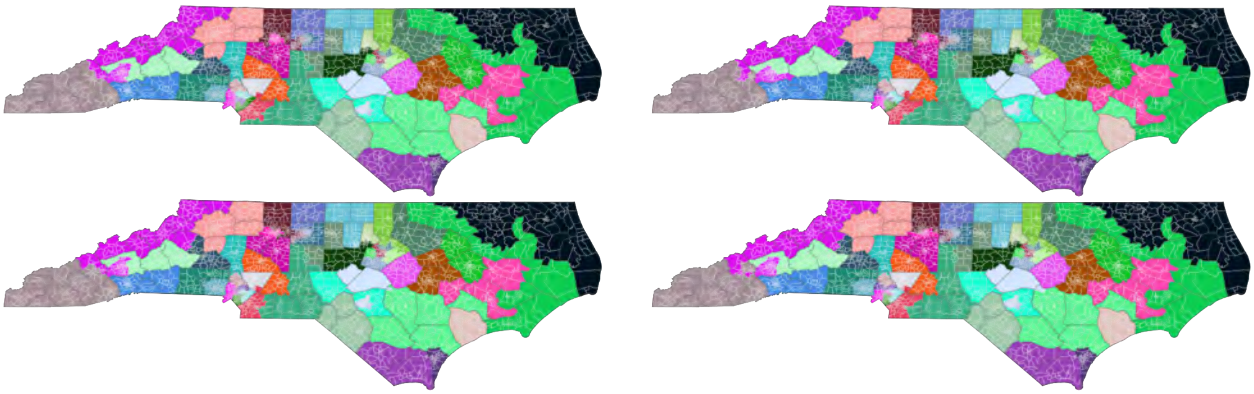


- **First level analysis:** In *every* run, the districting was in the most partisan 0.0000081% of districtings (in other words, 99.9999918% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000024% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999975% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0000081\%$.

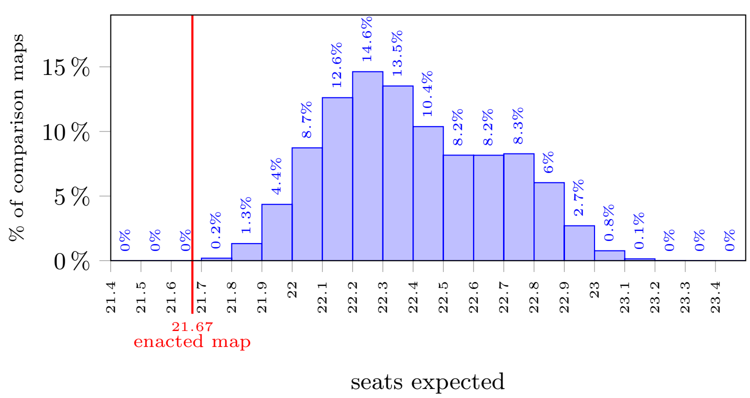
6.3 Senate districting

6.3.1 Comparison map examples



6.3.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.988%	9	99.9974%	17	99.9977%	25	99.998%
2	99.9988%	10	99.9958%	18	99.9987%	26	99.9948%
3	99.9938%	11	99.9985%	19	99.9988%	27	99.987%
4	99.9981%	12	99.9957%	20	99.978%	28	99.9988%
5	99.9929%	13	99.988%	21	99.9982%	29	99.9979%
6	99.9916%	14	99.989%	22	99.9978%	30	99.9981%
7	99.9957%	15	99.9974%	23	99.9976%	31	99.99914%
8	99.9973%	16	99.997%	24	99.9975%	32	99.9978%

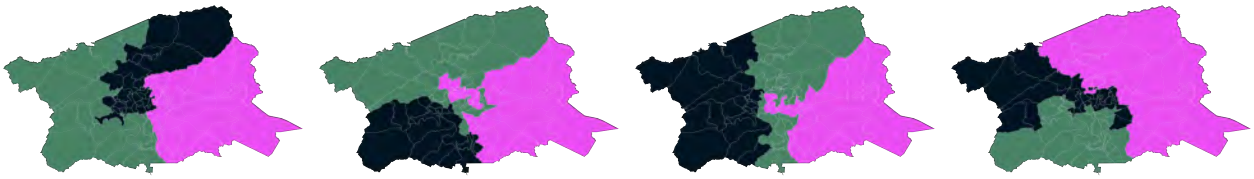


- **First level analysis:** In *every* run, the districting was in the most partisan 0.021% of districtings (in other words, 99.978% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.065% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.934% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.021\%$.

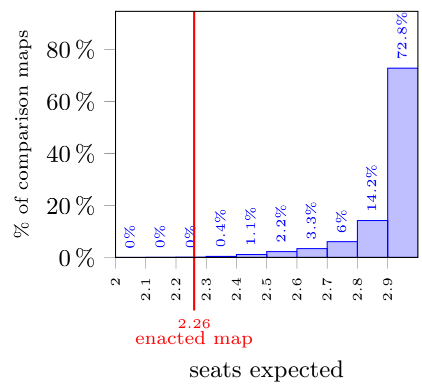
6.4 House Cluster: Buncombe

6.4.1 Comparison map examples



6.4.2 Results

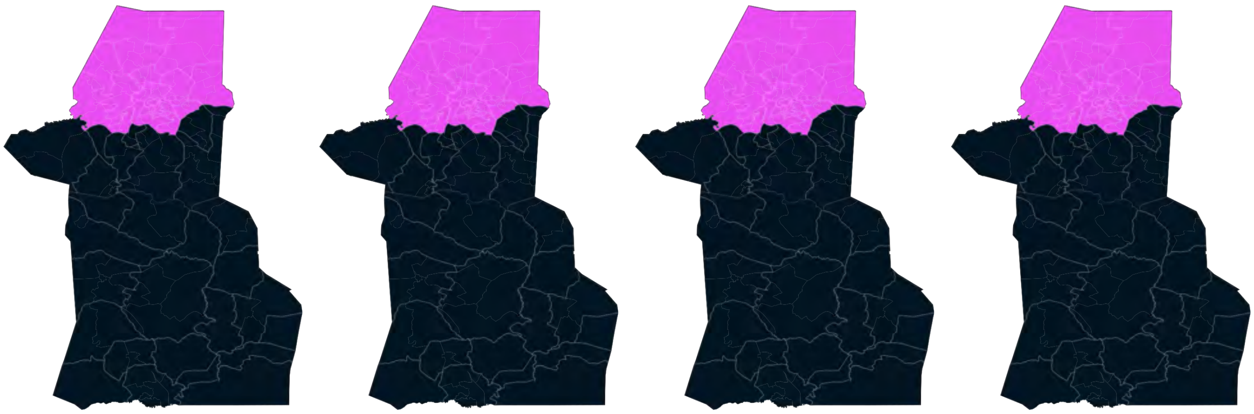
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.979%	9	99.979%	17	99.979%	25	99.98%
2	99.98%	10	99.98%	18	99.979%	26	99.979%
3	99.98%	11	99.98%	19	99.98%	27	99.979%
4	99.98%	12	99.98%	20	99.98%	28	99.98%
5	99.98%	13	99.98%	21	99.98%	29	99.98%
6	99.979%	14	99.98%	22	99.98%	30	99.98%
7	99.98%	15	99.98%	23	99.98%	31	99.979%
8	99.979%	16	99.98%	24	99.98%	32	99.979%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.020% of districtings (in other words, 99.979% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.061% of all alternative districtings satisfying my districting criteria (in other words, 99.938% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.020\%$.

6.5 House Cluster:Duplin/Wayne

6.5.1 Comparison map examples



6.5.2 Results

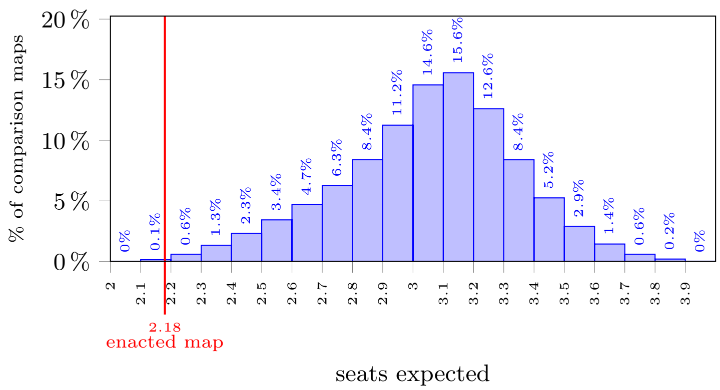
- For this cluster, my conservative approach (as discussed in Section 4.3.2) does not allow my algorithm to generate any comparison maps other than the map itself.

6.6 House Cluster: Forsyth-Stokes

6.6.1 Comparison map examples

6.6.2 Results

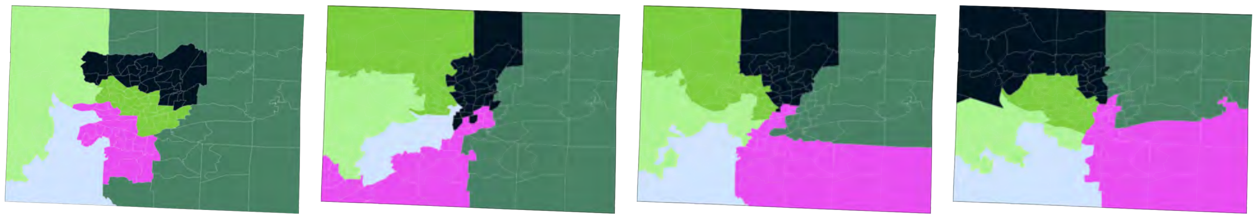
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.913%	9	99.912%	17	99.915%	25	99.914%
2	99.914%	10	99.914%	18	99.914%	26	99.913%
3	99.917%	11	99.912%	19	99.916%	27	99.914%
4	99.916%	12	99.912%	20	99.914%	28	99.912%
5	99.913%	13	99.914%	21	99.913%	29	99.915%
6	99.913%	14	99.914%	22	99.914%	30	99.914%
7	99.913%	15	99.912%	23	99.914%	31	99.917%
8	99.913%	16	99.916%	24	99.915%	32	99.915%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.087% of districtings (in other words, 99.912% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.26% of all alternative districtings satisfying my districting criteria (in other words, 99.73% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.087\%$.

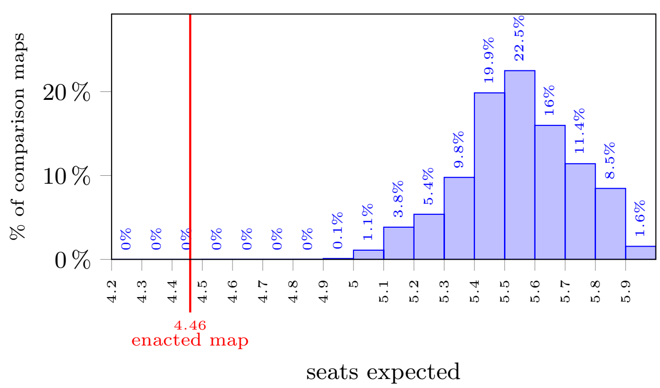
6.7 House Cluster: Guilford

6.7.1 Comparison map examples



6.7.2 Results

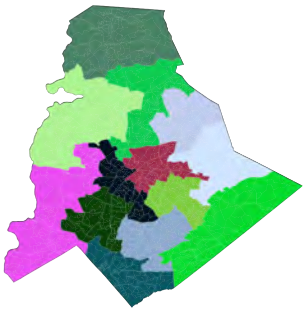
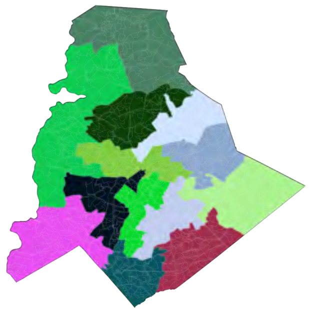
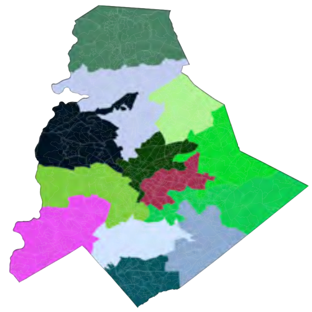
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999989%	9	99.999982%	17	99.999979%	25	99.999972%
2	99.999982%	10	99.999979%	18	99.999978%	26	99.999979%
3	99.999972%	11	99.999978%	19	99.999981%	27	99.999978%
4	99.999986%	12	99.999981%	20	99.999984%	28	99.999979%
5	99.999975%	13	99.999986%	21	99.999983%	29	99.999982%
6	99.999982%	14	99.99998%	22	99.999979%	30	99.999982%
7	99.999981%	15	99.99997%	23	99.999983%	31	99.999982%
8	99.999982%	16	99.999976%	24	99.999981%	32	99.999984%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.000029% of districtings (in other words, 99.99997% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000089% of all alternative districtings satisfying my districting criteria (in other words, 99.99991% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000029\%$.

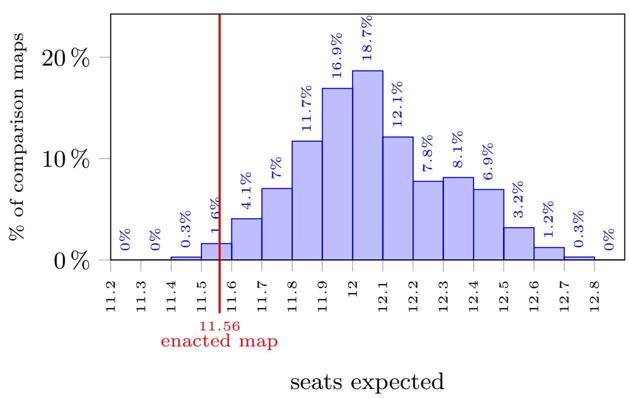
6.8 House Cluster: Mecklenburg

6.8.1 Comparison map examples

6.8.2 Results

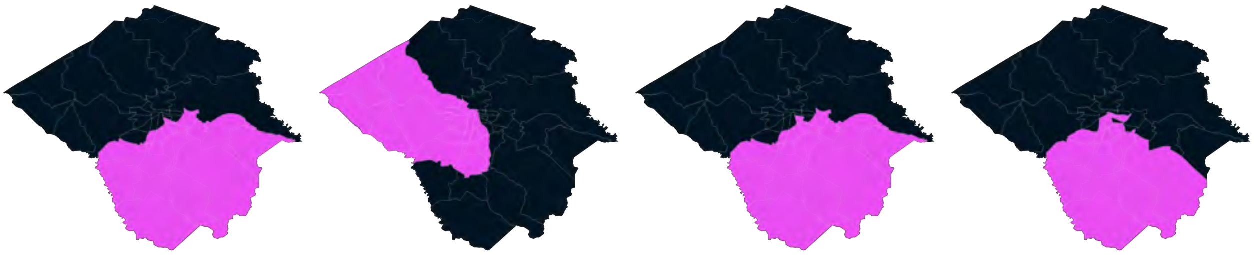
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	98.7%	9	98.6%	17	98.4%	25	98.9%
2	99.36%	10	99.15%	18	99.%	26	98.3%
3	98.7%	11	98.7%	19	98.4%	27	98.8%
4	99.14%	12	99.17%	20	99.17%	28	98.5%
5	98.4%	13	99.05%	21	98.8%	29	99.08%
6	99.33%	14	99.02%	22	98.9%	30	98.9%
7	98.5%	15	99.%	23	98.9%	31	99.12%
8	98.9%	16	99.17%	24	98.9%	32	99.2%



- **First level analysis:** In *every* run, the districting was in the most partisan 1.7% of districtings (in other words, 98.3% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 5.0% of all alternative districtings satisfying my districting criteria (in other words, 95.0% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 1.7\%$.

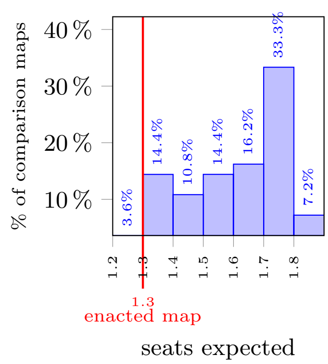
6.9 House Cluster: Pitt

6.9.1 Comparison map examples



6.9.2 Results

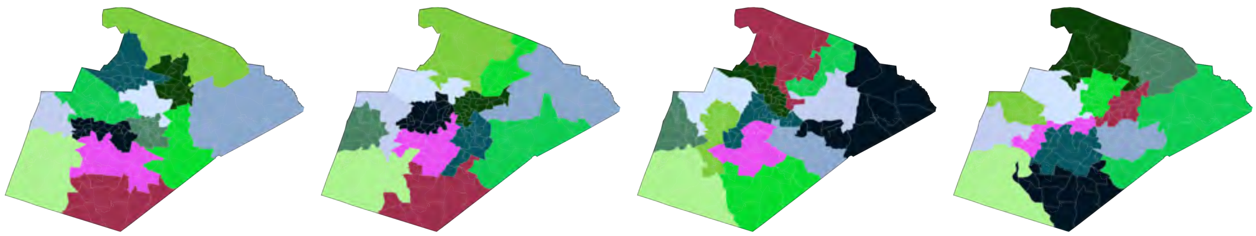
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	96.3%	9	96.4%	17	96.3%	25	96.4%
2	96.3%	10	96.3%	18	96.3%	26	96.3%
3	96.4%	11	96.4%	19	96.3%	27	96.4%
4	96.4%	12	96.4%	20	96.3%	28	96.3%
5	96.4%	13	96.4%	21	96.3%	29	96.4%
6	96.3%	14	96.3%	22	96.4%	30	96.3%
7	96.3%	15	96.3%	23	96.4%	31	96.4%
8	96.3%	16	96.4%	24	96.4%	32	96.4%



- **First level analysis:** In *every* run, the districting was in the most partisan 3.6% of districtings (in other words, 96.3% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 11% of all alternative districtings satisfying my districting criteria (in other words, 89.1% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 3.6\%$.

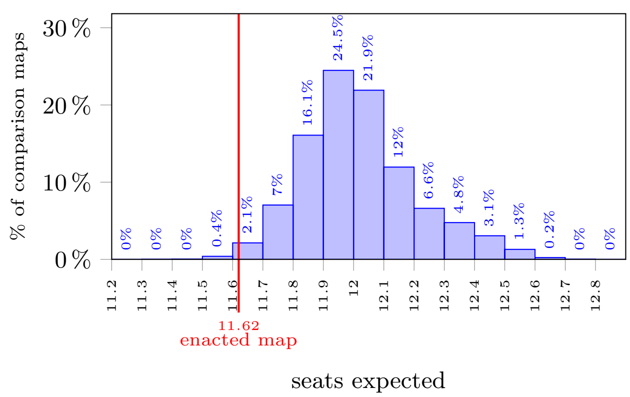
6.10 House Cluster: Wake

6.10.1 Comparison map examples



6.10.2 Results

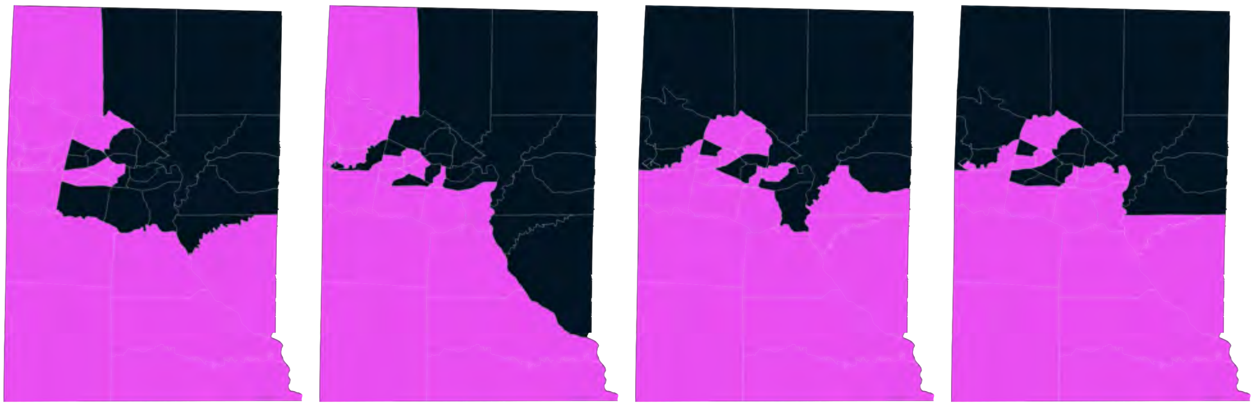
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.38%	9	99.34%	17	99.37%	25	99.35%
2	99.33%	10	99.35%	18	99.36%	26	99.36%
3	99.34%	11	99.33%	19	99.33%	27	99.34%
4	99.32%	12	99.34%	20	99.35%	28	99.33%
5	99.35%	13	99.34%	21	99.33%	29	99.35%
6	99.33%	14	99.27%	22	99.31%	30	99.36%
7	99.34%	15	99.34%	23	99.32%	31	99.36%
8	99.34%	16	99.36%	24	99.35%	32	99.35%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.72% of districtings (in other words, 99.27% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 2.2% of all alternative districtings satisfying my districting criteria (in other words, 97.8% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.72\%$.

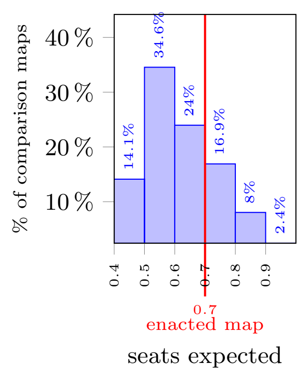
6.11 House Cluster: Alamance

6.11.1 Comparison map examples



6.11.2 Results

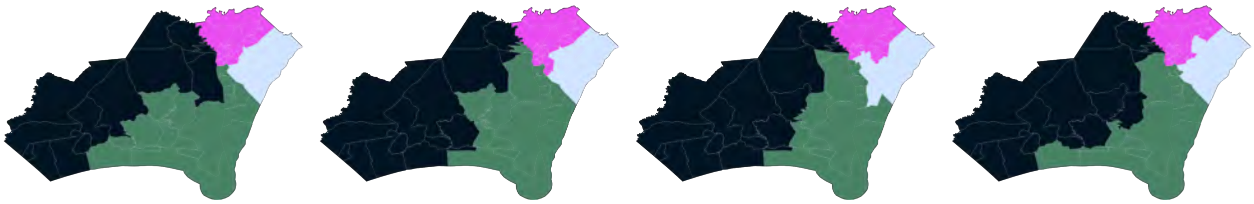
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	26.3%	9	26.4%	17	26.3%	25	26.4%
2	26.3%	10	26.3%	18	26.4%	26	26.3%
3	26.3%	11	26.3%	19	26.3%	27	26.3%
4	26.4%	12	26.3%	20	26.3%	28	26.3%
5	26.4%	13	26.4%	21	26.4%	29	26.3%
6	26.3%	14	26.3%	22	26.4%	30	26.4%
7	26.4%	15	26.3%	23	26.3%	31	26.3%
8	26.4%	16	26.4%	24	26.4%	32	26.4%



- **First level analysis:** In *every* run, the districting was in the most partisan 74% of districtings (in other words, 26.3% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

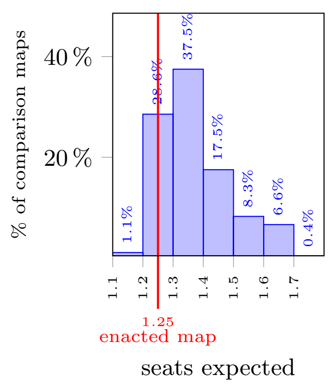
6.12 House Cluster: Brunswick/New Hanover

6.12.1 Comparison map examples



6.12.2 Results

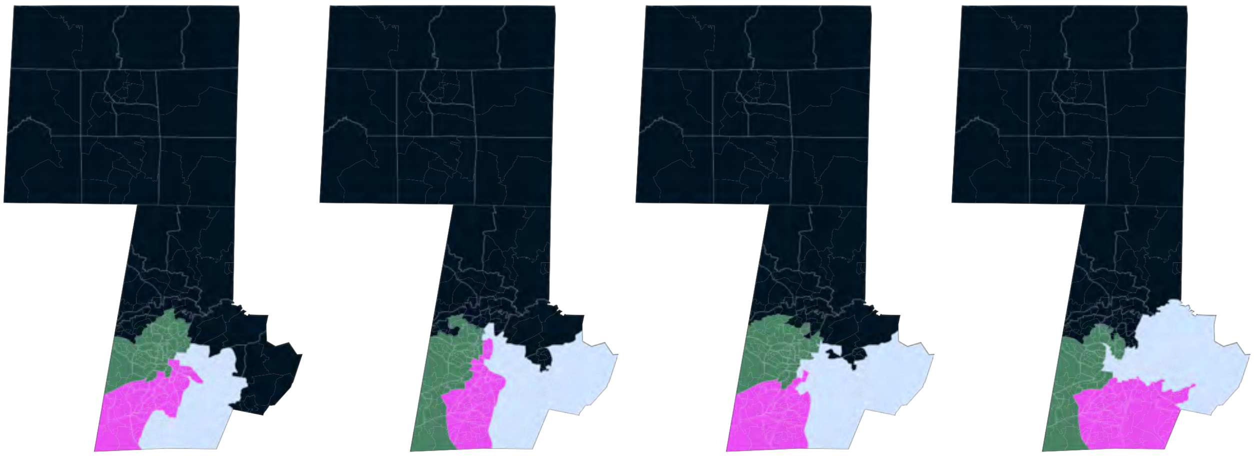
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	89.4%	9	89.5%	17	89.5%	25	89.5%
2	89.4%	10	89.5%	18	89.4%	26	89.5%
3	89.5%	11	89.5%	19	89.5%	27	89.4%
4	89.4%	12	89.4%	20	89.4%	28	89.5%
5	89.4%	13	89.5%	21	89.5%	29	89.5%
6	89.5%	14	89.6%	22	89.5%	30	89.4%
7	89.4%	15	89.5%	23	89.5%	31	89.5%
8	89.5%	16	89.4%	24	89.4%	32	89.5%



- **First level analysis:** In *every* run, the districting was in the most partisan 11% of districtings (in other words, 89.4% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

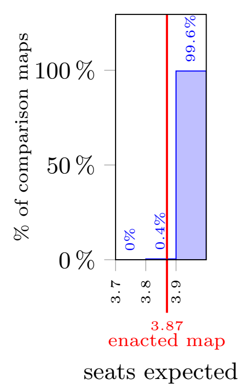
6.13 House Cluster: Durham/Person

6.13.1 Comparison map examples



6.13.2 Results

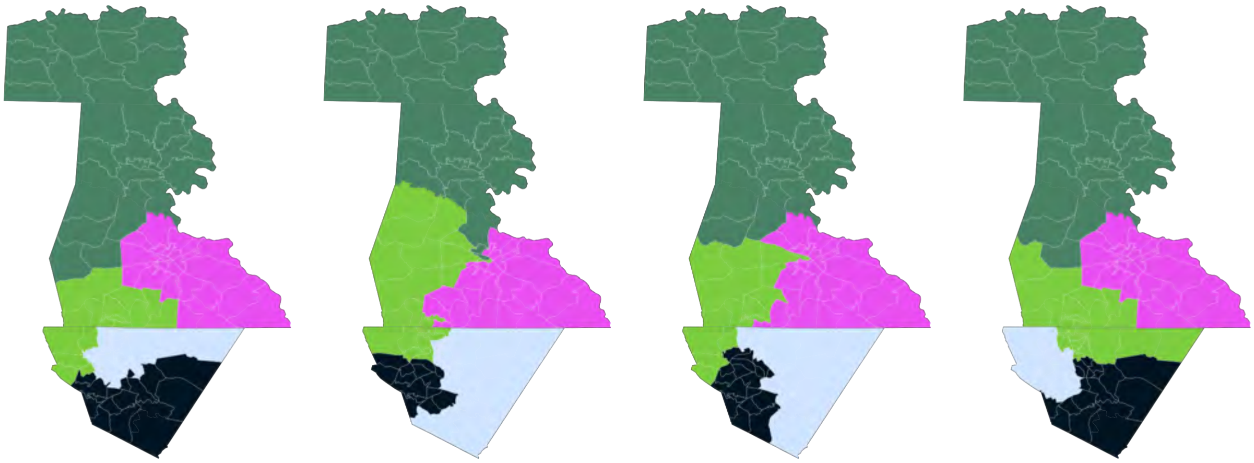
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.936%	9	99.935%	17	99.938%	25	99.935%
2	99.933%	10	99.937%	18	99.937%	26	99.933%
3	99.937%	11	99.94%	19	99.934%	27	99.939%
4	99.932%	12	99.933%	20	99.934%	28	99.936%
5	99.933%	13	99.936%	21	99.936%	29	99.937%
6	99.936%	14	99.935%	22	99.938%	30	99.933%
7	99.937%	15	99.933%	23	99.937%	31	99.94%
8	99.936%	16	99.936%	24	99.934%	32	99.934%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.067% of districtings (in other words, 99.932% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.20% of all alternative districtings satisfying my districting criteria (in other words, 99.79% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.067\%$.

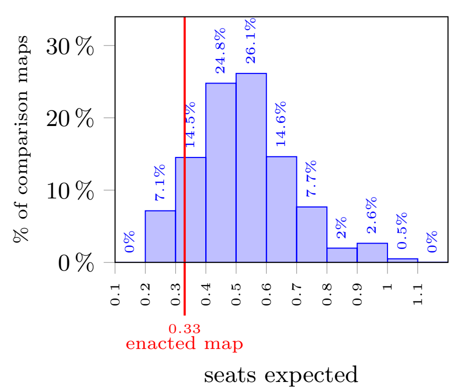
6.14 House Cluster: Cabarrus/Davie/Rowan/Yadkin

6.14.1 Comparison map examples



6.14.2 Results

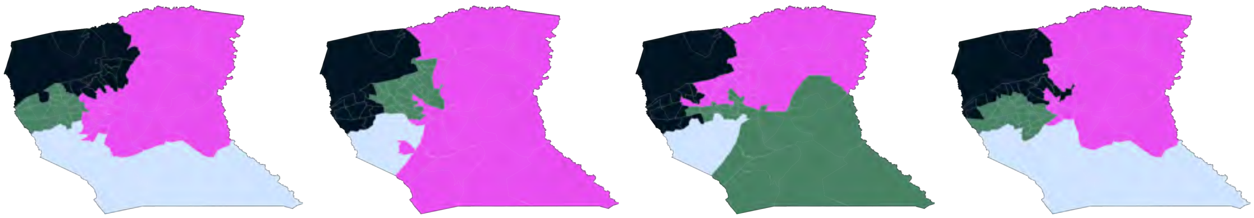
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	89.0%	9	90.0%	17	88.5%	25	89.9%
2	90.0%	10	88.9%	18	89.0%	26	88.6%
3	90.1%	11	88.7%	19	89.4%	27	89.9%
4	88.4%	12	89.8%	20	89.3%	28	88.9%
5	89.7%	13	89.4%	21	92.8%	29	89.5%
6	88.6%	14	89.2%	22	89.1%	30	87.7%
7	89.5%	15	88.8%	23	89.1%	31	90.2%
8	90.0%	16	90.0%	24	88.7%	32	90.4%



- **First level analysis:** In *every* run, the districting was in the most partisan 12% of districtings (in other words, 87.7% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

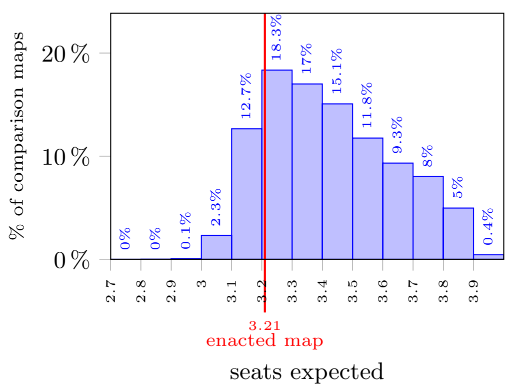
6.15 House Cluster: Cumberland

6.15.1 Comparison map examples



6.15.2 Results

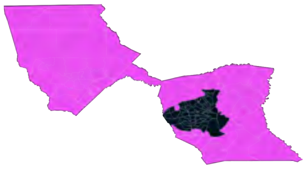
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	83.6%	9	83.8%	17	83.8%	25	84.0%
2	83.7%	10	83.9%	18	83.6%	26	83.5%
3	83.8%	11	83.8%	19	83.7%	27	83.8%
4	83.7%	12	83.6%	20	83.7%	28	83.8%
5	83.6%	13	83.7%	21	84.0%	29	83.7%
6	83.7%	14	83.6%	22	83.9%	30	83.6%
7	83.5%	15	83.8%	23	83.7%	31	83.9%
8	83.7%	16	83.8%	24	83.6%	32	83.9%

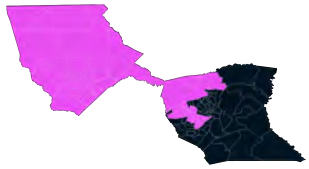


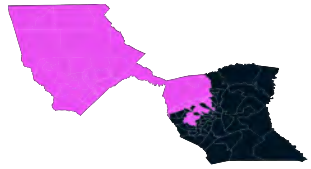
- **First level analysis:** In *every* run, the districting was in the most partisan 16% of districtings (in other words, 83.5% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

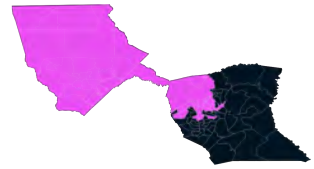
6.16 Senate Cluster: Cumberland Moore

6.16.1 Comparison map examples



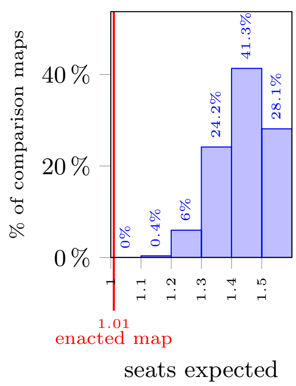






6.16.2 Results

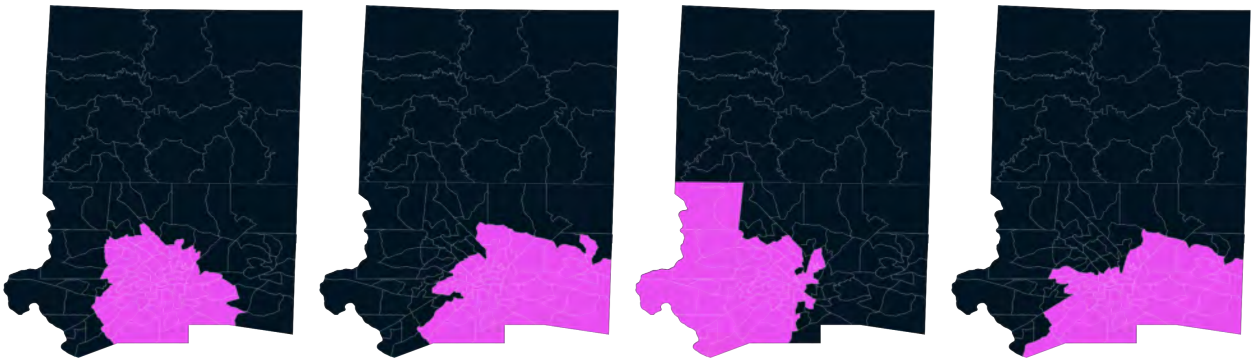
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999968%	9	99.9999962%	17	99.9999963%	25	99.9999954%
2	99.9999961%	10	99.9999965%	18	99.9999969%	26	99.9999955%
3	99.999998%	11	99.9999954%	19	99.9999967%	27	99.999997%
4	99.9999953%	12	99.9999961%	20	99.9999969%	28	99.9999952%
5	99.9999969%	13	99.9999957%	21	99.9999971%	29	99.9999959%
6	99.9999969%	14	99.9999949%	22	99.9999961%	30	99.9999956%
7	99.9999966%	15	99.9999964%	23	99.9999961%	31	99.9999961%
8	99.9999966%	16	99.9999959%	24	99.9999977%	32	99.9999965%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.0000050% of districtings (in other words, 99.9999949% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000015% of all alternative districtings satisfying my districting criteria (in other words, 99.999984% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0000050\%$.

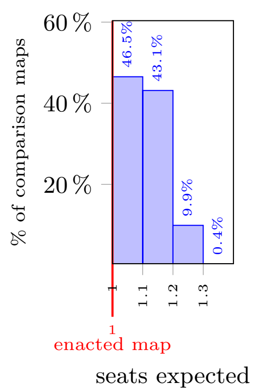
6.17 Senate Cluster: Forsyth-Stokes

6.17.1 Comparison map examples



6.17.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9983%	9	99.9983%	17	99.9983%	25	99.9983%
2	99.9984%	10	99.9984%	18	99.9984%	26	99.9983%
3	99.9982%	11	99.9983%	19	99.9984%	27	99.9983%
4	99.9982%	12	99.9984%	20	99.9983%	28	99.9984%
5	99.9983%	13	99.9983%	21	99.9983%	29	99.9983%
6	99.9984%	14	99.9983%	22	99.9983%	30	99.9984%
7	99.9984%	15	99.9983%	23	99.9983%	31	99.9984%
8	99.9984%	16	99.9984%	24	99.9984%	32	99.9983%

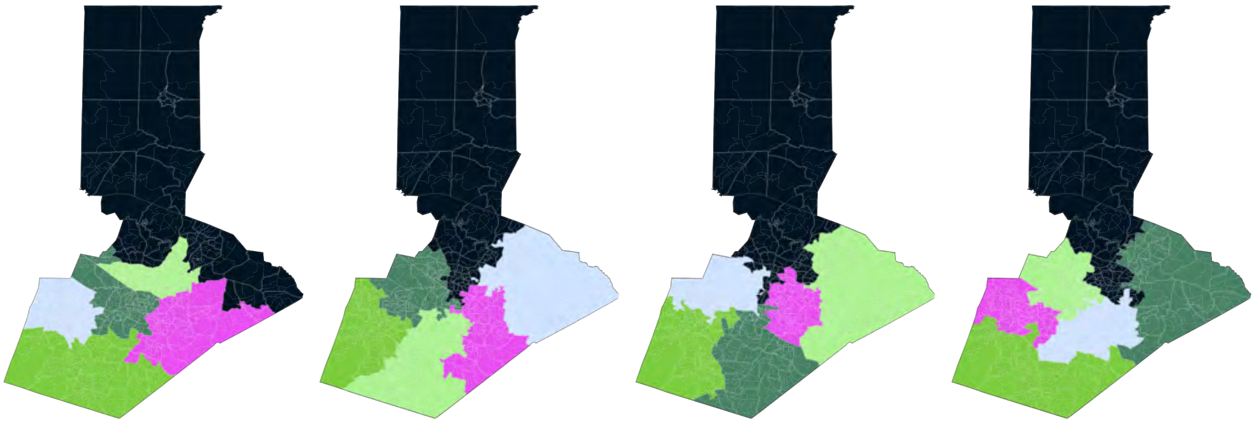


- **First level analysis:** In *every* run, the districting was in the most partisan 0.0016% of districtings (in other words, 99.9983% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.0051% of all alternative districtings satisfying my districting criteria (in other words, 99.9947% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0016\%$.

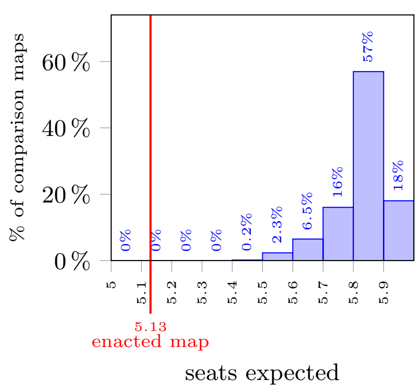
6.18 Senate Cluster: Granville-Wake

6.18.1 Comparison map examples



6.18.2 Results

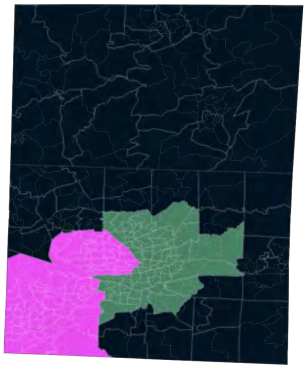
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999934%	9	99.99999921%	17	99.99999999936%	25	99.9999971%
2	99.9999984%	10	99.99999999936%	18	99.99999913%	26	99.9999975%
3	99.99999917%	11	99.99999966%	19	99.9999967%	27	99.99999909%
4	99.99999999945%	12	99.9999979%	20	99.99999963%	28	99.999989%
5	99.99999974%	13	99.9999989%	21	99.9999999984%	29	99.99999999954%
6	99.999999939%	14	99.9999976%	22	99.99999948%	30	99.9999968%
7	99.9999999982%	15	99.9999947%	23	99.9999984%	31	99.99999999945%
8	99.9999995%	16	99.99999969%	24	99.99999967%	32	99.99999971%

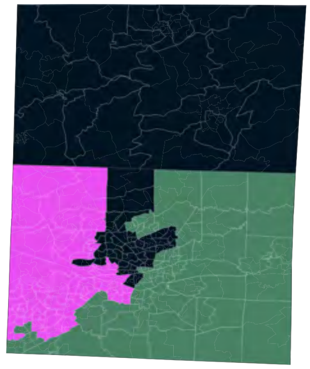


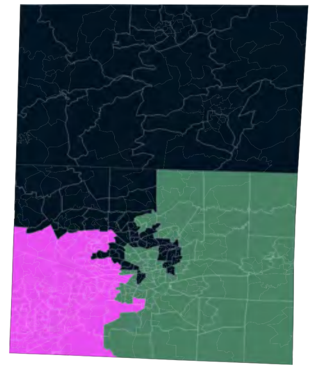
- **First level analysis:** In *every* run, the districting was in the most partisan 0.000010% of districtings (in other words, 99.999989% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000030% of all alternative districtings satisfying my districting criteria (in other words, 99.999969% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000010\%$.

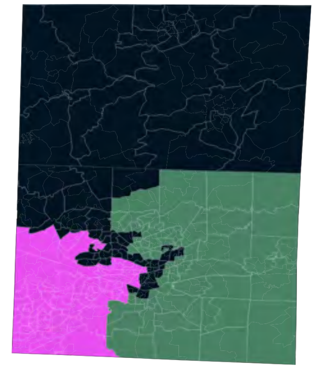
6.19 Senate Cluster: Guilford-Rockingham

6.19.1 Comparison map examples



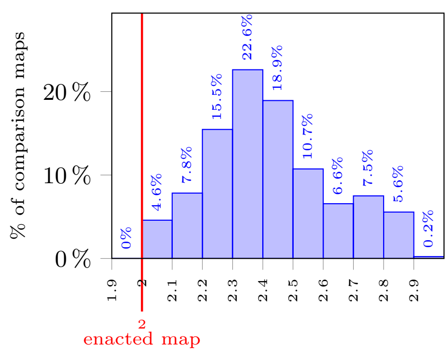






6.19.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999979%	9	99.9999971%	17	99.999989%	25	99.999984%
2	99.999975%	10	99.999999976%	18	99.9999929%	26	99.99999949%
3	99.9999991%	11	99.9999944%	19	99.999988%	27	99.999967%
4	99.999984%	12	99.99998%	20	99.99998%	28	99.999995%
5	99.999976%	13	99.9999978%	21	99.99996%	29	99.999957%
6	99.9999922%	14	99.999978%	22	99.999979%	30	99.9999999957%
7	99.9999997%	15	99.999986%	23	99.9999964%	31	99.9999935%
8	99.999967%	16	99.9999939%	24	99.999983%	32	99.9999984%

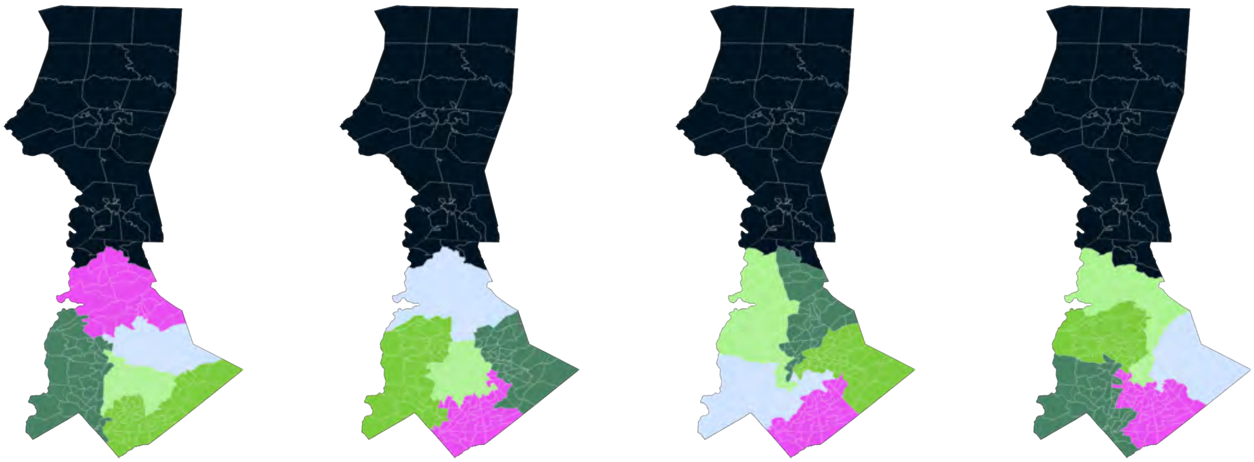


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000042% of districtings (in other words, 99.999957% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.00012% of all alternative districtings satisfying my districting criteria (in other words, 99.99987% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000042\%$.

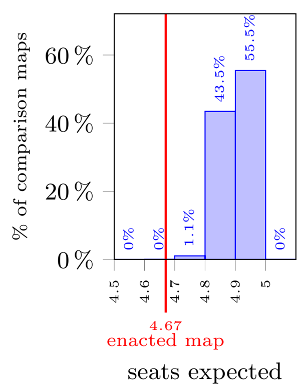
6.20 Senate Cluster: Iredell-Mecklenburg

6.20.1 Comparison map examples



6.20.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9981%	9	99.9983%	17	99.9982%	25	99.9982%
2	99.9982%	10	99.9983%	18	99.9982%	26	99.9983%
3	99.9982%	11	99.9981%	19	99.9981%	27	99.9981%
4	99.9982%	12	99.9982%	20	99.9982%	28	99.9982%
5	99.9981%	13	99.9982%	21	99.9982%	29	99.9982%
6	99.9983%	14	99.9982%	22	99.9982%	30	99.9982%
7	99.9982%	15	99.9982%	23	99.9982%	31	99.9982%
8	99.9982%	16	99.9982%	24	99.9982%	32	99.9981%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.0019% of districtings (in other words, 99.998% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.0057% of all alternative districtings satisfying my districting criteria (in other words, 99.9943% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0019\%$.

7 Seat preservation analyses

In this section I present analyses of clusters for which my main analysis does not achieve high confidence of gerrymandering with respect to the seats-expected metric. These are the districtings in the following House clusters:

- Alamance
- Brunswick/New Hanover
- Cabarrus/Davie/Rowan/Yadkin
- Cumberland

Note that the motivation for the seat-expected metric is to detect partisan gerrymandering aimed at maximizing the expected total number of seats belonging to one party in a representative body (Congress, the North Carolina house, or the North Carolina senate). But there may be other conceivable partisan goals, such as facilitating the re-election of particular representatives in particular districts, which may be orthogonal to or (at least not perfectly correlated with) the goal of maximizing expected representation from one party, and thus which would not be detected by the seats-expected metric.

The metric I use in this section to re-analyze these districtings is the *wave threshold* for a particular seat count. In particular, for a given number of seats x, the wave threshold for x is the smallest uniform swing which can be applied to election data (here, the 2020 Attorney General race) which would result in $x + 1$ Democratic seats. Put differently, this is the threshold such that for any smaller uniform swing, the Democrats will win at most x seats. Referring back to Figure 1, we see that for the enacted Congressional districting of North Carolina, the wave thresholds for $x = 3, 4, 5$, and 6 are -3.56%, 1.68%, 3.05%, and 5.82%, respectively. In particular, even in an election in which voter patterns mirror the 2020 Attorney General race **with all Democratic vote shares increased by an additional** 5.81 **percentage points**, the enacted Congressional districting would still produce only 6 Democrat representatives.

The wave threshold metric can capture partisan goals which may be washed out in the seats-expected metric. For example, if a 5-district cluster is proposed to be districted to optimize the chance that three Republican incumbents all can save their seats, this may or may not result in an increase in the seats-expected metric (for example, if the alternative was to have 4 lean-Republican competitive districts, the extent of the lean would determine how the proposed and alternative districtings would compare under the seats expected metric). But such a plan would be expected to stand out as being highly unusual with respect to the wave threshold for 2 Democratic seats, as it would be an extreme outlier with respect to how difficult it would be for Democrats to capture more than 2 seats in the cluster.

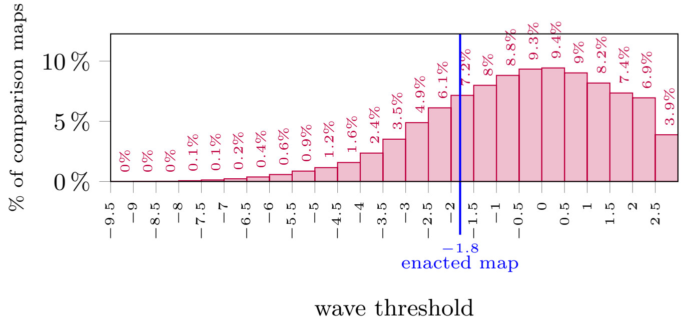
All wave-threshold histograms are shown with red bars, to visually distinguish them from the seats-expected histograms shown elsewhere in the report. Note that unlike for the seats-expected histograms, a Republican bias in the enacted map with respect to a particular wave threshold is indicated by the enacted map showing as an outlier on the righthand side of the plot.

[Report continues on next page for formatting reasons]

7.1 Alamance

The comparison maps generated by my algorithm were similar to the enacted map with respect to their wave threshold for both possible seat values (results here shown for the wave threshold for 0 seats):

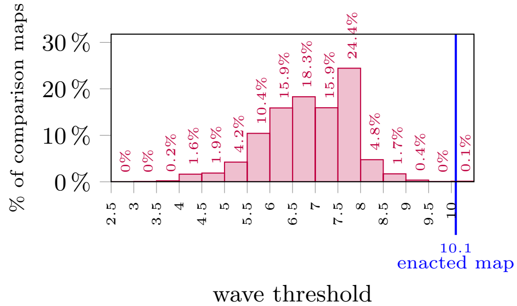
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	25.2%	9	25.2%	17	25.1%	25	25.2%
2	25.2%	10	25.0%	18	25.1%	26	25.2%
3	25.2%	11	25.1%	19	25.1%	27	25.2%
4	25.2%	12	25.2%	20	25.1%	28	25.2%
5	25.3%	13	25.2%	21	25.3%	29	25.3%
6	25.2%	14	25.2%	22	25.2%	30	25.2%
7	25.2%	15	25.1%	23	25.3%	31	25.3%
8	25.2%	16	25.2%	24	25.2%	32	25.2%



7.2 Brunswick/New Hanover

Despite the fact that my algorithm did not detect large differences between the enacted districting and comparison districtings of this cluster, the enacted map is an extreme outlier among the comparison maps generated by my algorithm with respect to the wave threshold for two seats. In particuliar, for the enacted map in this cluster, Democratic performance could increase by 10.1 percentage points in every district without Democrats capturing more than two seats. In every run of my algorithm, 99.72% of comparison maps would allow Democrats to capture a third seat with a smaller wave.

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.987%	9	99.94%	17	99.9956%	25	99.83%
2	99.99%	10	99.907%	18	99.9957%	26	99.79%
3	99.929%	11	99.85%	19	99.8%	27	99.975%
4	99.88%	12	99.9912%	20	99.922%	28	99.85%
5	99.86%	13	99.77%	21	99.961%	29	99.83%
6	99.934%	14	99.89%	22	99.952%	30	99.92%
7	99.73%	15	99.87%	23	99.97%	31	99.946%
8	99.96%	16	99.72%	24	99.911%	32	99.961%

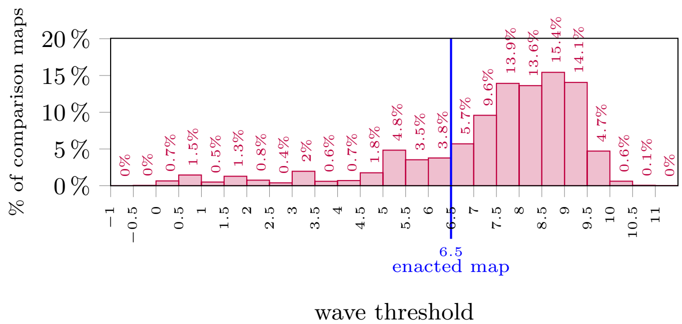


[Report continues on next page for formatting reasons]

7.3 Cabarrus/Davie/Rowan/Yadkin

The comparison maps generated by my algorithm were similar to the enacted map with respect to their wave threshold for all seat values (results here shown for the wave threshold for 1 seat):

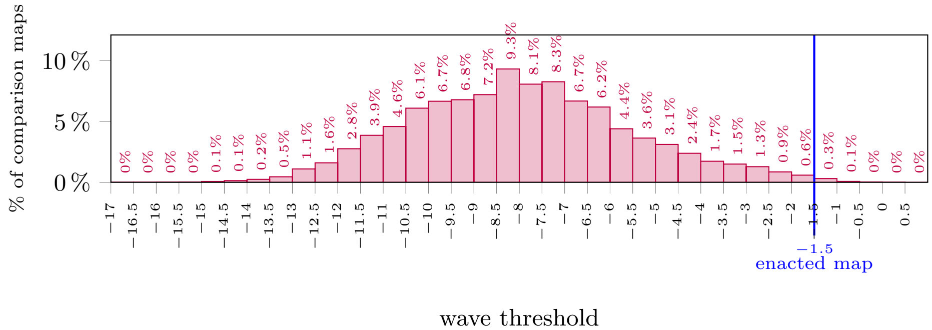
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	26.3%	9	20.7%	17	22.0%	25	22.3%
2	22.6%	10	23.1%	18	21.4%	26	20.8%
3	19.4%	11	27.6%	19	23.3%	27	20.2%
4	20.7%	12	21.2%	20	25.7%	28	22.0%
5	18.8%	13	23.4%	21	21.8%	29	22.1%
6	21.9%	14	25.4%	22	20.8%	30	22.3%
7	24.3%	15	20.0%	23	22.9%	31	22.4%
8	20.4%	16	19.9%	24	23.1%	32	23.8%



7.4 Cumberland

Despite the fact that my algorithm did not detect large differences between the enacted districting and comparison districtings of this cluster, the enacted map is an extreme outlier among the comparison maps generated by my algorithm with respect to the wave threshold for two seats.

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.61%	9	99.62%	17	99.62%	25	99.64%
2	99.64%	10	99.64%	18	99.62%	26	99.63%
3	99.61%	11	99.61%	19	99.61%	27	99.63%
4	99.62%	12	99.62%	20	99.63%	28	99.6%
5	99.59%	13	99.62%	21	99.64%	29	99.63%
6	99.61%	14	99.59%	22	99.63%	30	99.62%
7	99.61%	15	99.62%	23	99.62%	31	99.6%
8	99.61%	16	99.63%	24	99.62%	32	99.62%



Appendix A Multimoves / Precinct splits

As discussed in Section 5 my algorithm can be set to allow multiple changes to a map to occur in one step, when this is necessary to produce a sufficiently rich set of comparison maps.

Here I describe details of this technique so that technical experts can understand how precisely our method works. These details are not necessary to understand the basic mechanics of the method, which are simply that:

- Multiple changes may be made to a map in a single step,
- The result of the changes must always be a valid comparison map, in the sense that it complies with the districting criteria we consider in our report, and
- Our implementation of multiple moves does not bias the algorithm to any map or family of maps.

For technical experts: these multiple moves can be implemented with a Metropolis-Hastings approach. In particular, a score function based on the deviation of an invalid map from the compactness and population thresholds can be defined. *The score function is set to be equal for all maps satisfying the districting criteria.* With this choice, a uniform stationary distribution can be constructed on the space of maps satisfying the districting criteria. The Metropolis-Hastings chain will occasionally leave the feasible region of the map-space for some number of steps before returning to the feasible region. The collection of steps made outside the feasible region can be performed in a single step, to give a single multi-move which transforms one valid map into another valid map.

A related implementation detail concerns precinct splits. When operating at the geounit level but preserving the maximum number of precinct splits, I can allow the chain at intermediate points to have one more split than is allowed, while discarding these intermediate, invalid comparison maps. For example, in a map which currently splits two specific precincts, the chain is allowed to produce a valid comparison map by changing the district membership of another precinct. Note that this does not change the number of precinct splits, but viewed in terms of single geounit moves, it passes through a set of maps with a greater number of precinct splits. As in the case of multimoves discussed above, these intermediate maps are not part of the comparison set, and we can view the precinct swap as a single multimove of geounit swaps.

Finally, I note that when operating below the precinct level in House clusters with split precincts, my algorithm imposes an additional compactness-like constraint on any precinct splits, which is simply that the length of the precinct split is not large relative to the perimeter of the precinct itself. (The enacted plan satisfies this constraint in all cases.)

Appendix B Theorems

The second level analyses in my report are calculated using the theorems from [CFMP]; in particular, Theorem 1.5 from that manuscript suffices for all of my second-level findings here.

In plain language, that theorem says that if I conduct m runs, and observe that in every run the enacted plan is in the bottom ε fraction of comparison maps, then I can conclude that the enacted plan is among the most carefully crafted α fraction of *all* maps satisfying the districting criteria (not just those encountered by the algorithm), measured by their ε-fragility, at a statistical significance calculated with the formula

$$p = \left(\frac{2\varepsilon}{\alpha}\right)^{m/2}.$$

In this report, I frequently have $m = 32$ runs and choose α to simply be 3 times as big as ε. In this case, we see that we can conclude that the enacted plan is among the most carefully crafted 3ε of all maps, at a statistical significance of

$$p = \left(\frac{2}{3}\right)^{16} \approx .0015 < .002.$$

Note that, for example, if we used instead a threshold of $\alpha = 4\varepsilon$, this would give significance of

$$p = \left(\frac{2}{4}\right)^{16} \approx .000015,$$

and taking a threshold of $\alpha = 6\varepsilon$ would give

$$p = \left(\frac{2}{6}\right)^{16} \approx .00000002,$$

[Report continues on next page for formatting reasons]

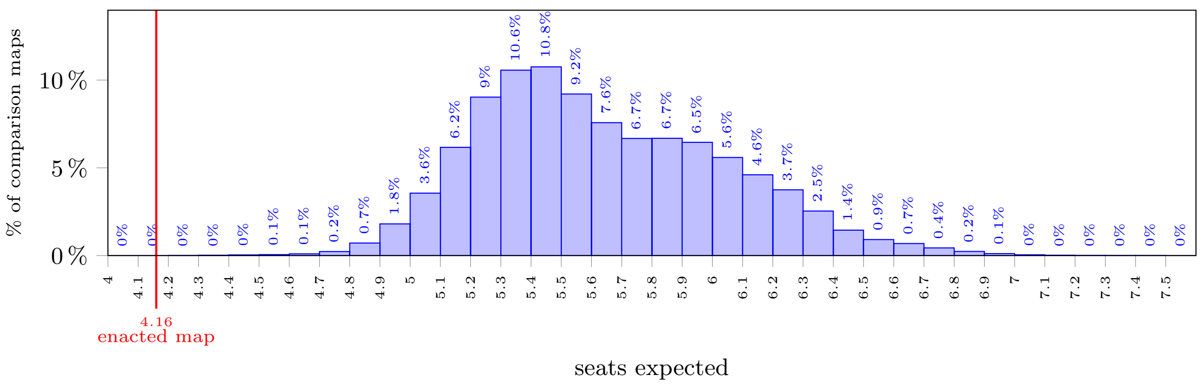
Appendix C Robustness Checks, Congressional districting

C.1 Robustness to election data

Here I show results when my analysis of the Congressional map is repeated with other elections in place of the 2020 Attorney General election as my proxy for partisan voting patterns.

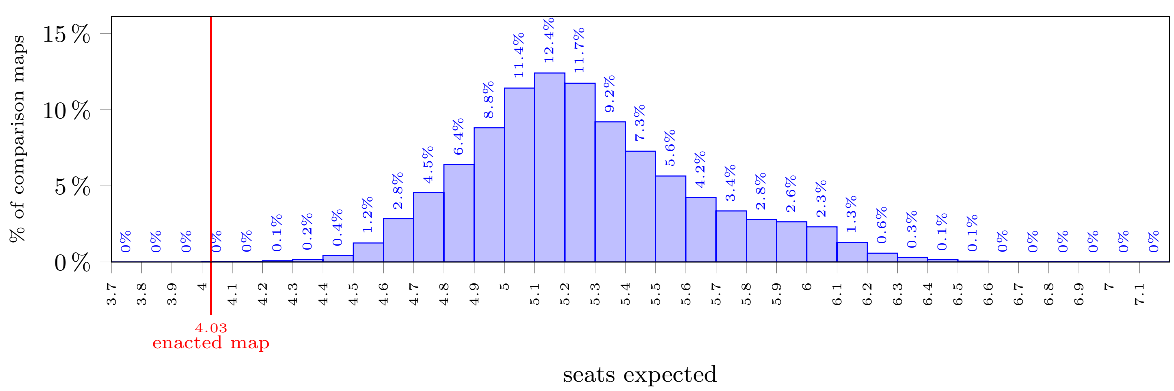
C.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999925%	5	99.999986%	9	99.9999908%	13	99.9999926%
2	99.999921%	6	99.999999968%	10	99.9999932%	14	99.999988%
3	99.9999955%	7	99.999984%	11	99.9999979%	15	99.9999989%
4	99.9999933%	8	99.99995%	12	99.9999999981%	16	99.999978%



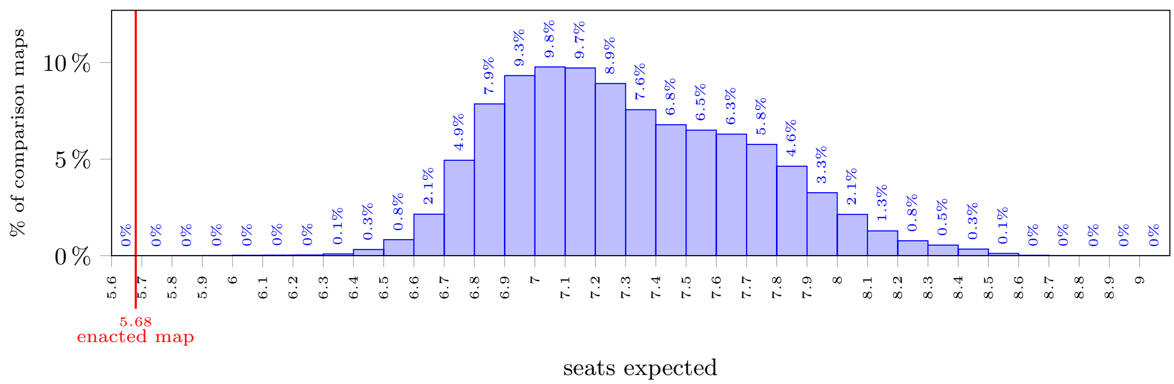
C.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999973%	5	99.999937%	9	99.999942%	13	99.999982%
2	99.99985%	6	99.9999964%	10	99.99901%	14	99.999978%
3	99.999905%	7	99.99954%	11	99.9999928%	15	99.999934%
4	99.999964%	8	99.99975%	12	99.9995%	16	99.9998%



C.1.3 Results with 2020 Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999989%	5	99.9999979%	9	99.9999975%	13	99.99999923%
2	99.9999914%	6	99.9999999922%	10	99.99999974%	14	99.99999968%
3	99.9999996%	7	99.999999934%	11	99.999999994%	15	99.999999982%
4	99.999999966%	8	99.9999982%	12	99.9999999981%	16	99.999999961%

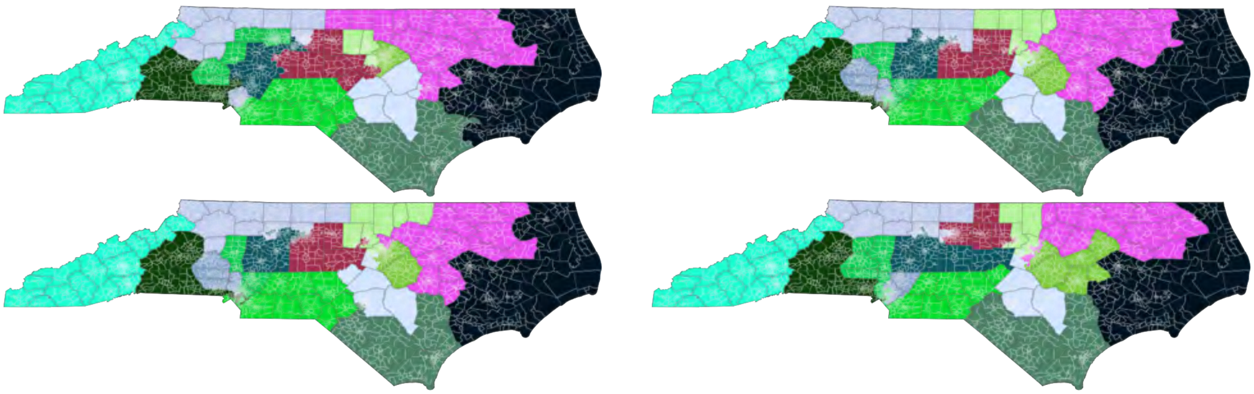


[Report continues on next page for formatting reasons]

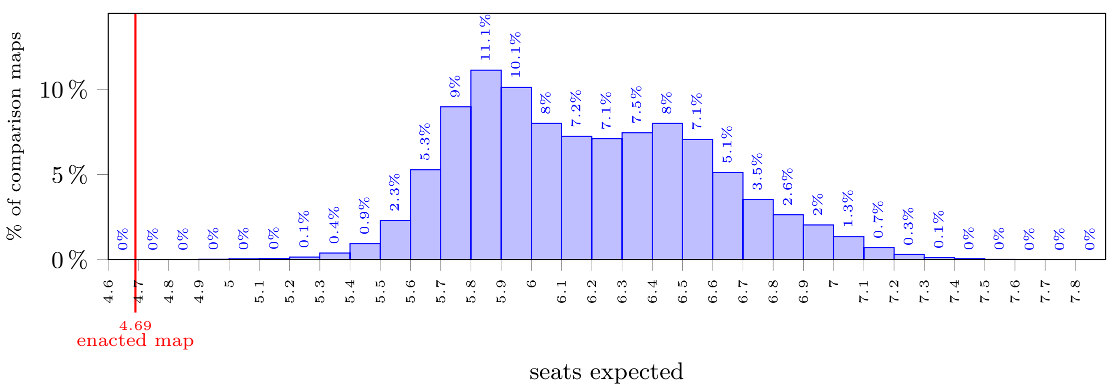
C.2 Robustness to incumbency protection

Here I show results when my analysis of the Congressional map is repeated without ensuring the protection of incumbents.

C.2.1 Comparison map examples



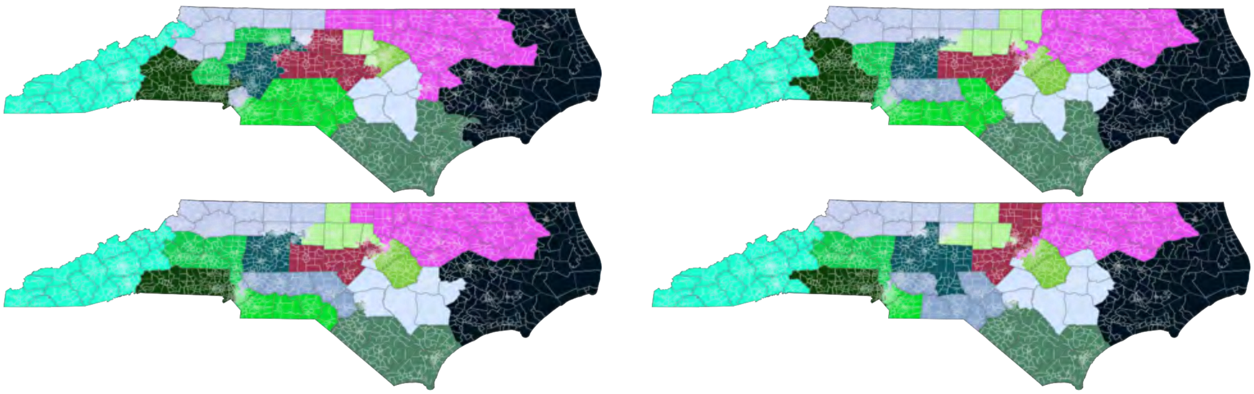
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999998%	5	99.99999918%	9	99.9999976%	13	99.999982%
2	99.999999901%	6	99.9999978%	10	99.999989%	14	99.99999901%
3	99.9999986%	7	99.999999961%	11	99.9999967%	15	99.99999977%
4	99.9999967%	8	99.9999954%	12	99.9999999981%	16	99.9999986%



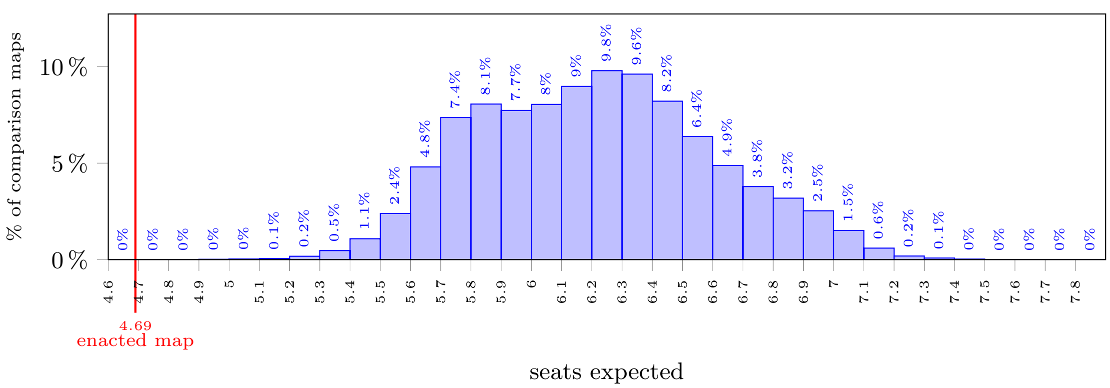
C.3 Robustness to compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the Congressional map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

C.3.1 Comparison map examples



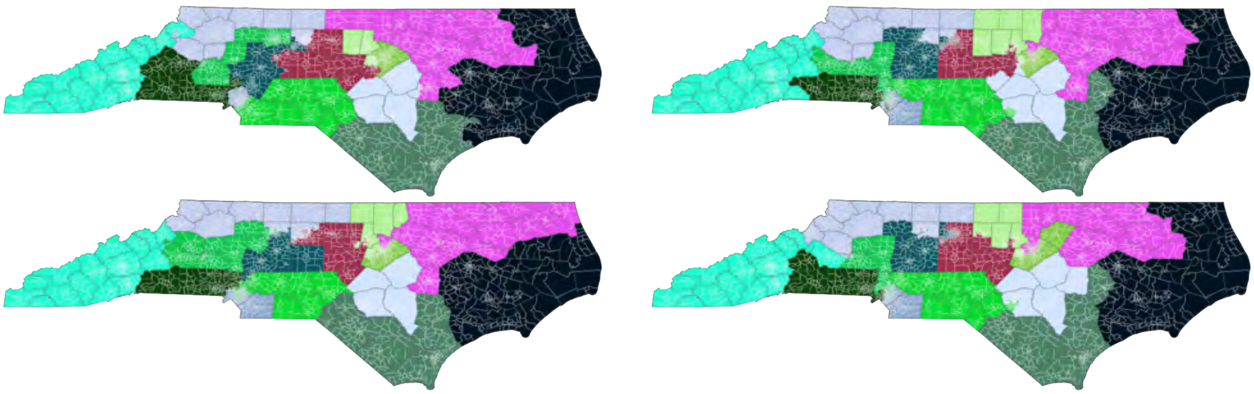
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999989%	5	99.9999997%	9	99.9999975%	13	99.999979%
2	99.9999984%	6	99.99999983%	10	99.9999968%	14	99.9999968%
3	99.9999933%	7	99.9999962%	11	99.9999968%	15	99.9999983%
4	99.999986%	8	99.9999983%	12	99.99999954%	16	99.9999984%



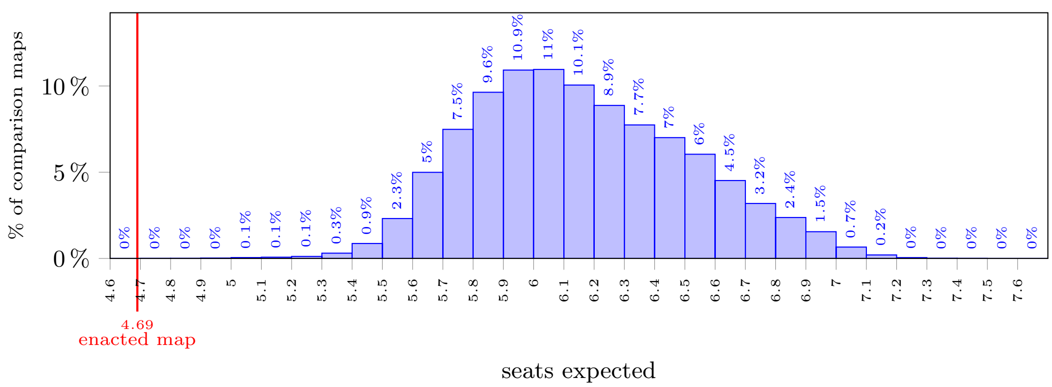
C.4 Robustness to compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the Congressional map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

C.4.1 Comparison map examples



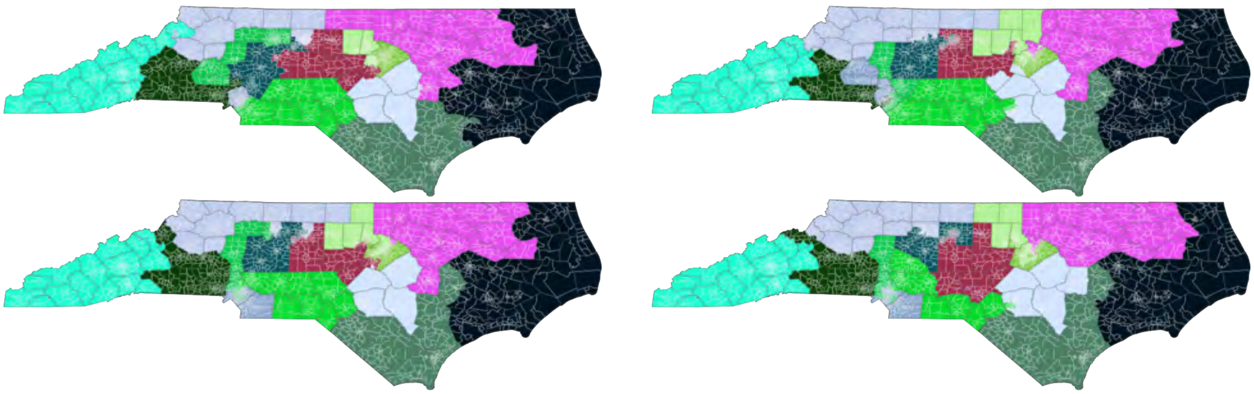
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999988%	5	99.9999974%	9	99.999982%	13	99.9999976%
2	99.9999989%	6	99.9999989%	10	99.9999954%	14	99.9999985%
3	99.9999961%	7	99.999999946%	11	99.9999965%	15	99.99999983%
4	99.99999981%	8	99.9999973%	12	99.9999999981%	16	99.99999985%



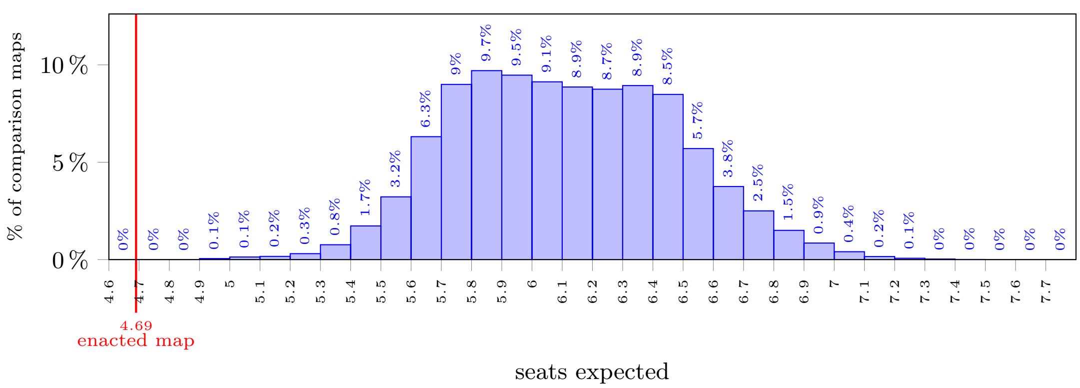
C.5 Robustness to compactness 5% Perimeter compactness

Here I show results when my analysis of the Congressional map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

C.5.1 Comparison map examples



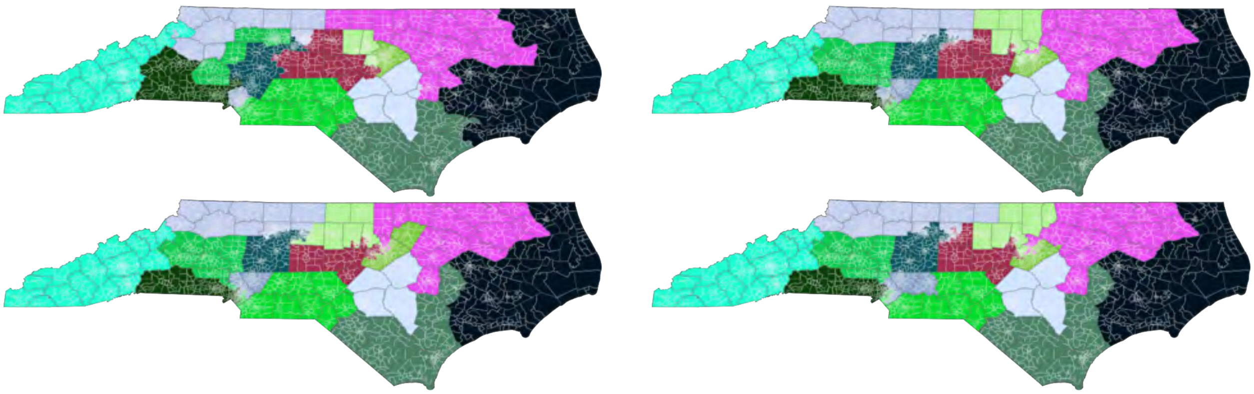
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999988%	5	99.9999968%	9	99.999998%	13	99.9999976%
2	99.99999948%	6	99.9999949%	10	99.9999978%	14	99.9999986%
3	99.99999941%	7	99.9999999976%	11	99.999982%	15	99.99999983%
4	99.99999981%	8	99.9999906%	12	99.9999999981%	16	99.9999963%



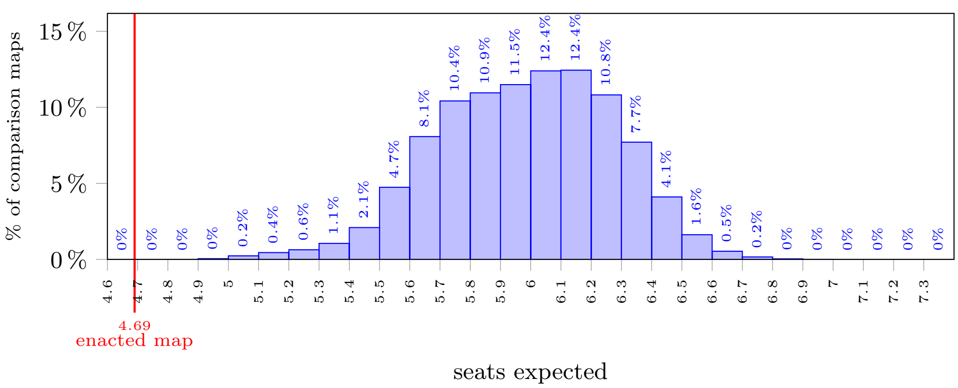
C.6 Robustness to 1% population deviation

Here I show results when my analysis of the Congressional map is repeated with a 1% population deviation constraint instead of a 2% population deviation constraint.

C.6.1 Comparison map examples



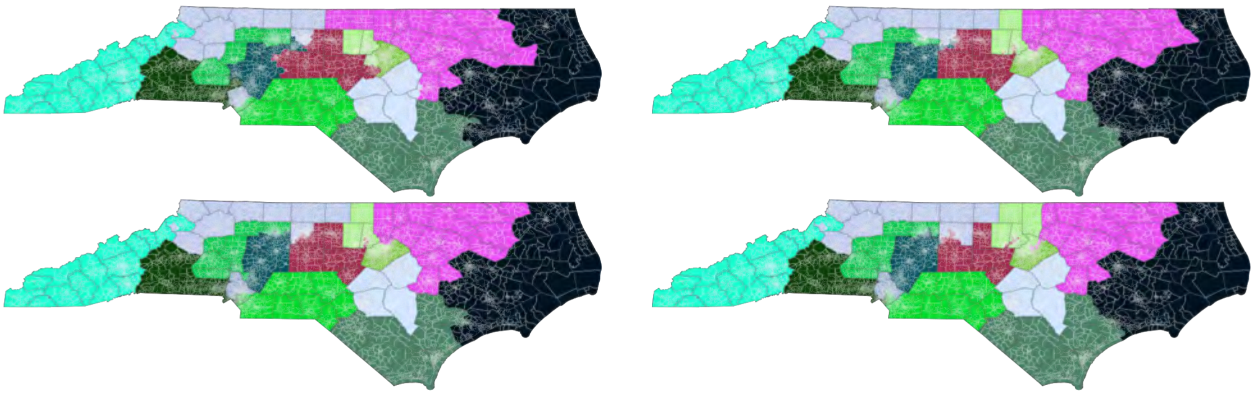
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999911%	5	99.999999907%	9	99.9999983%	13	99.999914%
2	99.9999966%	6	99.99999999945%	10	99.99978%	14	99.9999988%
3	99.999949%	7	99.9999986%	11	99.999989%	15	99.999971%
4	99.9999935%	8	99.999951%	12	99.999934%	16	99.999997%



C.7 Geounit analysis

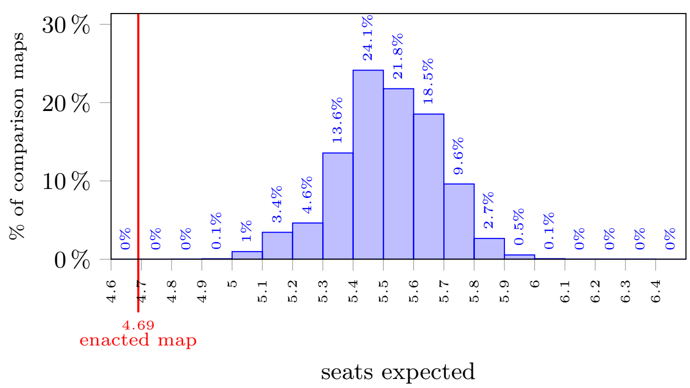
Here I show results when my analysis of the Congressional map is repeated at the geounit level, with a 0.5% population deviation constraint.

C.7.1 Comparison map examples



C.7.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999952%	5	99.999987%	9	99.999962%	13	99.9999952%
2	99.999989%	6	99.999986%	10	99.9999964%	14	99.9999962%
3	99.999967%	7	99.9999924%	11	99.999974%	15	99.999926%
4	99.999964%	8	99.999996%	12	99.999977%	16	99.9999935%

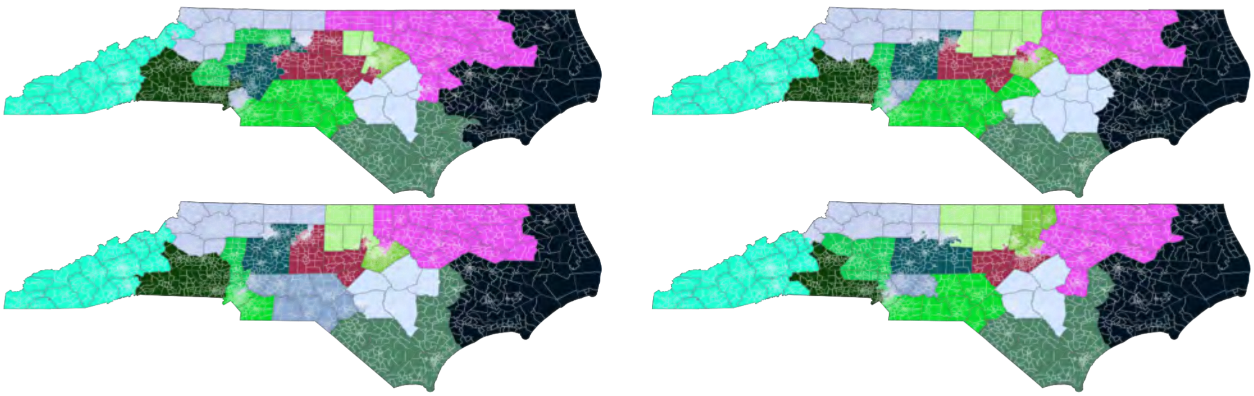


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000073% of districtings (in other words, 99.999926% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.00022% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.99977% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000073\%$.

C.8 Analysis of VTD-level blueprint

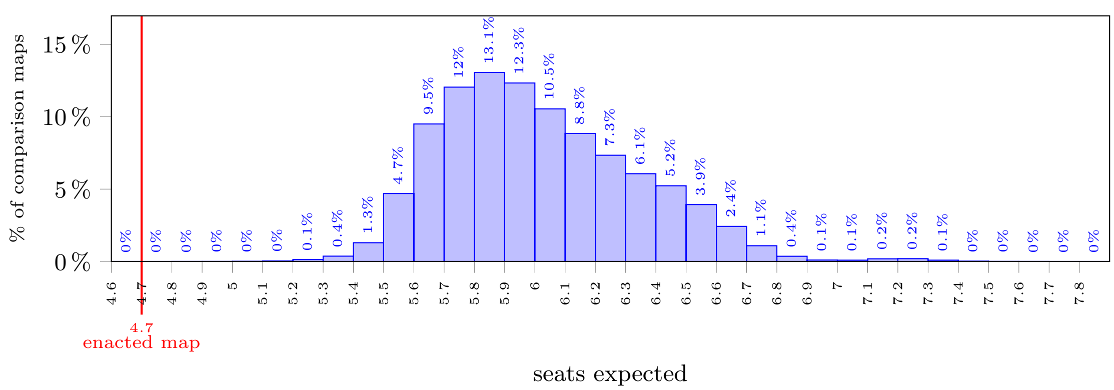
Here I show results when my analysis of the Congressional map is performed not on the precise enacted map, but a whole-VTD-level blueprint for the enacted map obtained by assigning each split VTD to the district it has the greatest intersection with.

C.8.1 Comparison map examples



C.8.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999982%	9	99.99999969%	17	99.9999991%	25	99.9999986%
2	99.99999947%	10	99.9999952%	18	99.99999944%	26	99.9999998%
3	99.9999957%	11	99.999986%	19	99.999978%	27	99.9999977%
4	99.9999907%	12	99.999979%	20	99.9999959%	28	99.9999976%
5	99.9999981%	13	99.9999986%	21	99.99999946%	29	99.99999958%
6	99.99999954%	14	99.999984%	22	99.9999971%	30	99.999986%
7	99.9999917%	15	99.9999977%	23	99.9999974%	31	99.9999969%
8	99.9999917%	16	99.9999961%	24	99.9999942%	32	99.9999958%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.000021% of districtings (in other words, 99.999978% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000064% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999935% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000021\%$.

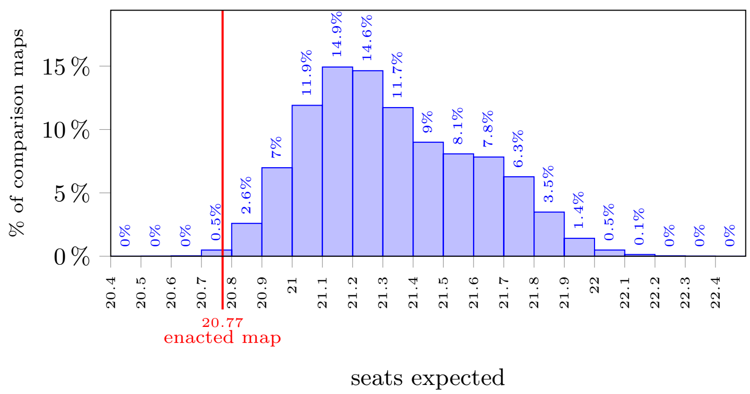
Appendix D Robustness Checks, Senate districting

D.1 Robustness to election data

Here I show results when my analysis of the Senate map is repeated with other elections in place of the 2020 Attorney General election as my proxy for partisan voting patterns.

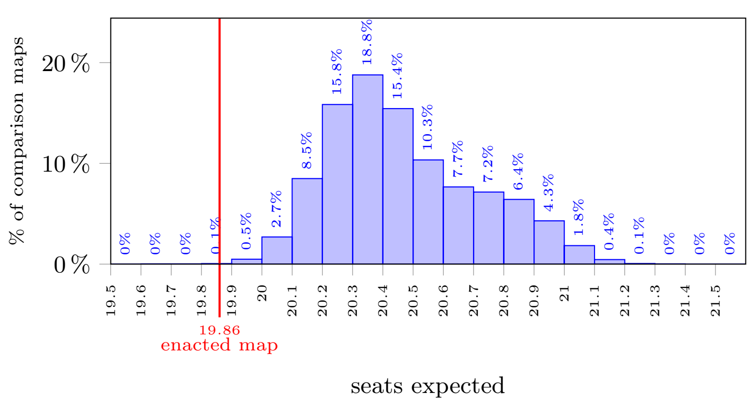
D.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.65%	5	99.78%	9	99.79%	13	99.8%
2	99.81%	6	99.79%	10	99.82%	14	99.73%
3	99.75%	7	99.79%	11	99.81%	15	99.66%
4	99.8%	8	99.75%	12	99.75%	16	99.81%



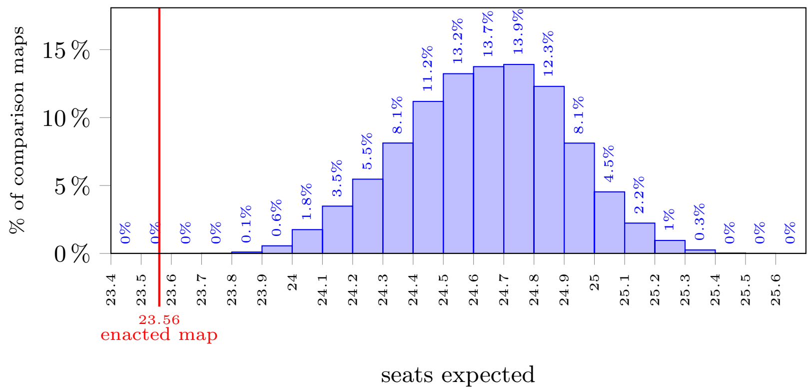
D.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.943%	5	99.987%	9	99.9912%	13	99.9911%
2	99.996%	6	99.982%	10	99.9955%	14	99.977%
3	99.973%	7	99.994%	11	99.9958%	15	99.944%
4	99.9927%	8	99.983%	12	99.89%	16	99.995%



D.1.3 Results with 2020 Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999936%	5	99.9999996%	9	99.9999998%	13	99.999999973%
2	99.999999949%	6	99.9999974%	10	99.9999987%	14	99.9999985%
3	99.99999978%	7	99.9999999929%	11	99.9999998%	15	99.999999961%
4	99.9999989%	8	99.9999999969%	12	99.999999973%	16	99.9999985%

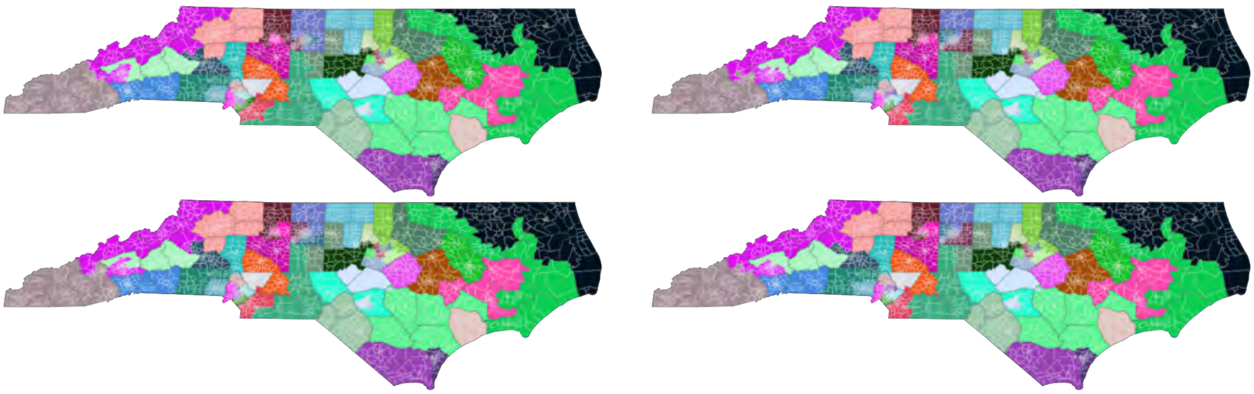


[Report continues on next page for formatting reasons]

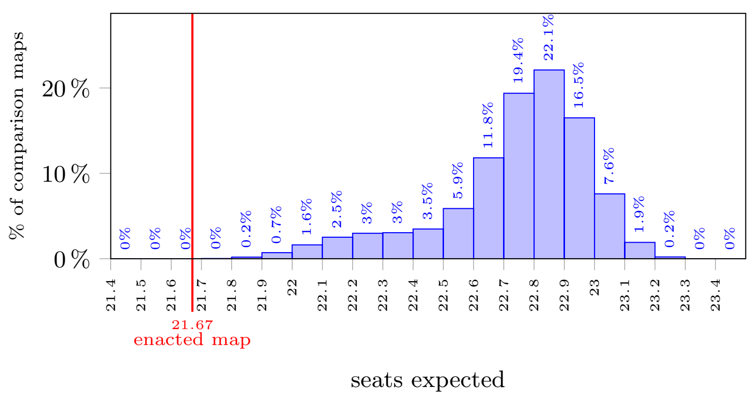
D.2 Robustness to incumbency protection

Here I show results when my analysis of the Senate map is repeated without ensuring the protection of incumbents.

D.2.1 Comparison map examples



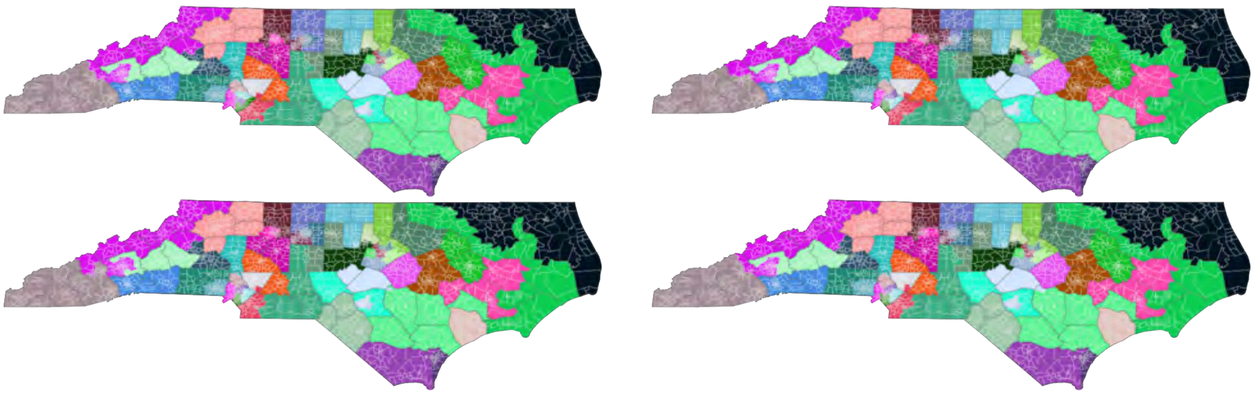
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9998%	5	99.9993%	9	99.99989%	13	99.99906%
2	99.99988%	6	99.99985%	10	99.99968%	14	99.9987%
3	99.99971%	7	99.999907%	11	99.9998%	15	99.99928%
4	99.99922%	8	99.9985%	12	99.99976%	16	99.9943%



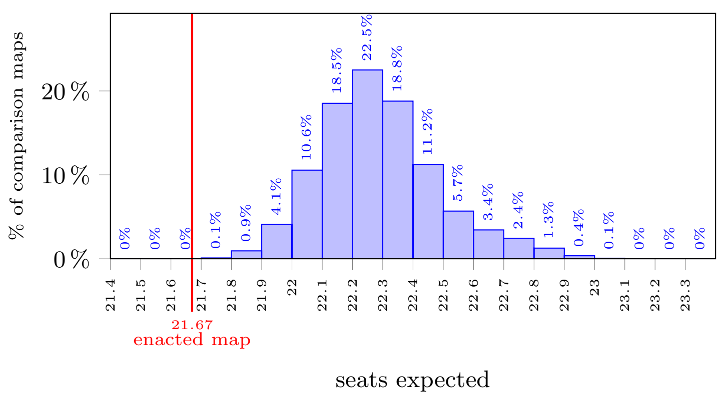
D.3 Compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the Senate map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

D.3.1 Comparison map examples



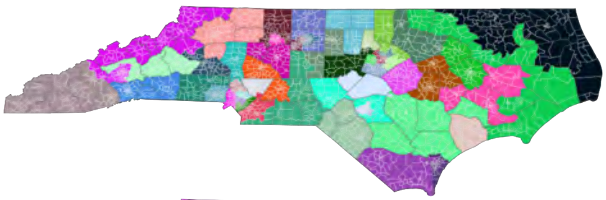
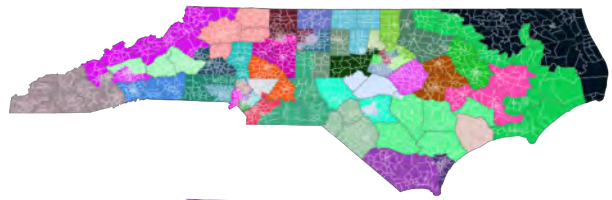
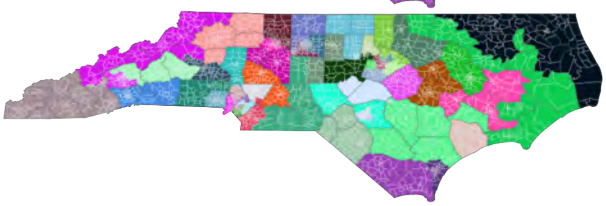
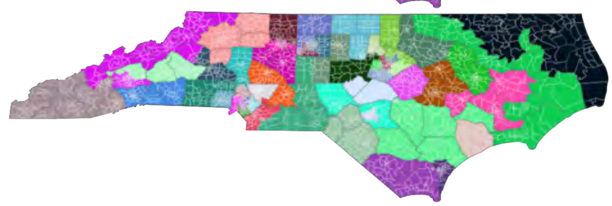
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9979%	5	99.9978%	9	99.995%	13	99.9986%
2	99.99909%	6	99.9968%	10	99.9982%	14	99.9989%
3	99.9968%	7	99.99933%	11	99.9987%	15	99.9973%
4	99.99927%	8	99.9979%	12	99.99923%	16	99.9976%



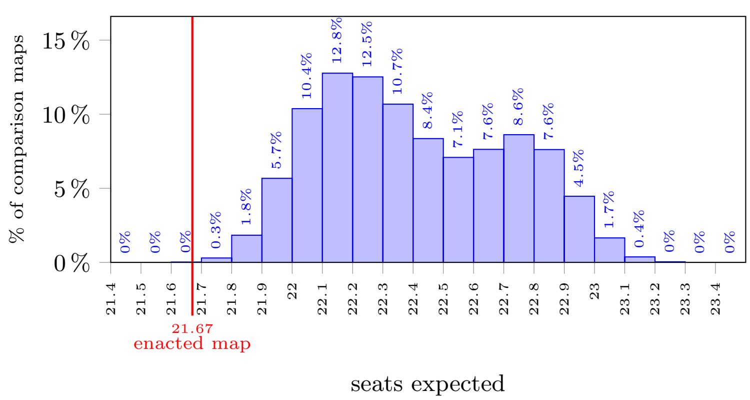
D.4 Compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the Senate map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

D.4.1 Comparison map examples

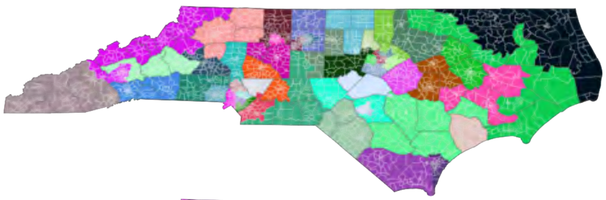
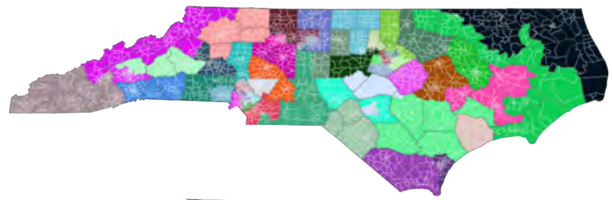
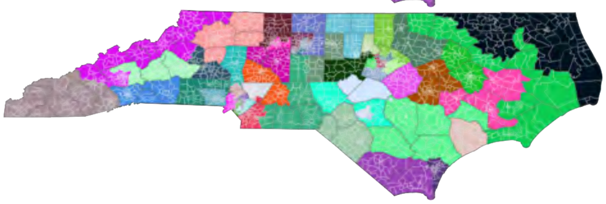
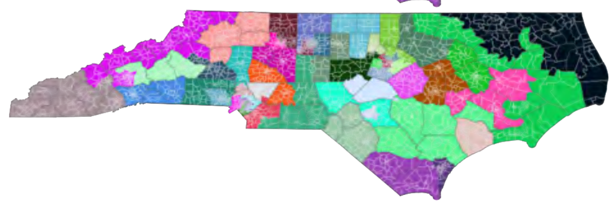
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9963%	5	99.992%	9	99.971%	13	99.98%
2	99.9928%	6	99.986%	10	99.985%	14	99.9917%
3	99.988%	7	99.993%	11	99.9924%	15	99.978%
4	99.987%	8	99.9957%	12	99.9908%	16	99.9969%



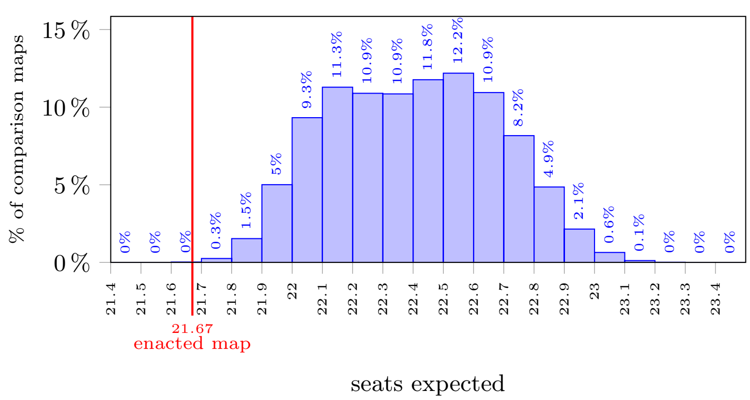
D.5 Compactness 5% Perimeter compactness

Here I show results when my analysis of the Senate map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

D.5.1 Comparison map examples

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9913%	5	99.985%	9	99.988%	13	99.9907%
2	99.9907%	6	99.989%	10	99.988%	14	99.982%
3	99.9949%	7	99.9929%	11	99.986%	15	99.981%
4	99.989%	8	99.989%	12	99.987%	16	99.9919%



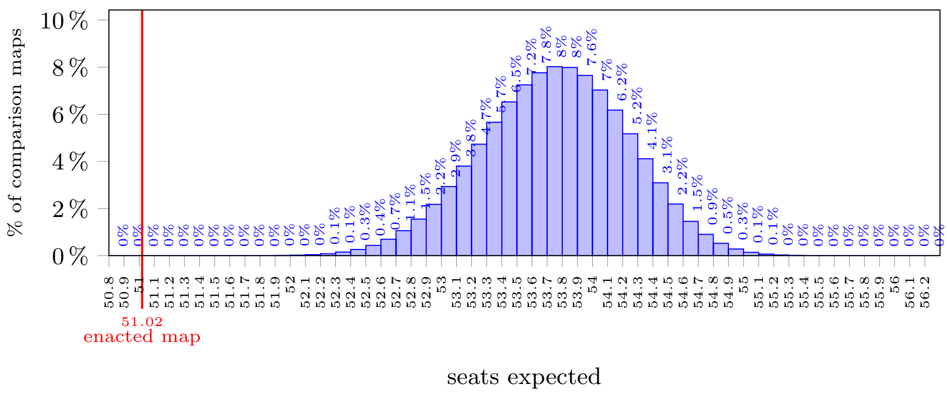
Appendix E Robustness Checks, House districting

E.1 Robustness to election data

Here I show results when my analysis of the House map is repeated with other elections in place of the 2020 Attorney General election as my proxy for partisan voting patterns.

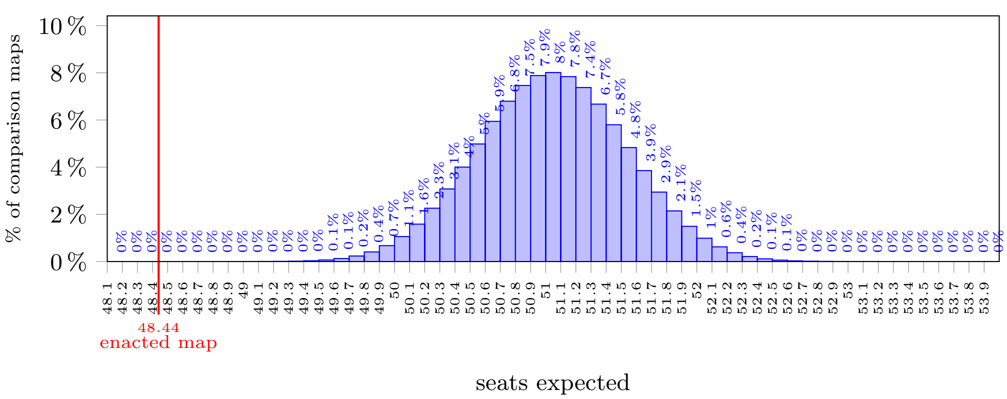
E.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999999985%	5	99.99999945%	9	99.9999986%	13	99.99999986%
2	99.999999981%	6	99.99999948%	10	99.99999912%	14	99.999999976%
3	99.99999997%	7	99.999999963%	11	99.99999986%	15	99.99999984%
4	99.9999969%	8	99.9999981%	12	99.9999985%	16	99.9999989%



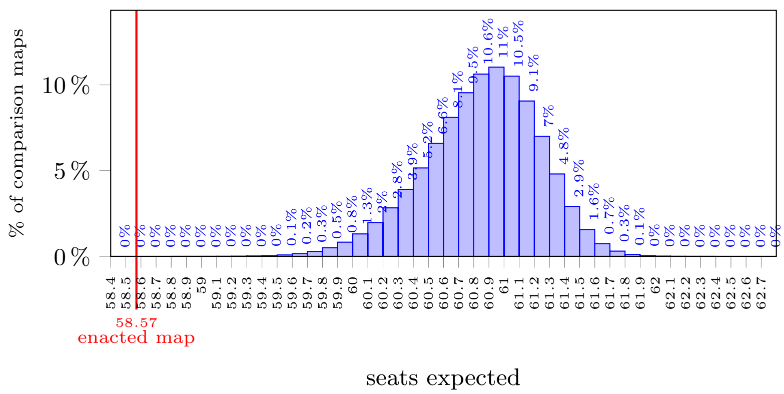
E.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999988%	5	99.9999983%	9	99.999997%	13	99.9999957%
2	99.999981%	6	99.9999926%	10	99.9999979%	14	99.9999905%
3	99.99999907%	7	99.9999927%	11	99.9999974%	15	99.99999914%
4	99.9999969%	8	99.999993%	12	99.9999981%	16	99.99999924%



E.1.3 Results with 2020 Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999985%	5	99.999999931%	9	99.999999975%	13	99.99999986%
2	99.999999984%	6	99.9999994%	10	99.9999986%	14	99.99999988%
3	99.99999997%	7	99.999999986%	11	99.9999998%	15	99.99999948%
4	99.9999985%	8	99.99999985%	12	99.99999914%	16	99.99999989%

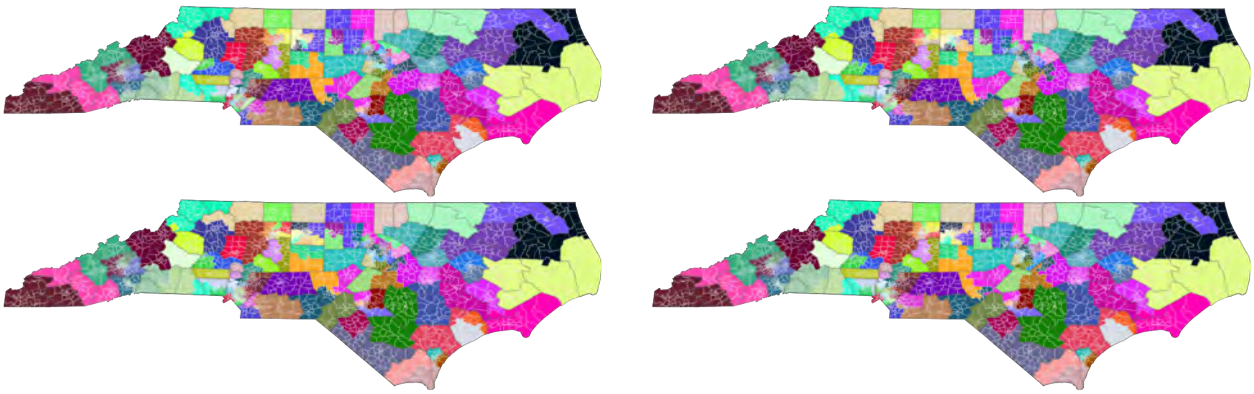


[Report continues on next page for formatting reasons]

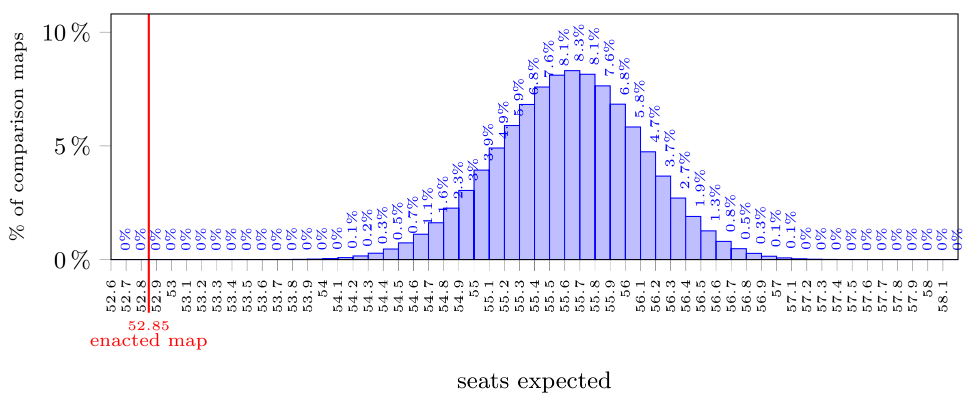
E.2 Robustness to incumbency protection

Here I show results when my analysis of the House map is repeated without ensuring the protection of incumbents.

E.2.1 Comparison map examples



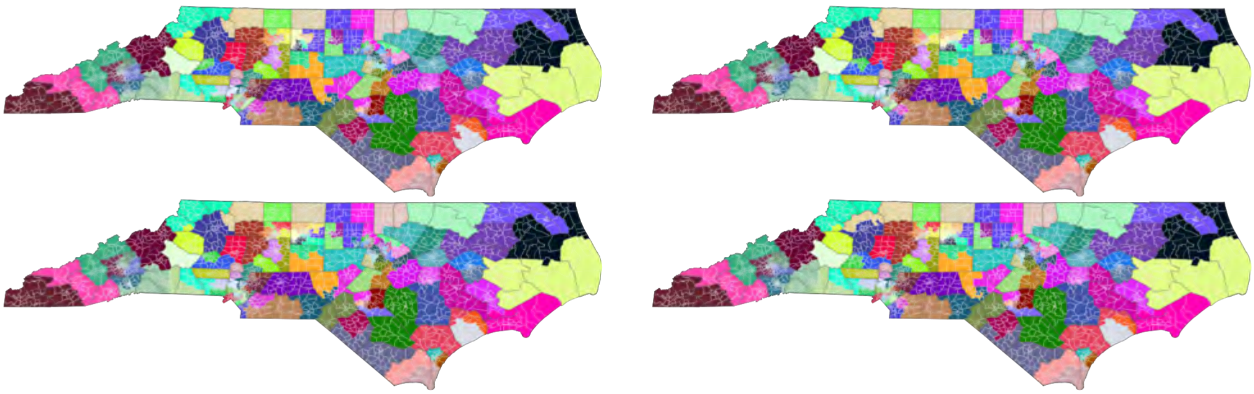
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999987%	5	99.9999933%	9	99.99999967%	13	99.99999989%
2	99.999999981%	6	99.9999962%	10	99.99999944%	14	99.99999981%
3	99.99999997%	7	99.9999968%	11	99.9999944%	15	99.99999%
4	99.999999908%	8	99.99999961%	12	99.999999963%	16	99.99999947%



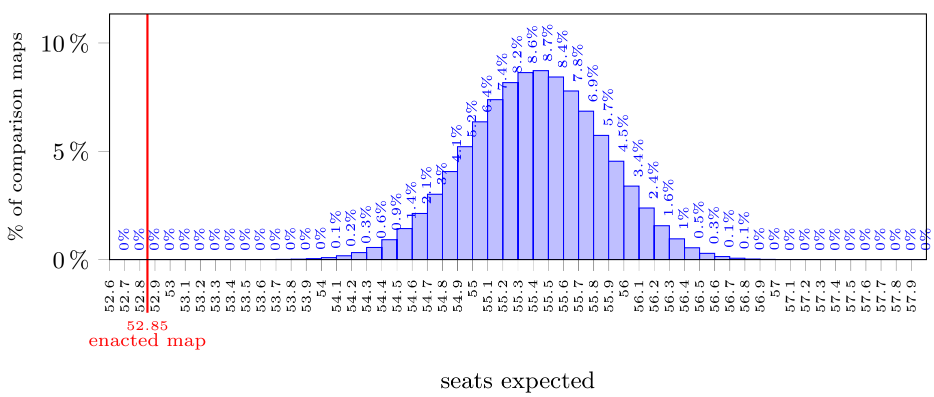
E.3 Compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the House map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

E.3.1 Comparison map examples



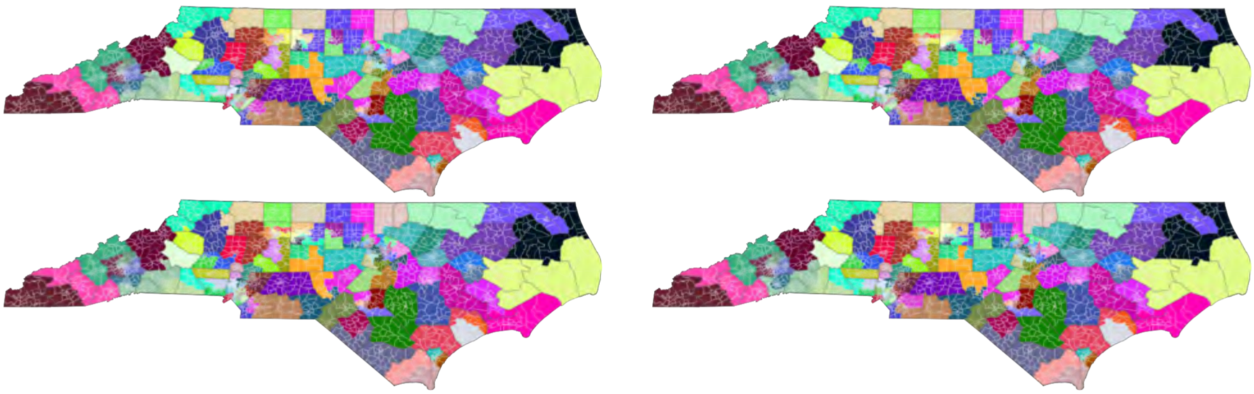
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999996%	5	99.99999927%	9	99.9999987%	13	99.9999978%
2	99.99999982%	6	99.999999941%	10	99.9999966%	14	99.9999986%
3	99.999987%	7	99.9999971%	11	99.9999963%	15	99.99999975%
4	99.9999912%	8	99.9999988%	12	99.99999928%	16	99.9999968%



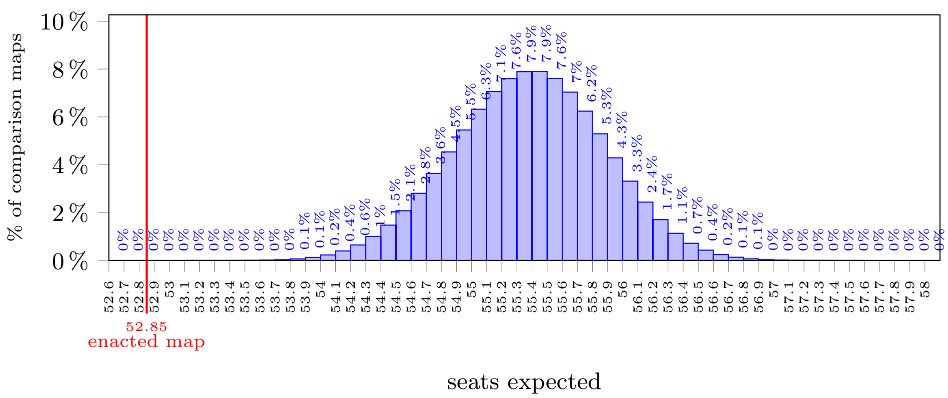
E.4 Compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the House map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

E.4.1 Comparison map examples



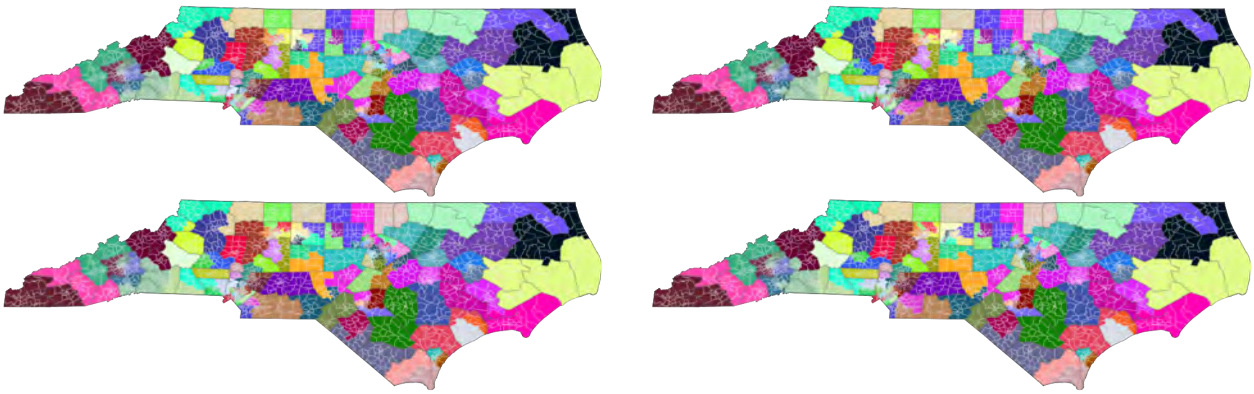
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999904%	5	99.9999989%	9	99.999999917%	13	99.9999983%
2	99.999999957%	6	99.9999971%	10	99.9999983%	14	99.99999989%
3	99.9999948%	7	99.9999999916%	11	99.999988%	15	99.99999962%
4	99.9999987%	8	99.9999955%	12	99.9999922%	16	99.9999974%



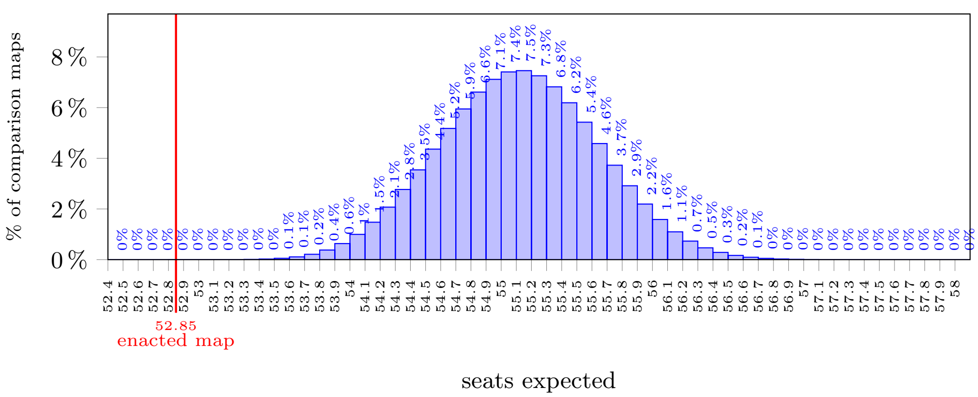
E.5 Compactness 5% Perimeter compactness

Here I show results when my analysis of the House map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

E.5.1 Comparison map examples



Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99985%	5	99.999957%	9	99.999988%	13	99.999953%
2	99.999977%	6	99.999976%	10	99.999978%	14	99.99991%
3	99.99988%	7	99.9999904%	11	99.999968%	15	99.999981%
4	99.999978%	8	99.999951%	12	99.999925%	16	99.99995%



I hereby certify that the foregoing statements are true and correct to the best of my knowledge, information, and belief.

Wesley Pegden
12/23/2021

Wesley Pegden

Contact Information

Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15217

office: 412 268 9782
cell: 412 708 3772
wes@math.cmu.edu
http://math.cmu.edu/~wes

Current Position

Carnegie Mellon University, Pittsburgh, PA

Associate Professor, 2017–present
Assistant Professor, 2013–2017

Postdoctoral

Courant Institute, New York, NY

NSF Postdoctoral Fellow, 2010–2013

Education

Rutgers University, New Brunswick, NJ

Ph.D., May 2010.
Advisor: József Beck
Thesis: "Graphs, games and geometry".

Budapest Semesters in Mathematics, Budapest, Hungary: 2004–2005

University of Chicago: 2001–2004.
BA in Mathematics, with Honors.

Grants, Fellowships, and Awards

National Academy of Science Kavli Fellow
NSF Grant DMS-1700365 (2017–2020)
Sloan Fellowship (2016–2018)
NSF Grant DMS-1363136 (2014–2017)
NSF Postdoctoral Research Fellowship (2010–2013)
Torrey Fellow (Rutgers, 2005–2007)

Publications

Separating effect from significance in Markov chain tests, with M. Chikina, A. Frieze, J. Mattingly.
in *Statistics and Public Policy* **7** (2020) 101-114.

On random multi-dimensional assignment problems, with A. Frieze, T. Tkocz
in *Discrete Applied Mathematics*, **287** 1-9 (2020).

Stability of patterns in the Abelian sandpile, with C. Smart.
Annales Henri Poincaré **21**, pages 1383–1399 (2020).

A note on the localization number of random graphs: diameter two case, with A. Frieze, A. Dudek.
Discrete Applied Mathematics **254** (2019), 107-112.

On the rank of a random binary matrix, with A. Frieze, C. Cooper.
Proceedings of SODA 2019.

A note on dispersing particles on a line, with A. Frieze.
Random Structures & Algorithms **53** (2018) 586-591.

On the distribution of the minimum weight clique, with A. Frieze, G. Sorkin
SIAM Journal on Discrete Mathematics **32**, 2115-2133.

Diffusion limited aggregation on the Boolean lattice, with A. Frieze.
Annals of Applied Probability **28** (2018), 3528-3557.

Online purchasing under uncertainty, with A. Frieze.
Random Structures & Algorithms **53** (2018).

Constraining the clustering transition for colorings of sparse random graphs, with M. Anastos, A. Frieze.
Electronic Journal of Combinatorics **25** (2018)

Assessing significance in a Markov chain without mixing, with M. Chikina and A. Frieze.
Proceedings of the National Academy of Sciences **114** (2017) 2860-2864.

Looking for vertex number one, with A. Frieze.
Annals of Applied Probability **27** (2017) 582-630.

The Apollonian structure of integer superharmonic matrices, with L. Levine and C. Smart.
Annals of Mathematics **186** (2017) 1-67.

Traveling in randomly embedded random graphs, with A. Frieze.
RANDOM 2017. Preprint at `http://arxiv.org/abs/1411.6596`

Apollonian structure in the Abelian Sandpile, with L. Levine and C. Smart.
GAFA **26** (2016) 306-336.

Separating subadditive Euclidean functionals, with A. Frieze.
STOC 2016. Journal version is *Random Structures & Algorithms* **51** (2017) 375-403.

Between 2- and 3-colorability, with A. Frieze.
Electronic Journal of Combinatorics **22** #P1.34 (2015).

Walker-Breaker games, with L. Espig, A. Frieze, and M. Krivelevich.
SIAM J. Discrete Math. **29** (2015).

The topology of competitively constructed graphs, with A. Frieze.
The Electronic Journal of Combinatorics **21** #P2.26.

Convergence of the Abelian Sandpile, with Charles K. Smart.
Duke Mathematical Journal **162** (2013) 627–642.

An extension of the Moser-Tardos algorithmic Local Lemma.
SIAM J. Discrete Math. **28** (2014)

Critical graphs without triangles: an optimum density construction.
Combinatorica **33** (2013) 495–512

The Lefthanded Local Lemma characterizes chordal dependency graphs.
Random Structures & Algorithms **41** (2012) 546–556

Highly nonrepetitive sequences: winning strategies from the Local Lemma.
Random Structures & Algorithms **38** (2011)

Sets resilient to erosion.
Advances in Geometry **11** (2011) 201–224.

The Hales-Jewett number is exponential, with J. Beck and S. Vijay.
Analytic Number Theory: Essays in Honour of Klaus Roth (Eds: W.W.L. Chen, W.T. Gowers, H. Halberstam, W.M. Schmidt, R.C. Vaughan), Cambridge University Press 2009.

A finite goal set in the plane which is not a winner.
Discrete Mathematics **308** (2008) 6546–6551.

Distance Sequences in Locally Infinite Vertex-Transitive Digraphs.
Combinatorica **26** (2006) 577–585.

PREPRINTS

Extremal Collections of k-Uniform Vectors, with J. Briggs
Submitted. Preprint at `https://arxiv.org/abs/1801.09609`.

A partisan districting protocol with provably nonpartisan outcomes, with A. D. Procaccia, D. Yu.
Submitted. Preprint at `https://arxiv.org/abs/1710.08781`.

On the cover time of the emerging giant, with A. Frieze, T. Tkocz
Submitted. Preprint at `https://arxiv.org/abs/1808.09608`.

Minors of a random binary matroid, with C. Cooper and A. Frieze.
Submitted. Preprint at `https://arxiv.org/abs/1612.02084`

Scalefree hardness of average-case Euclidean TSP approximation, with A. Frieze.
Preprint at `http://arxiv.org/abs/1604.04549`

COVERAGE IN POPULAR MEDIA

The amazing, autotuning sandpile. Article in *Nautilus* magazine (2015) by J. Ellenberg.
`http://nautil.us/issue/23/dominoes/the-amazing-autotuning-sandpile.`

Das Wahnsinnsamt, Sandhäufchen und Apollonische Dreiecke. Article in *Spektrum der Wissenschaft* (German-language edition of Scientific American) by C. Pöppe, pages 67–71, August 2015.

Piling on and on and on.... Article and podcast interview by M. Breen, at `http://www.ams.org/samplings/mathmoments/mm117-sandpiles-podcast` (2015).

On sandpiles. Coverage in *AMS Math in the media* column, by Allyn Jackson (April 2015).

Math and the gerrymander. Coverage in *AMS Math in the media* column, by Tony Phillips (March 2017).

Study: Math proves Pennsylvania's congressional districts 'almost certainly' gerrymandered. Story in *Philly Voice* by Daniel Craig (March 1, 2017).

Where Allegheny County's Harrisburg delegation stands on redistricting reform. Story in *The Incline* by Sarah Anne Hughes (March 23, 2017).

Groups sue Pennsylvania over congressional district map, citing gerrymandering. Story in the *Pittsburgh Post-Gazette* by Chris Potter (June 15, 2017).

Cake-cutting game theory trick could stop gerrymandering. Story in the *New Scientist* by Timothy Revell (November 1, 2017).

Top 100 Science stories of 2017 (**#17: fighting politics with math**). *Discover Magazine*, by Stephen Ornes, January-February 2018 issue.

TALKS

Social Science Applications Forum, Center of Mathematical Sciences and Applications, Harvard University, March 11, 2019.

The Statistical and Applied Mathematical Sciences Institute at Duke University, October 8 2018.

[*Colloquium*] University of Wisconsin, February 9 2018.

[*Colloquium*] University of Toronto, January 17 2018.

[*Colloquium*] Duke University, January 11 2018.

SIAM annual meeting, Random Structures mini-session, April 20, 2017.

Ohio State Discrete Math Seminar, April 20, 2017, at Ohio State University.

Atlanta Lecture Series in Combinatorics and Graph Theory XVIII, October 22-23, 2016, at Emory University.

Princeton Discrete Mathematics Seminar, October 13, 2016, at Princeton University.

[*Conference*]*STOC 2016*, June 19 2016, in Boston, MA.

Princeton Discrete Mathematics Seminar, March 10, 2016, at Princeton University.

University of Chicago Theory Seminar, October 20, 2015 at the University of Chicago.

CMU CS Theory Seminar, May 14, 2015.

[*Colloquium*] University of Geneva, March 5 2015.

Ohio State Discrete Math Seminar, November 6 2014, at Ohio State University.

[*Conference*] *SIAM DM14, Special Session on Combinatorics and Statistical Mechanics*, June 18 2014 in Minneapolis, MN (2 talks)

Princeton Discrete Math Seminar, March 13 2014, at Princeton University.

AIM workshop: Generalizations of chip-firing and the critical group, July 2013 at AIM.

[*Conference*] *Special Session on Combinatorics and Classical Integrability* at the AMS Spring Eastern Sectional Meeting, April, 2013 at Boston College.

[*Colloquium*] University of Illinois at Urbana-Champaign, January 30, 2013.

[*Colloquium*] CMU, January 16, 2013.

[*Colloquium*] University of Illinois at Chicago, December 5, 2012.

Cornell Workshop on Sandpiles and Number Theory, October 2012 at Cornell University in Ithaca, NY.

MIT Combinatorics Seminar, April 27, 2012, at MIT.

UPenn seminar on Combinatorics and Probability, February 21, 2012, at the University of Pennsylvania.

Rutgers Discrete Math Seminar, February 7, 2012, at Rutgers University in New Brunswick.

Princeton Discrete Math Seminar, September 27, 2012 at Princeton University.

Probabilistic Combinatorics Mini-symposium of SIAM DM12, June 19, 2012 in Halifax, Nova Scotia.

[*Conference*] *the 15th conference on Random Structures & Algorithms*, May 24, 2012, at Emory University.

Columbia Discrete Math Seminar, February 14, 2012 at Columbia University in New York, NY.

Rutgers Discrete Math Seminar, February 1, 2011 at Rutgers University in New Brunswick.

New York Number Theory Seminar, November 4, 2010.

Columbia Discrete Math Seminar, October 27, 2009 at Columbia University in New York, NY.

Princeton Discrete Math Seminar, October 22, 2009 at Princeton University.

The 14th International Conference on Random Structures and Algorithms, in Poznań, Poland, August 2009.

[*Conference*] *Special Session on Probabilistic and Extremal Combinatorics*, at the 2009 AMS Spring Sectional Meeting, UIUC in Urbana-Champaign, IL.

Rutgers Discrete Mathematics Seminar, April 28 at Rutgers University in New Brunswick.

[*Conference*] *National AMS meeting*, in Washington, DC, January 2009.

Rutgers Experimental Mathematics Seminar, February 7 at Rutgers University in New Brunswick.

Princeton Discrete Mathematics Seminar, in December 2007 at Princeton University.

[*Conference*] *Workshop on Extremal Combinatorics*, Alfred Renyi Institute of Mathematics, in Budapest, Hungary, June 2007.

4.1 First level analysis

The first level of my analysis simply uses the above procedure to generate a large set of comparison districtings against which one can compare the enacted plan. For example, for the Congressional districting, I conducted 32 runs of the above procedure. A "run" in this context consists of a single consecutive sequence of small random changes to the enacted plan, producing a set of comparison districtings. For example, for the Congressional districting, each run consisted of carrying out Steps 1 and 2 in the procedure above $2^{40} \approx 1$ trillion times. As discussed in later sections, these comparison maps adhere to districting criteria in ways that constrain them to be similar in several respects to the enacted map being evaluated. For example, the comparison districtings will preserve the same counties and municipalities preserved by the enacted plan.

In total for this districting, I conducted 32 such runs. I then show the results of these runs in a table, like this:

Congressional districting

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999947%	9	99.9999909%	17	99.9999955%	25	99.999995%
2	99.999968%	10	99.99999966%	18	99.9999973%	26	99.9999961%
3	99.9999988%	11	99.9999943%	19	99.99999972%	27	99.99999977%
4	99.99999931%	12	99.999988%	20	99.9999999981%	28	99.99999979%
5	99.99999999927%	13	99.999988%	21	99.9999999962%	29	99.9999981%
6	99.9999959%	14	99.9999987%	22	99.99999919%	30	99.9999941%
7	99.99999984%	15	99.999996%	23	99.9999908%	31	99.99999901%
8	99.9999999947%	16	99.999985%	24	99.999981%	32	99.9999969%

The random choice of my uniform swing is made from a normal distribution whose standard deviation is 4 percentage points, which is roughly the standard deviation of the swing in the past five North Carolina gubernatorial elections. The Figure 1 visualizes the probabilities that this distribution assigns to the various seat splits which would arise from the enacted Congressional map under uniform swings of the 2020 Attorney General election:

– Ex. 4591 –

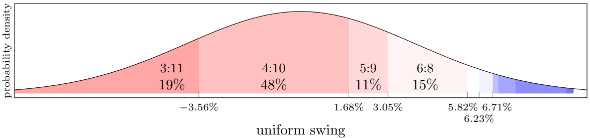
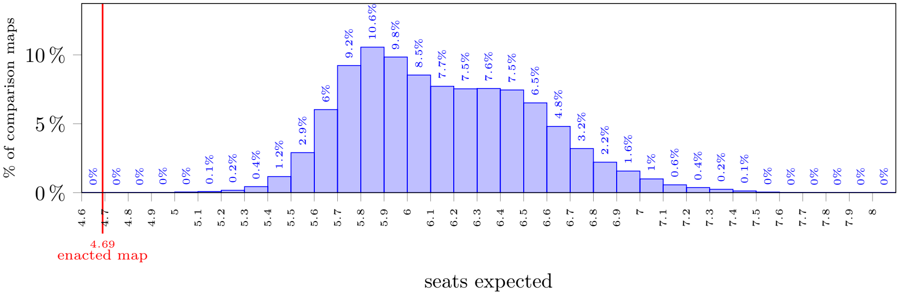


Figure 1: A normally distributed uniform swing applied to the enacted Congressional districting.

In particular, we can list the probability of any number of Democratic seats for the enacted Congressional plan according to this uniform swing model using the 2020 Attorney General race:

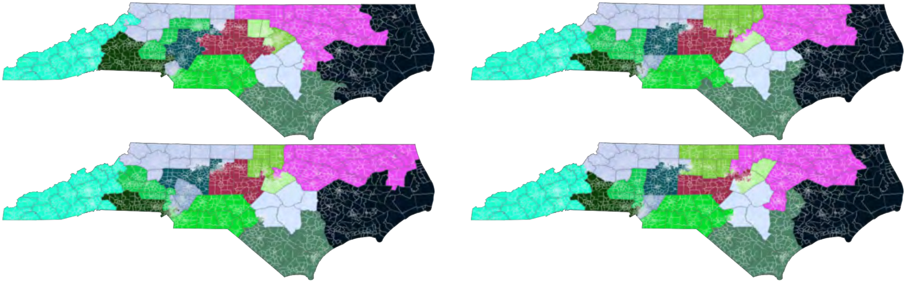
– Ex. 4592 –

This "seats expected" number for the Congressional plan shows up in our analysis page for the Congressional districting (page 13), in a histogram we reproduce here for the purpose of illustration:



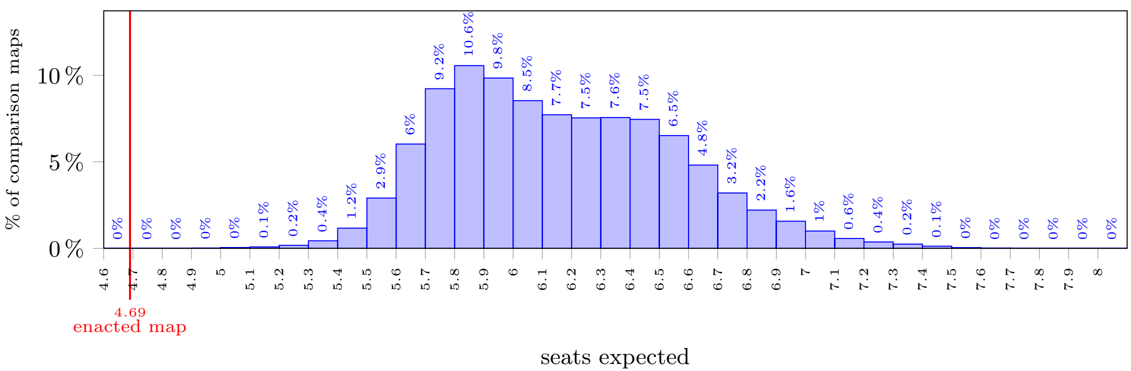
6.1 Congressional districting

6.1.1 Comparison map examples



6.1.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999947%	9	99.9999909%	17	99.9999955%	25	99.999995%
2	99.999968%	10	99.99999966%	18	99.9999973%	26	99.9999961%
3	99.9999988%	11	99.9999943%	19	99.99999972%	27	99.99999977%
4	99.99999931%	12	99.999988%	20	99.9999999981%	28	99.99999979%
5	99.99999999927%	13	99.999988%	21	99.9999999962%	29	99.9999981%
6	99.9999959%	14	99.9999987%	22	99.99999919%	30	99.9999941%
7	99.99999984%	15	99.999996%	23	99.9999908%	31	99.99999901%
8	99.9999999947%	16	99.999985%	24	99.999981%	32	99.9999969%

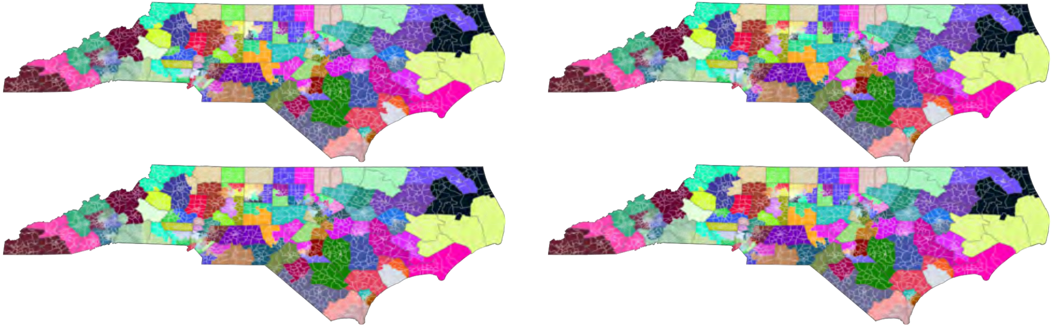


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000031% of districtings (in other words, 99.999968% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted House districting is among the most optimized-for-partisanship 0.000094% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999905% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000031\%$.

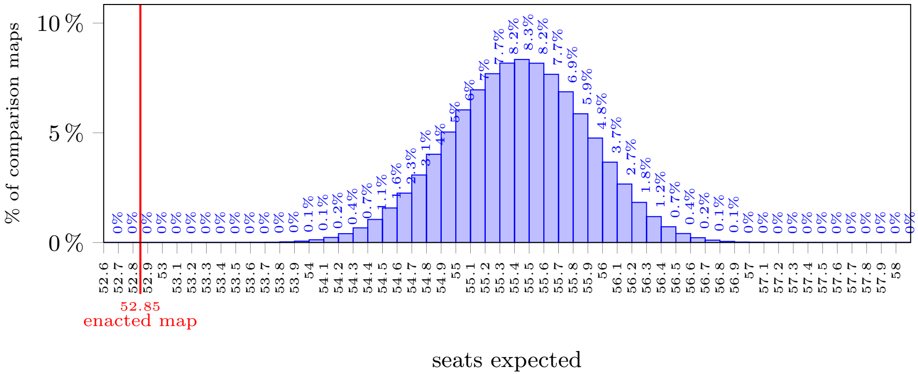
6.2 House districting

6.2.1 Comparison map examples



6.2.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999999985%	9	99.99999957%	17	99.9999989%	25	99.9999989%
2	99.99999942%	10	99.99999904%	18	99.99999966%	26	99.9999918%
3	99.99999997%	11	99.9999984%	19	99.99999982%	27	99.99999984%
4	99.9999969%	12	99.9999986%	20	99.9999986%	28	99.9999988%
5	99.9999975%	13	99.99999989%	21	99.9999935%	29	99.99999987%
6	99.9999999959%	14	99.99999996%	22	99.9999999967%	30	99.99999908%
7	99.999999985%	15	99.9999984%	23	99.9999975%	31	99.9999966%
8	99.999999951%	16	99.99999954%	24	99.999999939%	32	99.999999939%

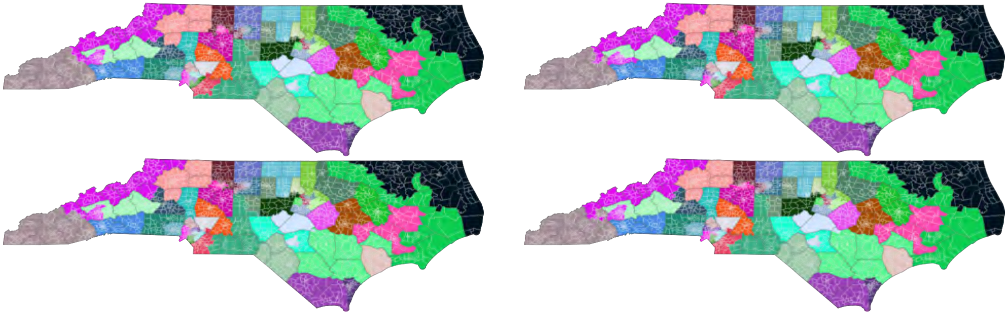


- **First level analysis:** In *every* run, the districting was in the most partisan 0.0000081% of districtings (in other words, 99.9999918% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000024% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999975% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0000081\%$.

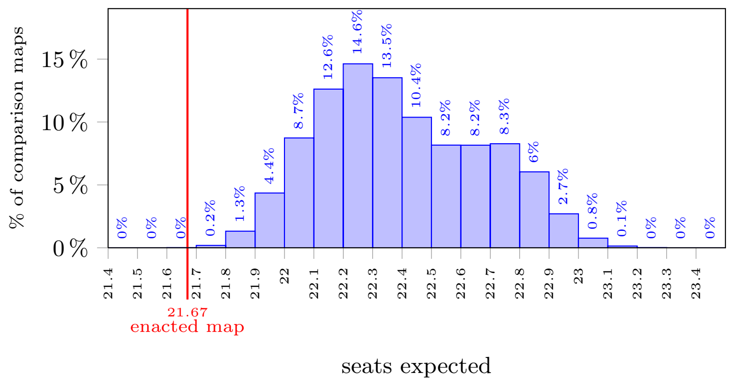
6.3 Senate districting

6.3.1 Comparison map examples



6.3.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.988%	9	99.9974%	17	99.9977%	25	99.998%
2	99.9988%	10	99.9958%	18	99.9987%	26	99.9948%
3	99.9938%	11	99.9985%	19	99.9988%	27	99.987%
4	99.9981%	12	99.9957%	20	99.978%	28	99.9988%
5	99.9929%	13	99.988%	21	99.9982%	29	99.9979%
6	99.9916%	14	99.989%	22	99.9978%	30	99.9981%
7	99.9957%	15	99.9974%	23	99.9976%	31	99.99914%
8	99.9973%	16	99.997%	24	99.9975%	32	99.9978%

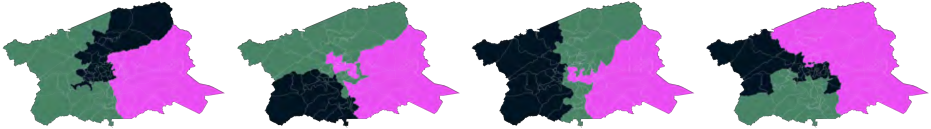


- **First level analysis:** In *every* run, the districting was in the most partisan 0.021% of districtings (in other words, 99.978% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.065% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.934% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.021\%$.

– Ex. 4599 –

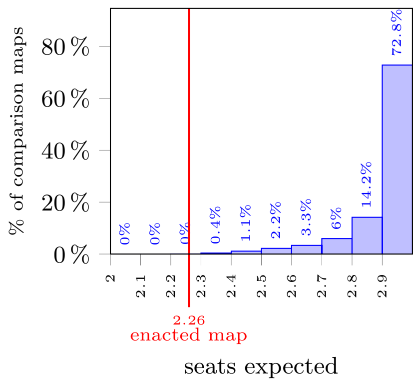
6.4 House Cluster: Buncombe

6.4.1 Comparison map examples



6.4.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.979%	9	99.979%	17	99.979%	25	99.98%
2	99.98%	10	99.98%	18	99.979%	26	99.979%
3	99.98%	11	99.98%	19	99.98%	27	99.979%
4	99.98%	12	99.98%	20	99.98%	28	99.98%
5	99.98%	13	99.98%	21	99.98%	29	99.98%
6	99.979%	14	99.98%	22	99.98%	30	99.98%
7	99.98%	15	99.98%	23	99.98%	31	99.979%
8	99.979%	16	99.98%	24	99.98%	32	99.979%

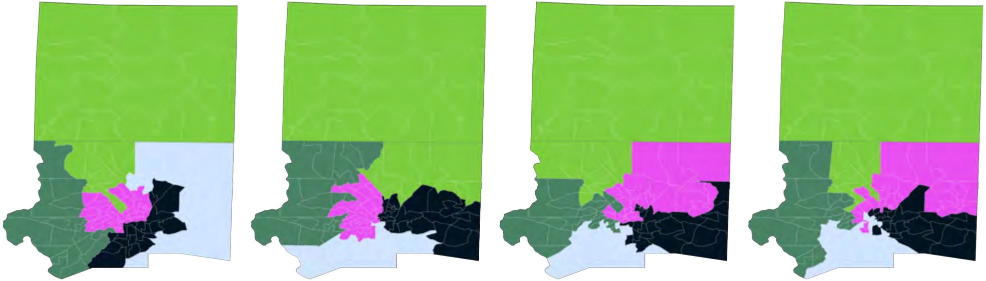


- **First level analysis:** In *every* run, the districting was in the most partisan 0.020% of districtings (in other words, 99.979% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.061% of all alternative districtings satisfying my districting criteria (in other words, 99.938% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.020\%$.

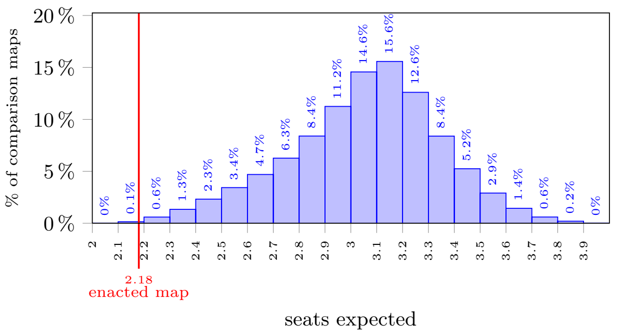
6.6 House Cluster: Forsyth-Stokes

6.6.1 Comparison map examples



6.6.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.913%	9	99.912%	17	99.915%	25	99.914%
2	99.914%	10	99.914%	18	99.914%	26	99.913%
3	99.917%	11	99.912%	19	99.916%	27	99.914%
4	99.916%	12	99.912%	20	99.914%	28	99.912%
5	99.913%	13	99.914%	21	99.913%	29	99.915%
6	99.913%	14	99.914%	22	99.914%	30	99.914%
7	99.913%	15	99.912%	23	99.914%	31	99.917%
8	99.913%	16	99.916%	24	99.915%	32	99.915%

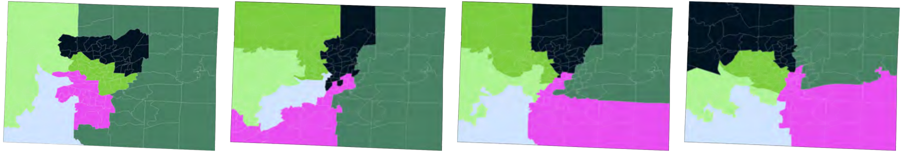


- **First level analysis:** In *every* run, the districting was in the most partisan 0.087% of districtings (in other words, 99.912% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.26% of all alternative districtings satisfying my districting criteria (in other words, 99.73% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.087\%$.

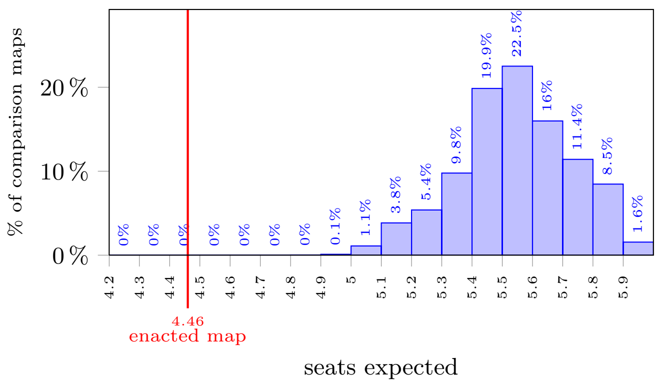
6.7 House Cluster: Guilford

6.7.1 Comparison map examples



6.7.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999989%	9	99.999982%	17	99.999979%	25	99.999972%
2	99.999982%	10	99.999979%	18	99.999978%	26	99.999979%
3	99.999972%	11	99.999978%	19	99.999981%	27	99.999978%
4	99.999986%	12	99.999981%	20	99.999984%	28	99.999979%
5	99.999975%	13	99.999986%	21	99.999983%	29	99.999982%
6	99.999982%	14	99.99998%	22	99.999979%	30	99.999982%
7	99.999981%	15	99.99997%	23	99.999983%	31	99.999982%
8	99.999982%	16	99.999976%	24	99.999981%	32	99.999984%

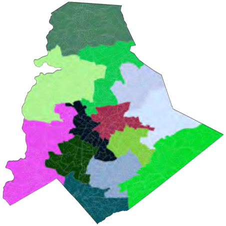
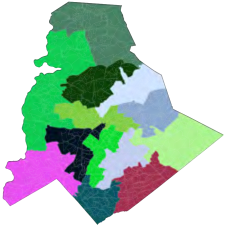
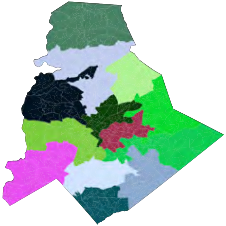


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000029% of districtings (in other words, 99.99997% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000089% of all alternative districtings satisfying my districting criteria (in other words, 99.99991% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000029\%$.

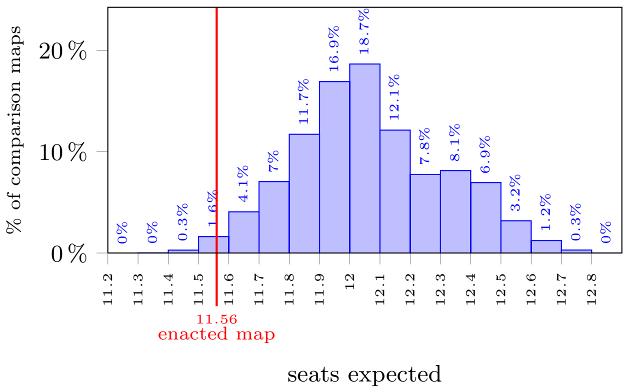
6.8 House Cluster: Mecklenburg

6.8.1 Comparison map examples


6.8.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	98.7%	9	98.6%	17	98.4%	25	98.9%
2	99.36%	10	99.15%	18	99.%	26	98.3%
3	98.7%	11	98.7%	19	98.4%	27	98.8%
4	99.14%	12	99.17%	20	99.17%	28	98.5%
5	98.4%	13	99.05%	21	98.8%	29	99.08%
6	99.33%	14	99.02%	22	98.9%	30	98.9%
7	98.5%	15	99.%	23	98.9%	31	99.12%
8	98.9%	16	99.17%	24	98.9%	32	99.2%

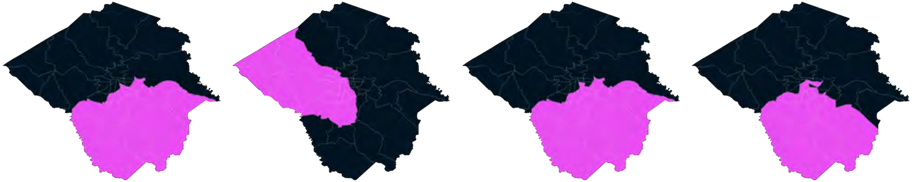


- **First level analysis:** In *every* run, the districting was in the most partisan 1.7% of districtings (in other words, 98.3% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 5.0% of all alternative districtings satisfying my districting criteria (in other words, 95.0% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 1.7\%$.

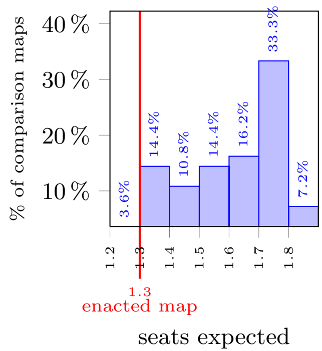
6.9 House Cluster: Pitt

6.9.1 Comparison map examples



6.9.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	96.3%	9	96.4%	17	96.3%	25	96.4%
2	96.3%	10	96.3%	18	96.3%	26	96.3%
3	96.4%	11	96.4%	19	96.3%	27	96.4%
4	96.4%	12	96.4%	20	96.3%	28	96.3%
5	96.4%	13	96.4%	21	96.3%	29	96.4%
6	96.3%	14	96.3%	22	96.4%	30	96.3%
7	96.3%	15	96.3%	23	96.4%	31	96.4%
8	96.3%	16	96.4%	24	96.4%	32	96.4%

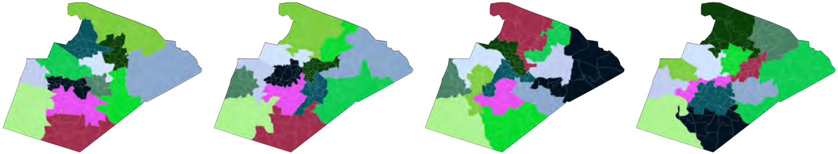


- **First level analysis:** In *every* run, the districting was in the most partisan 3.6% of districtings (in other words, 96.3% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 11% of all alternative districtings satisfying my districting criteria (in other words, 89.1% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 3.6\%$.

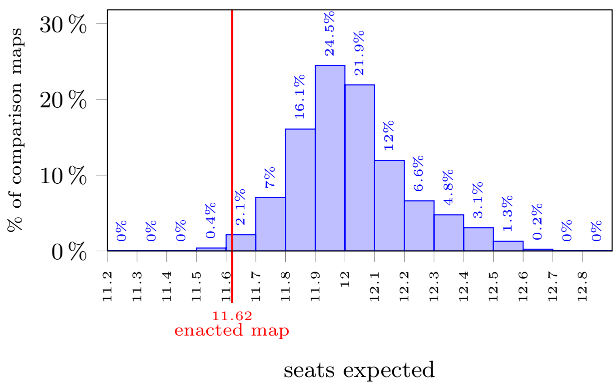
6.10 House Cluster: Wake

6.10.1 Comparison map examples



6.10.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.38%	9	99.34%	17	99.37%	25	99.35%
2	99.33%	10	99.35%	18	99.36%	26	99.36%
3	99.34%	11	99.33%	19	99.33%	27	99.34%
4	99.32%	12	99.34%	20	99.35%	28	99.33%
5	99.35%	13	99.34%	21	99.33%	29	99.35%
6	99.33%	14	99.27%	22	99.31%	30	99.36%
7	99.34%	15	99.34%	23	99.32%	31	99.36%
8	99.34%	16	99.36%	24	99.35%	32	99.35%

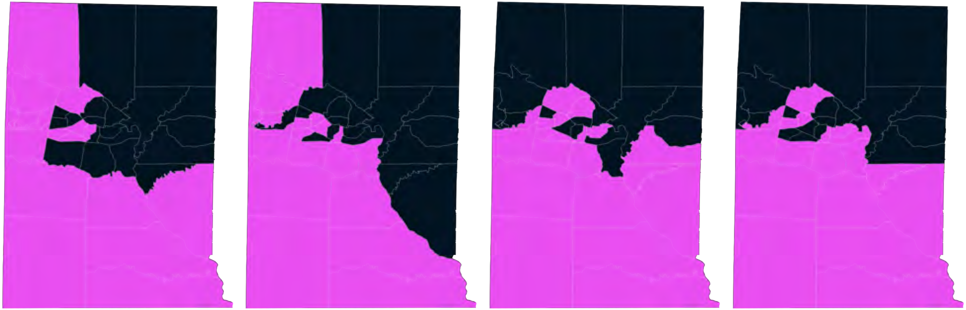


- **First level analysis:** In *every* run, the districting was in the most partisan 0.72% of districtings (in other words, 99.27% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 2.2% of all alternative districtings satisfying my districting criteria (in other words, 97.8% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.72\%$.

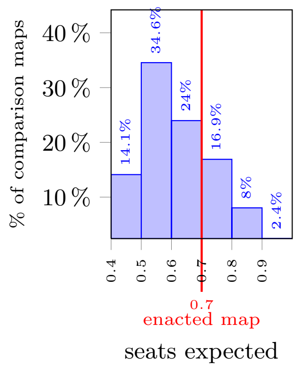
6.11 House Cluster: Alamance

6.11.1 Comparison map examples



6.11.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	26.3%	9	26.4%	17	26.3%	25	26.4%
2	26.3%	10	26.3%	18	26.4%	26	26.3%
3	26.3%	11	26.3%	19	26.3%	27	26.3%
4	26.4%	12	26.3%	20	26.3%	28	26.3%
5	26.4%	13	26.4%	21	26.4%	29	26.3%
6	26.3%	14	26.3%	22	26.4%	30	26.4%
7	26.4%	15	26.3%	23	26.3%	31	26.3%
8	26.4%	16	26.4%	24	26.4%	32	26.4%

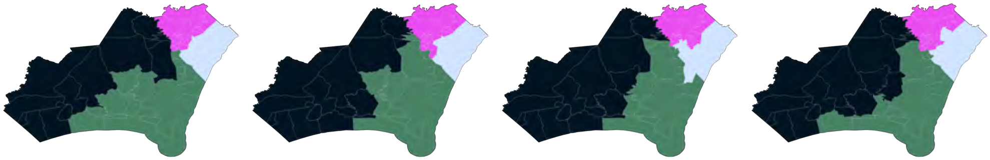


- **First level analysis:** In *every* run, the districting was in the most partisan 74% of districtings (in other words, 26.3% were less partisan, in *every* run).

- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

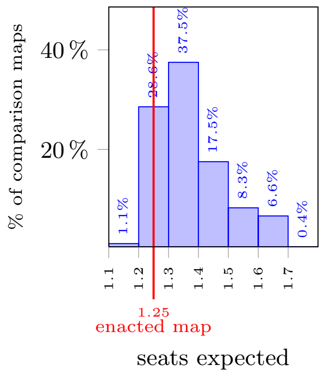
6.12 House Cluster: Brunswick/New Hanover

6.12.1 Comparison map examples



6.12.2 Results

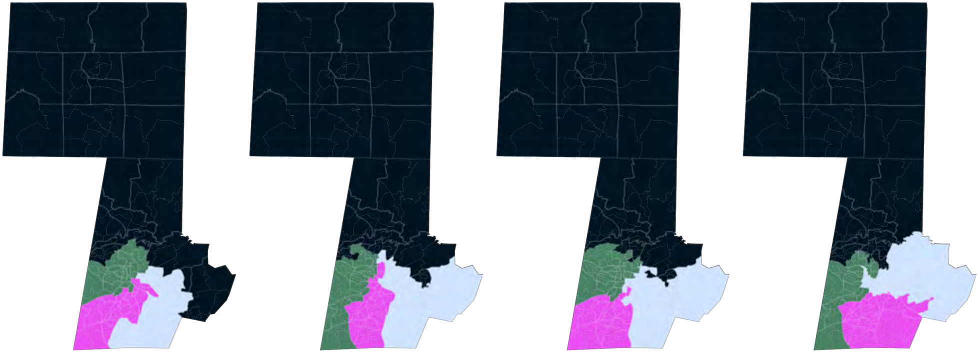
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	89.4%	9	89.5%	17	89.5%	25	89.5%
2	89.4%	10	89.5%	18	89.4%	26	89.5%
3	89.5%	11	89.5%	19	89.5%	27	89.4%
4	89.4%	12	89.4%	20	89.4%	28	89.5%
5	89.4%	13	89.5%	21	89.5%	29	89.5%
6	89.5%	14	89.6%	22	89.5%	30	89.4%
7	89.4%	15	89.5%	23	89.5%	31	89.5%
8	89.5%	16	89.4%	24	89.4%	32	89.5%



- **First level analysis:** In *every* run, the districting was in the most partisan 11% of districtings (in other words, 89.4% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

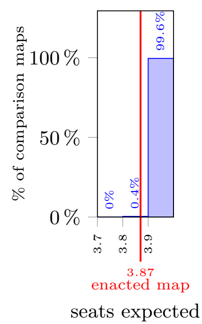
6.13 House Cluster: Durham/Person

6.13.1 Comparison map examples



6.13.2 Results

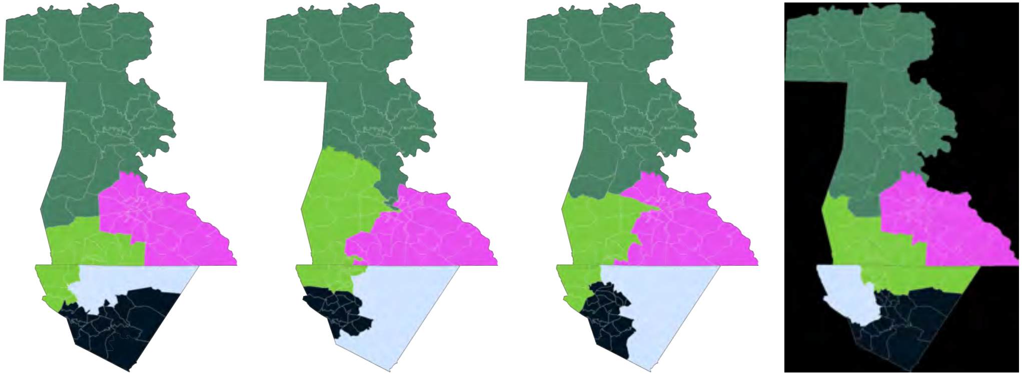
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.936%	9	99.935%	17	99.938%	25	99.935%
2	99.933%	10	99.937%	18	99.937%	26	99.933%
3	99.937%	11	99.94%	19	99.934%	27	99.939%
4	99.932%	12	99.933%	20	99.934%	28	99.936%
5	99.933%	13	99.936%	21	99.936%	29	99.937%
6	99.936%	14	99.935%	22	99.938%	30	99.933%
7	99.937%	15	99.933%	23	99.937%	31	99.94%
8	99.936%	16	99.936%	24	99.934%	32	99.934%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.067% of districtings (in other words, 99.932% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.20% of all alternative districtings satisfying my districting criteria (in other words, 99.79% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.067\%$.

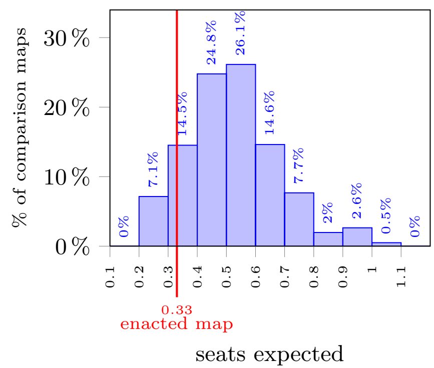
6.14 House Cluster: Cabarrus/Davie/Rowan/Yadkin

6.14.1 Comparison map examples



6.14.2 Results

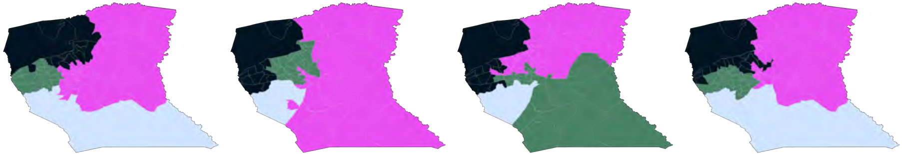
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	89.0%	9	90.0%	17	88.5%	25	89.9%
2	90.0%	10	88.9%	18	89.0%	26	88.6%
3	90.1%	11	88.7%	19	89.4%	27	89.9%
4	88.4%	12	89.8%	20	89.3%	28	88.9%
5	89.7%	13	89.4%	21	92.8%	29	89.5%
6	88.6%	14	89.2%	22	89.1%	30	87.7%
7	89.5%	15	88.8%	23	89.1%	31	90.2%
8	90.0%	16	90.0%	24	88.7%	32	90.4%



- **First level analysis:** In *every* run, the districting was in the most partisan 12% of districtings (in other words, 87.7% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

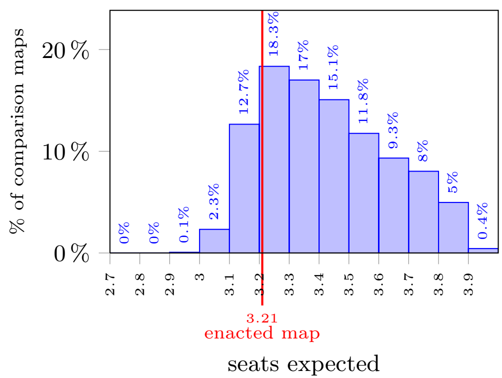
6.15 House Cluster: Cumberland

6.15.1 Comparison map examples



6.15.2 Results

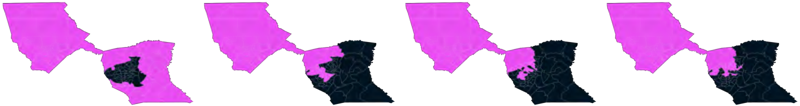
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	83.6%	9	83.8%	17	83.8%	25	84.0%
2	83.7%	10	83.9%	18	83.6%	26	83.5%
3	83.8%	11	83.8%	19	83.7%	27	83.8%
4	83.7%	12	83.6%	20	83.7%	28	83.8%
5	83.6%	13	83.7%	21	84.0%	29	83.7%
6	83.7%	14	83.6%	22	83.9%	30	83.6%
7	83.5%	15	83.8%	23	83.7%	31	83.9%
8	83.7%	16	83.8%	24	83.6%	32	83.9%



- **First level analysis:** In *every* run, the districting was in the most partisan 16% of districtings (in other words, 83.5% were less partisan, in *every* run).
- **Second level analysis:** The enacted map is not unusual enough in the first-level analysis to enable a statistically significant second-level analysis of this cluster.

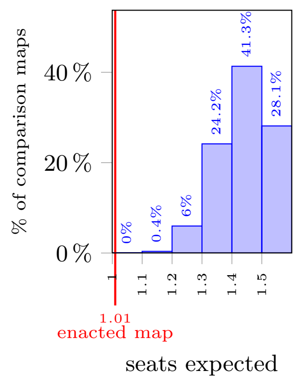
6.16 Senate Cluster: Cumberland Moore

6.16.1 Comparison map examples



6.16.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999968%	9	99.9999962%	17	99.9999963%	25	99.9999954%
2	99.9999961%	10	99.9999965%	18	99.9999969%	26	99.9999955%
3	99.999998%	11	99.9999954%	19	99.9999967%	27	99.999997%
4	99.9999953%	12	99.9999961%	20	99.9999969%	28	99.9999952%
5	99.9999969%	13	99.9999957%	21	99.9999971%	29	99.9999959%
6	99.9999969%	14	99.9999949%	22	99.9999961%	30	99.9999956%
7	99.9999966%	15	99.9999964%	23	99.9999961%	31	99.9999961%
8	99.9999966%	16	99.9999959%	24	99.9999977%	32	99.9999965%

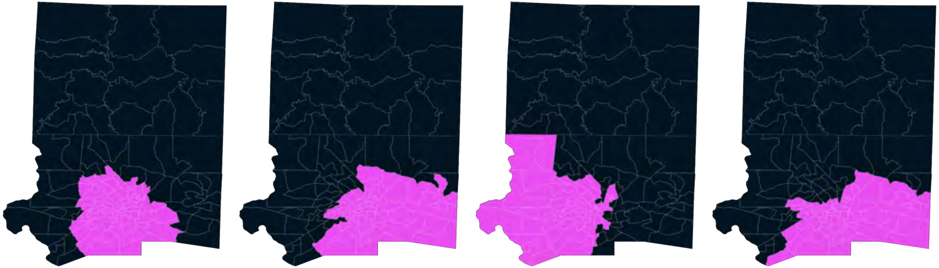


- **First level analysis:** In *every* run, the districting was in the most partisan 0.0000050% of districtings (in other words, 99.9999949% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000015% of all alternative districtings satisfying my districting criteria (in other words, 99.999984% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0000050\%$.

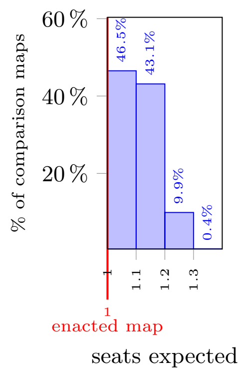
6.17 Senate Cluster: Forsyth-Stokes

6.17.1 Comparison map examples



6.17.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9983%	9	99.9983%	17	99.9983%	25	99.9983%
2	99.9984%	10	99.9984%	18	99.9984%	26	99.9983%
3	99.9982%	11	99.9983%	19	99.9984%	27	99.9983%
4	99.9982%	12	99.9984%	20	99.9983%	28	99.9984%
5	99.9983%	13	99.9983%	21	99.9983%	29	99.9983%
6	99.9984%	14	99.9983%	22	99.9983%	30	99.9984%
7	99.9984%	15	99.9983%	23	99.9983%	31	99.9984%
8	99.9984%	16	99.9984%	24	99.9984%	32	99.9983%

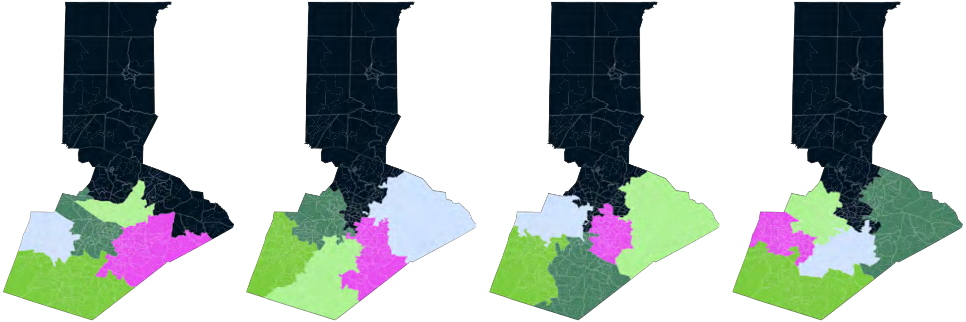


- **First level analysis:** In *every* run, the districting was in the most partisan 0.0016% of districtings (in other words, 99.9983% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.0051% of all alternative districtings satisfying my districting criteria (in other words, 99.9947% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0016\%$.

– Ex. 4625 –

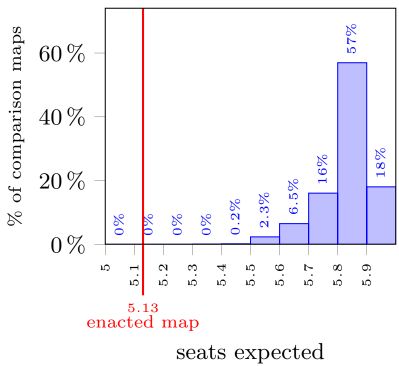
6.18 Senate Cluster: Granville-Wake

6.18.1 Comparison map examples



6.18.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999934%	9	99.99999921%	17	99.99999999936%	25	99.9999971%
2	99.9999984%	10	99.99999999936%	18	99.99999913%	26	99.9999975%
3	99.99999917%	11	99.99999966%	19	99.9999967%	27	99.99999909%
4	99.99999999945%	12	99.9999979%	20	99.99999963%	28	99.999989%
5	99.99999974%	13	99.9999989%	21	99.9999999984%	29	99.99999999954%
6	99.999999939%	14	99.9999976%	22	99.99999948%	30	99.9999968%
7	99.9999999982%	15	99.9999947%	23	99.9999984%	31	99.99999999945%
8	99.9999995%	16	99.99999969%	24	99.99999967%	32	99.99999971%

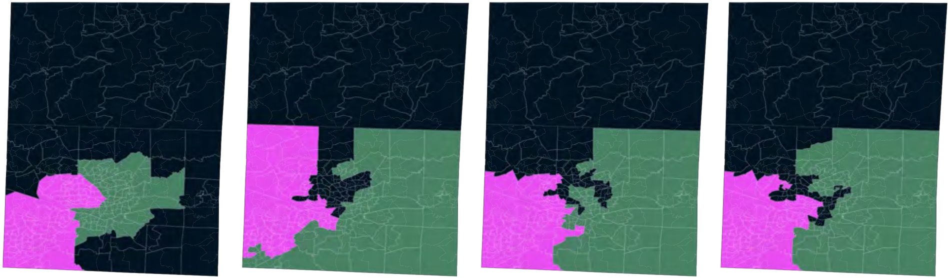


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000010% of districtings (in other words, 99.999989% were less partisan, in *every* run).

- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000030% of all alternative districtings satisfying my districting criteria (in other words, 99.999969% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000010\%$.

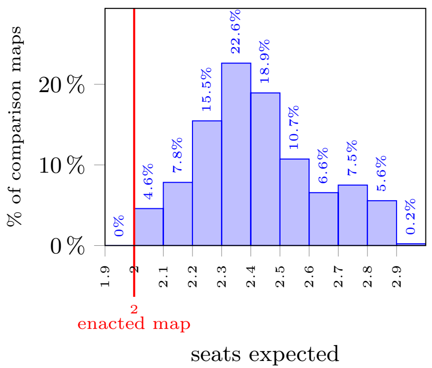
6.19 Senate Cluster: Guilford-Rockingham

6.19.1 Comparison map examples



6.19.2 Results

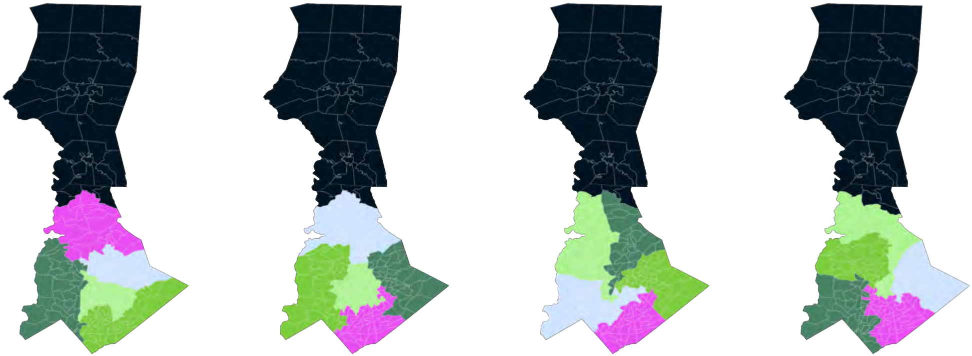
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999979%	9	99.9999971%	17	99.999989%	25	99.999984%
2	99.999975%	10	99.999999976%	18	99.9999929%	26	99.99999949%
3	99.9999991%	11	99.9999944%	19	99.999988%	27	99.999967%
4	99.999984%	12	99.99998%	20	99.99998%	28	99.999995%
5	99.999976%	13	99.9999978%	21	99.99996%	29	99.999957%
6	99.9999922%	14	99.999978%	22	99.999979%	30	99.9999999957%
7	99.9999997%	15	99.999986%	23	99.9999964%	31	99.9999935%
8	99.999967%	16	99.9999939%	24	99.999983%	32	99.9999984%



- **First level analysis:** In *every* run, the districting was in the most partisan 0.000042% of districtings (in other words, 99.999957% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.00012% of all alternative districtings satisfying my districting criteria (in other words, 99.99987% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000042\%$.

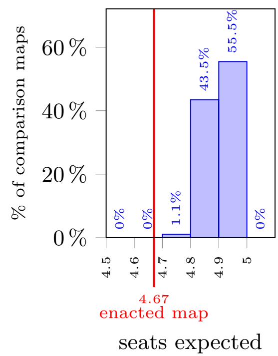
6.20 Senate Cluster: Iredell-Mecklenburg

6.20.1 Comparison map examples



6.20.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9981%	9	99.9983%	17	99.9982%	25	99.9982%
2	99.9982%	10	99.9983%	18	99.9982%	26	99.9983%
3	99.9982%	11	99.9981%	19	99.9981%	27	99.9981%
4	99.9982%	12	99.9982%	20	99.9982%	28	99.9982%
5	99.9981%	13	99.9982%	21	99.9982%	29	99.9982%
6	99.9983%	14	99.9982%	22	99.9982%	30	99.9982%
7	99.9982%	15	99.9982%	23	99.9982%	31	99.9982%
8	99.9982%	16	99.9982%	24	99.9982%	32	99.9981%

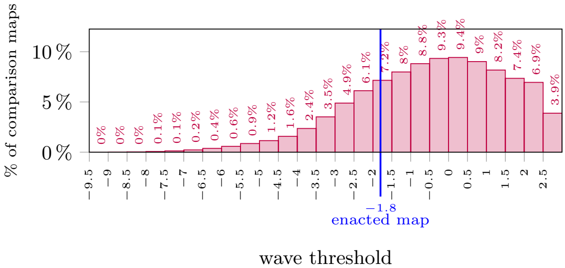


- **First level analysis:** In *every* run, the districting was in the most partisan 0.0019% of districtings (in other words, 99.998% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.0057% of all alternative districtings satisfying my districting criteria (in other words, 99.9943% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.0019\%$.

7.1 Alamance

The comparison maps generated by my algorithm were similar to the enacted map with respect to their wave threshold for both possible seat values (results here shown for the wave threshold for 0 seats):

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	25.2%	9	25.2%	17	25.1%	25	25.2%
2	25.2%	10	25.0%	18	25.1%	26	25.2%
3	25.2%	11	25.1%	19	25.1%	27	25.2%
4	25.2%	12	25.2%	20	25.1%	28	25.2%
5	25.3%	13	25.2%	21	25.3%	29	25.3%
6	25.2%	14	25.2%	22	25.2%	30	25.2%
7	25.2%	15	25.1%	23	25.3%	31	25.3%
8	25.2%	16	25.2%	24	25.2%	32	25.2%



7.2 Brunswick/New Hanover

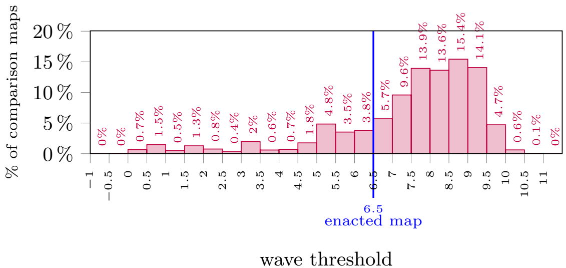
Despite the fact that my algorithm did not detect large differences between the enacted districting and comparison districtings of this cluster, the enacted map is an extreme outlier among the comparison maps generated by my algorithm with respect to the wave threshold for two seats. In particuliar, for the enacted map in this cluster, Democratic performance could increase by 10.1 percentage points in every district without Democrats capturing more than two seats. In every run of my algorithm, 99.72% of comparison maps would allow Democrats to capture a third seat with a smaller wave.

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.987%	9	99.94%	17	99.9956%	25	99.83%
2	99.99%	10	99.907%	18	99.9957%	26	99.79%
3	99.929%	11	99.85%	19	99.8%	27	99.975%
4	99.88%	12	99.9912%	20	99.922%	28	99.85%
5	99.86%	13	99.77%	21	99.961%	29	99.83%
6	99.934%	14	99.89%	22	99.952%	30	99.92%
7	99.73%	15	99.87%	23	99.97%	31	99.946%
8	99.96%	16	99.72%	24	99.911%	32	99.961%

7.3 Cabarrus/Davie/Rowan/Yadkin

The comparison maps generated by my algorithm were similar to the enacted map with respect to their wave threshold for all seat values (results here shown for the wave threshold for 1 seat):

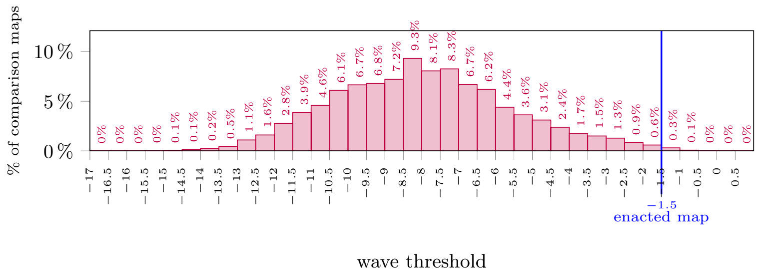
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	26.3%	9	20.7%	17	22.0%	25	22.3%
2	22.6%	10	23.1%	18	21.4%	26	20.8%
3	19.4%	11	27.6%	19	23.3%	27	20.2%
4	20.7%	12	21.2%	20	25.7%	28	22.0%
5	18.8%	13	23.4%	21	21.8%	29	22.1%
6	21.9%	14	25.4%	22	20.8%	30	22.3%
7	24.3%	15	20.0%	23	22.9%	31	22.4%
8	20.4%	16	19.9%	24	23.1%	32	23.8%



7.4 Cumberland

Despite the fact that my algorithm did not detect large differences between the enacted districting and comparison districtings of this cluster, the enacted map is an extreme outlier among the comparison maps generated by my algorithm with respect to the wave threshold for two seats.

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.61%	9	99.62%	17	99.62%	25	99.64%
2	99.64%	10	99.64%	18	99.62%	26	99.63%
3	99.61%	11	99.61%	19	99.61%	27	99.63%
4	99.62%	12	99.62%	20	99.63%	28	99.6%
5	99.59%	13	99.62%	21	99.64%	29	99.63%
6	99.61%	14	99.59%	22	99.63%	30	99.62%
7	99.61%	15	99.62%	23	99.62%	31	99.6%
8	99.61%	16	99.63%	24	99.62%	32	99.62%



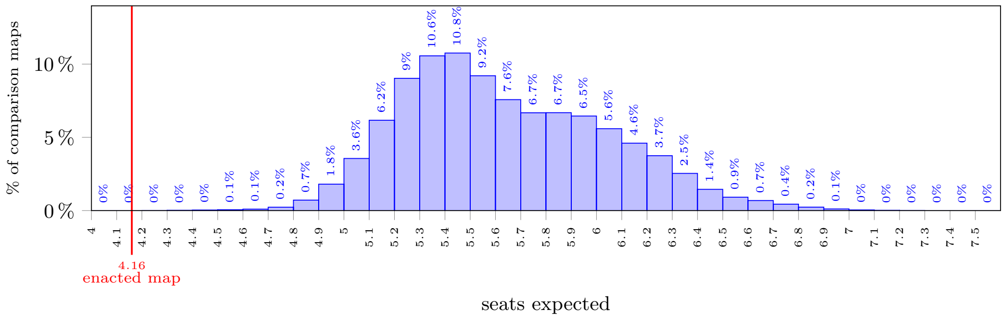
Appendix C Robustness Checks, Congressional districting

C.1 Robustness to election data

Here I show results when my analysis of the Congressional map is repeated with other elections in place of the 2020 Attorney General election as my proxy for partisan voting patterns.

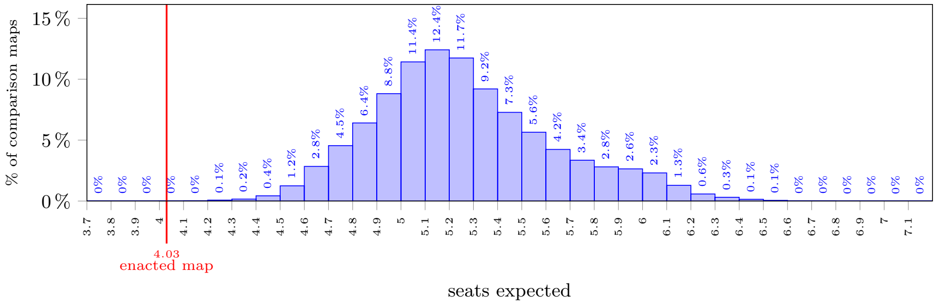
C.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999925%	5	99.999986%	9	99.9999908%	13	99.9999926%
2	99.999921%	6	99.999999968%	10	99.9999932%	14	99.999988%
3	99.9999955%	7	99.999984%	11	99.9999979%	15	99.9999989%
4	99.9999933%	8	99.99995%	12	99.9999999981%	16	99.999978%



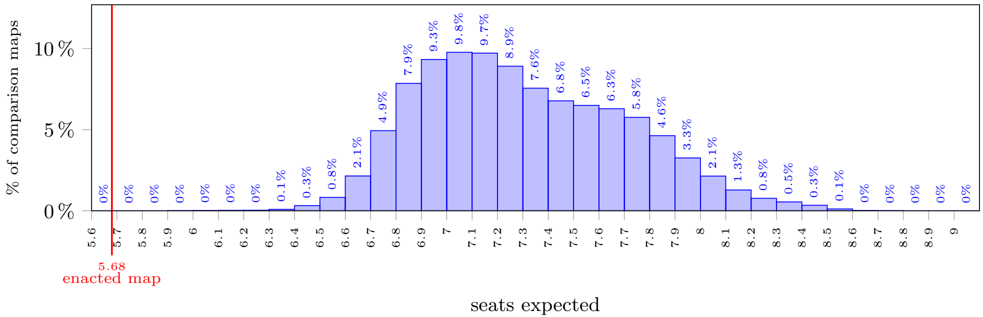
C.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999973%	5	99.999937%	9	99.999942%	13	99.999982%
2	99.99985%	6	99.9999964%	10	99.99901%	14	99.999978%
3	99.999905%	7	99.99954%	11	99.9999928%	15	99.999934%
4	99.999964%	8	99.99975%	12	99.9995%	16	99.9998%



C.1.3 Results with 2020 Governor election

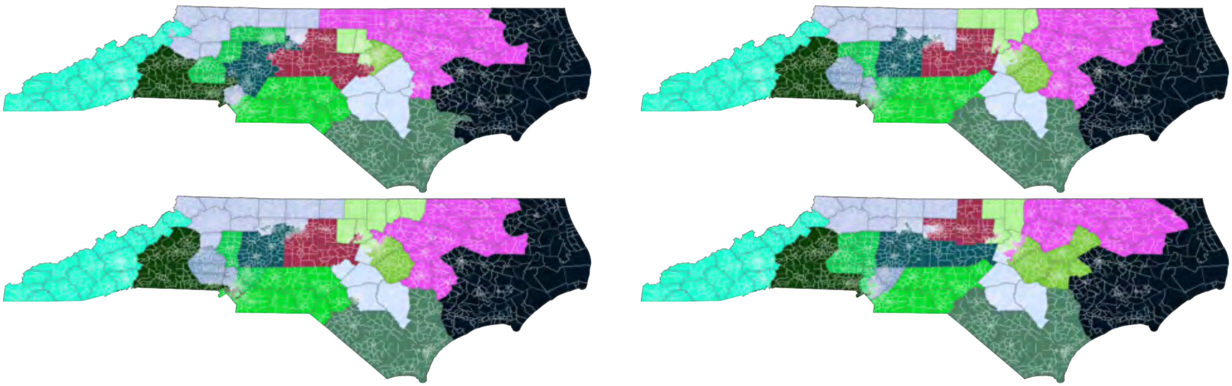
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999989%	5	99.9999979%	9	99.9999975%	13	99.99999923%
2	99.9999914%	6	99.9999999922%	10	99.99999974%	14	99.99999968%
3	99.9999996%	7	99.999999934%	11	99.999999994%	15	99.999999982%
4	99.999999966%	8	99.9999982%	12	99.9999999981%	16	99.999999961%



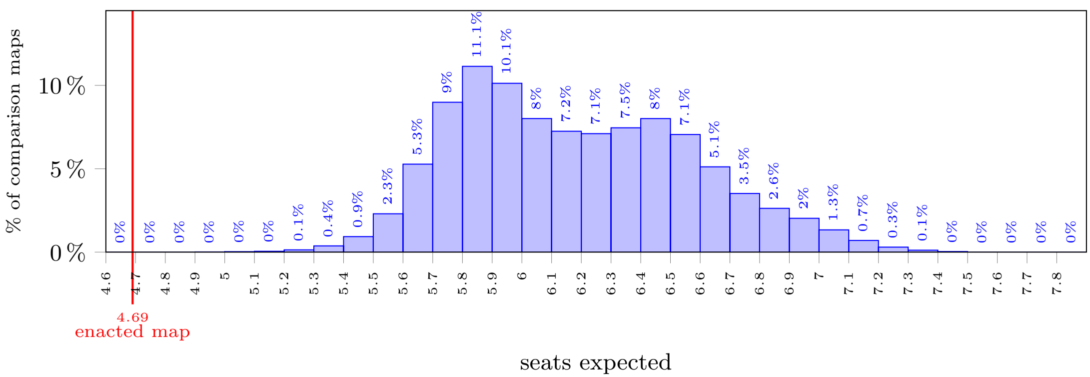
C.2 Robustness to incumbency protection

Here I show results when my analysis of the Congressional map is repeated without ensuring the protection of incumbents.

C.2.1 Comparison map examples



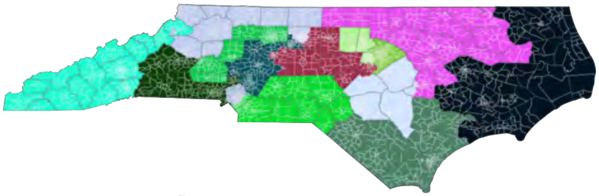
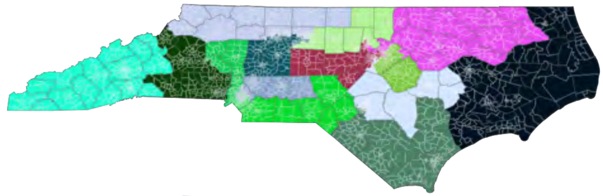
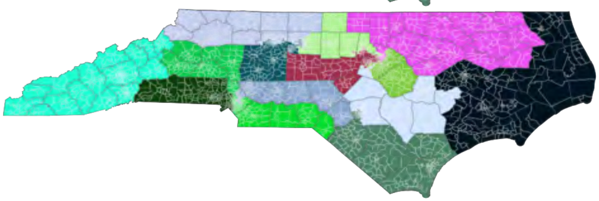
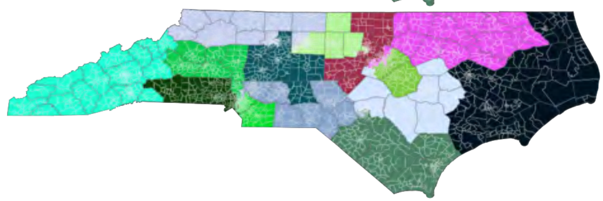
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999998%	5	99.99999918%	9	99.9999976%	13	99.999982%
2	99.999999901%	6	99.9999978%	10	99.999989%	14	99.99999901%
3	99.9999986%	7	99.999999961%	11	99.9999967%	15	99.99999977%
4	99.9999967%	8	99.9999954%	12	99.9999999981%	16	99.9999986%



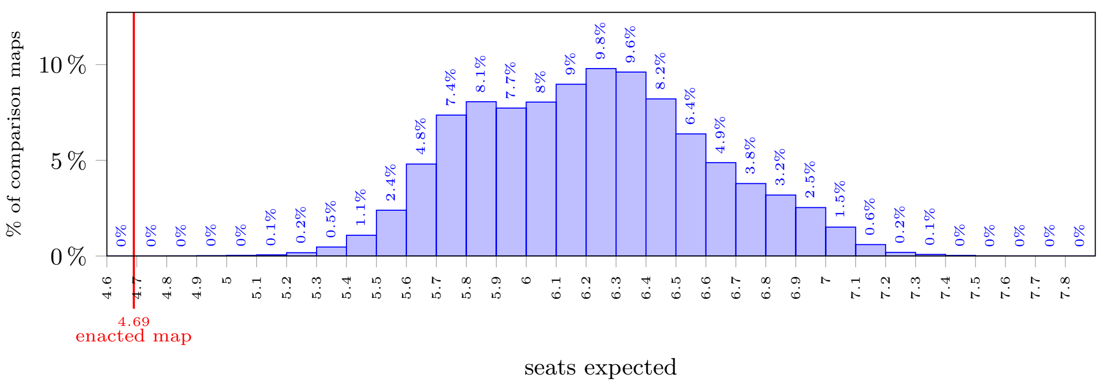
C.3 Robustness to compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the Congressional map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

C.3.1 Comparison map examples

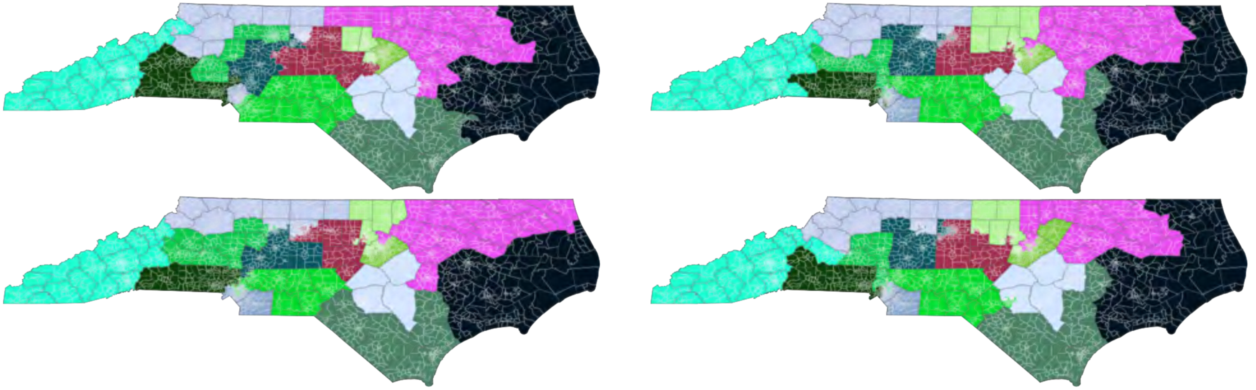
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999989%	5	99.9999997%	9	99.9999975%	13	99.999979%
2	99.9999984%	6	99.99999983%	10	99.9999968%	14	99.9999968%
3	99.9999933%	7	99.9999962%	11	99.9999968%	15	99.9999983%
4	99.999986%	8	99.9999983%	12	99.99999954%	16	99.9999984%



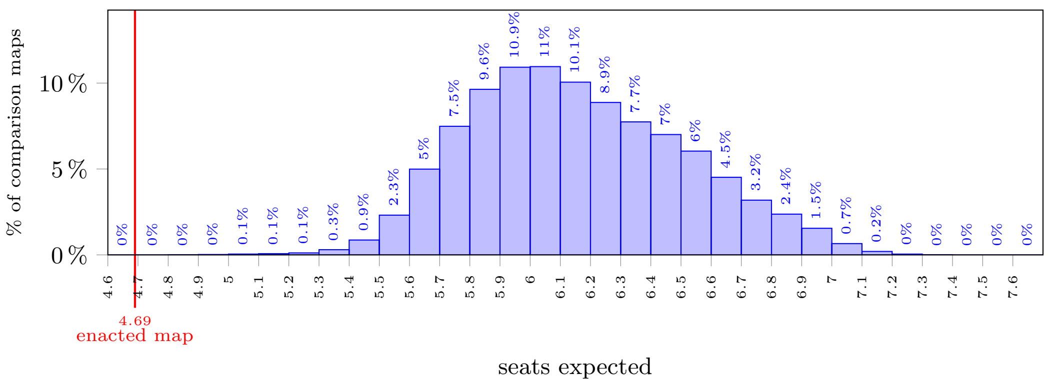
C.4 Robustness to compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the Congressional map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

C.4.1 Comparison map examples



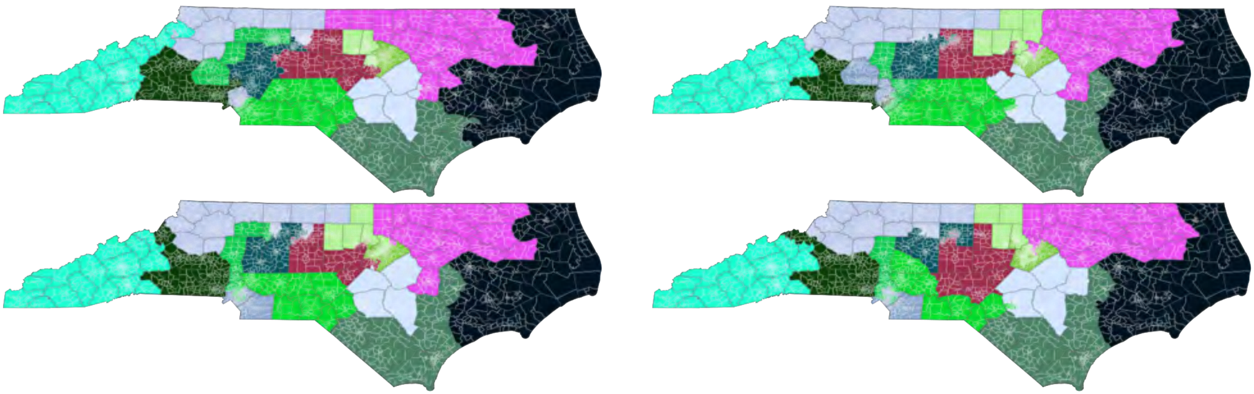
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999988%	5	99.9999974%	9	99.999982%	13	99.9999976%
2	99.9999989%	6	99.9999989%	10	99.9999954%	14	99.9999985%
3	99.9999961%	7	99.999999946%	11	99.9999965%	15	99.99999983%
4	99.99999981%	8	99.9999973%	12	99.9999999981%	16	99.99999985%



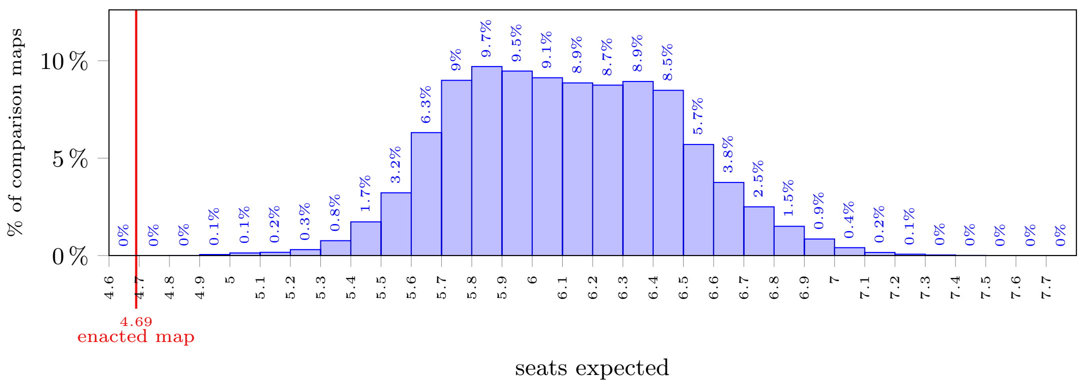
C.5 Robustness to compactness 5% Perimeter compactness

Here I show results when my analysis of the Congressional map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

C.5.1 Comparison map examples



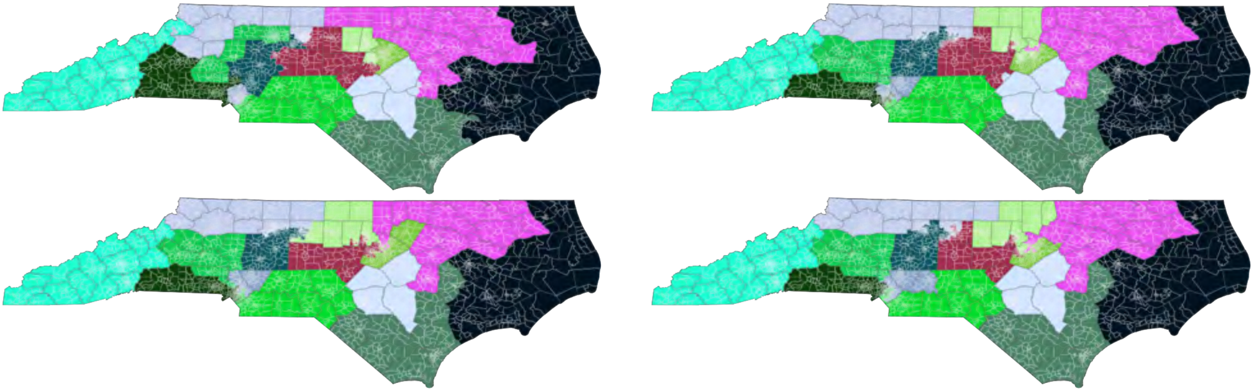
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999988%	5	99.9999968%	9	99.999998%	13	99.9999976%
2	99.99999948%	6	99.9999949%	10	99.9999978%	14	99.9999986%
3	99.99999941%	7	99.9999999976%	11	99.999982%	15	99.99999983%
4	99.99999981%	8	99.9999906%	12	99.9999999981%	16	99.9999963%



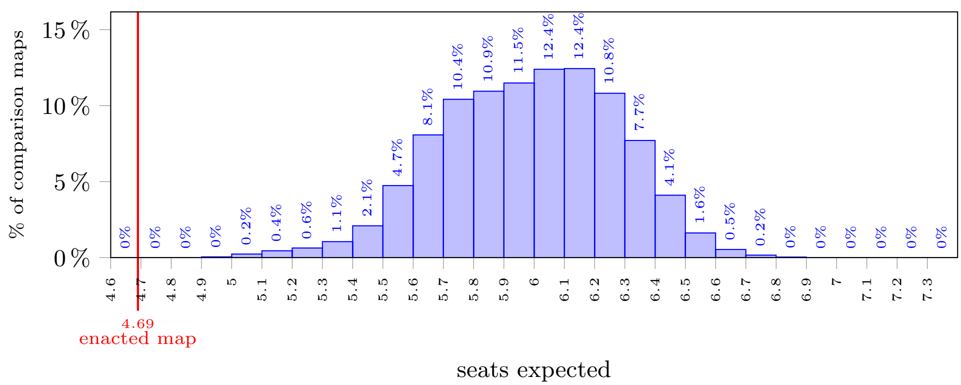
C.6 Robustness to 1% population deviation

Here I show results when my analysis of the Congressional map is repeated with a 1% population deviation constraint instead of a 2% population deviation constraint.

C.6.1 Comparison map examples



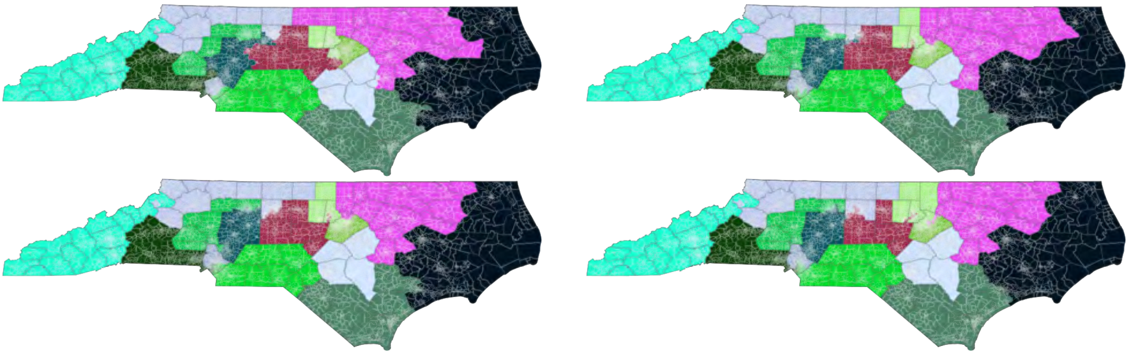
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999911%	5	99.999999907%	9	99.9999983%	13	99.999914%
2	99.9999966%	6	99.99999999945%	10	99.99978%	14	99.9999988%
3	99.999949%	7	99.9999986%	11	99.999989%	15	99.999971%
4	99.9999935%	8	99.999951%	12	99.999934%	16	99.999997%



C.7 Geounit analysis

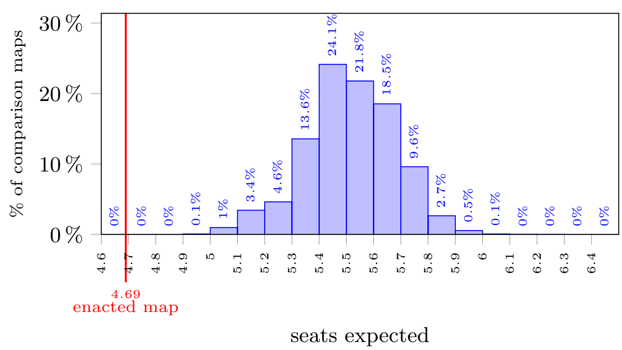
Here I show results when my analysis of the Congressional map is repeated at the geounit level, with a 0.5% population deviation constraint.

C.7.1 Comparison map examples



C.7.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999952%	5	99.999987%	9	99.999962%	13	99.9999952%
2	99.999989%	6	99.999986%	10	99.9999964%	14	99.9999962%
3	99.999967%	7	99.9999924%	11	99.999974%	15	99.999926%
4	99.999964%	8	99.999996%	12	99.999977%	16	99.9999935%

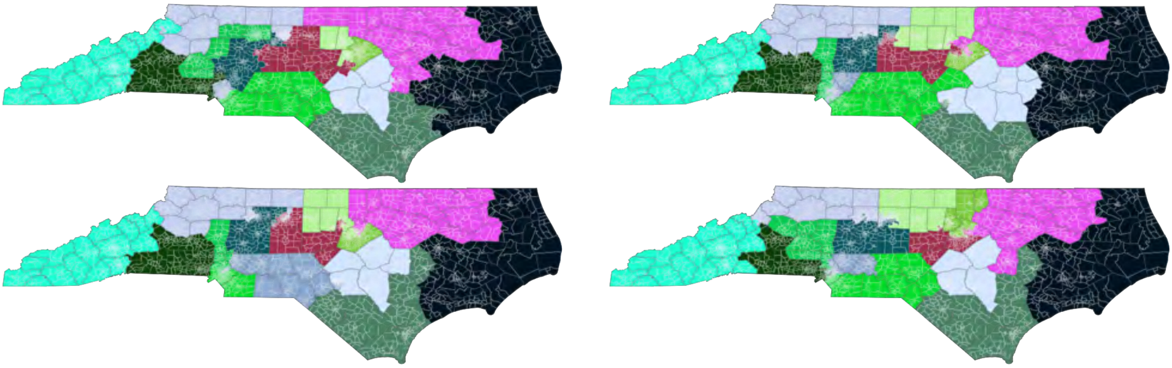


- **First level analysis:** In *every* run, the districting was in the most partisan 0.000073% of districtings (in other words, 99.999926% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.00022% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.99977% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000073\%$.

C.8 Analysis of VTD-level blueprint

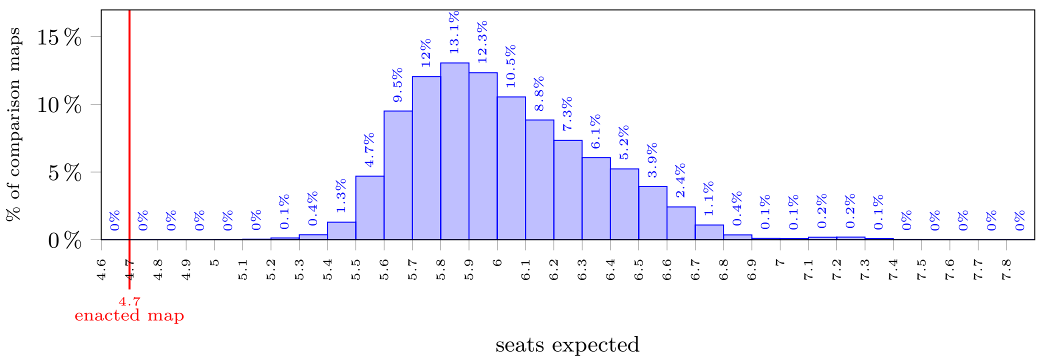
Here I show results when my analysis of the Congressional map is performed not on the precise enacted map, but a whole-VTD-level blueprint for the enacted map obtained by assigning each split VTD to the district it has the greatest intersection with.

C.8.1 Comparison map examples



C.8.2 Results

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999982%	9	99.99999969%	17	99.9999991%	25	99.9999986%
2	99.99999947%	10	99.9999952%	18	99.99999944%	26	99.9999998%
3	99.9999957%	11	99.999986%	19	99.999978%	27	99.9999977%
4	99.9999907%	12	99.999979%	20	99.9999959%	28	99.9999976%
5	99.9999981%	13	99.9999986%	21	99.99999946%	29	99.99999958%
6	99.99999954%	14	99.999984%	22	99.9999971%	30	99.999986%
7	99.9999917%	15	99.9999977%	23	99.9999974%	31	99.9999969%
8	99.9999917%	16	99.9999961%	24	99.9999942%	32	99.9999958%



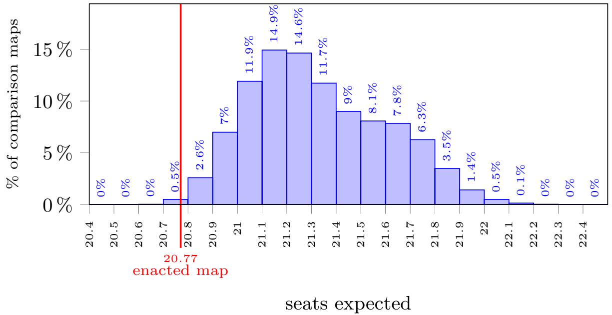
- **First level analysis:** In *every* run, the districting was in the most partisan 0.000021% of districtings (in other words, 99.999978% were less partisan, in *every* run).
- **Second level analysis:** My theorems imply that the enacted districting is among the most optimized-for-partisanship 0.000064% of all alternative districtings of North Carolina satisfying my districting criteria (in other words, 99.999935% are less optimized-for-partisanship), measured by their ε-fragility for $\varepsilon = 0.000021\%$.

Here I show results when my analysis of the Senate map is repeated with other elections in place of the 2020 Attorney General election as my proxy for partisan voting patterns.

– Ex. 4645 –

D.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.65%	5	99.78%	9	99.79%	13	99.8%
2	99.81%	6	99.79%	10	99.82%	14	99.73%
3	99.75%	7	99.79%	11	99.81%	15	99.66%
4	99.8%	8	99.75%	12	99.75%	16	99.81%



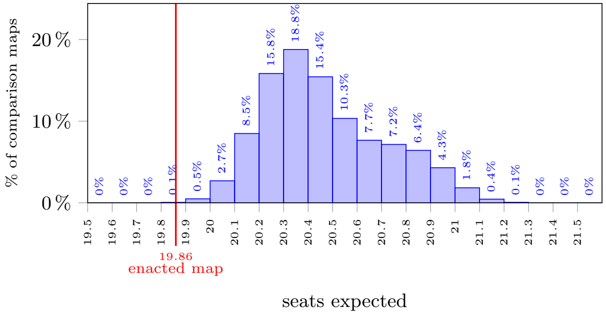
D.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of	Run	Percentage of	Run	Percentage of	Run	Percentage of

– Ex. 4646 –

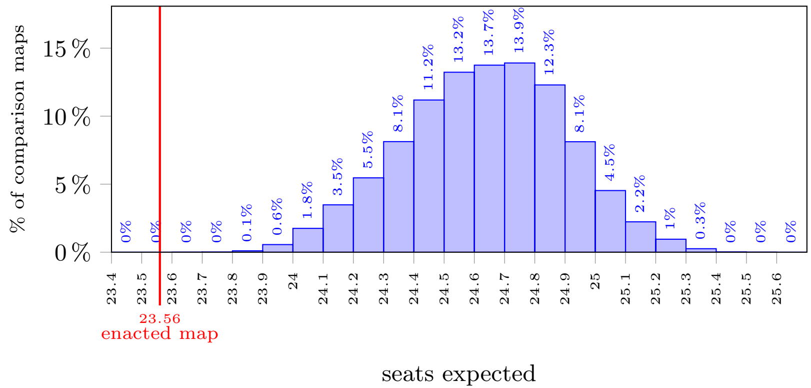
D.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.943%	5	99.987%	9	99.9912%	13	99.9911%
2	99.996%	6	99.982%	10	99.9955%	14	99.977%
3	99.973%	7	99.994%	11	99.9958%	15	99.944%
4	99.9927%	8	99.983%	12	99.89%	16	99.995%



D.1.3 Results with 2020 Governor election

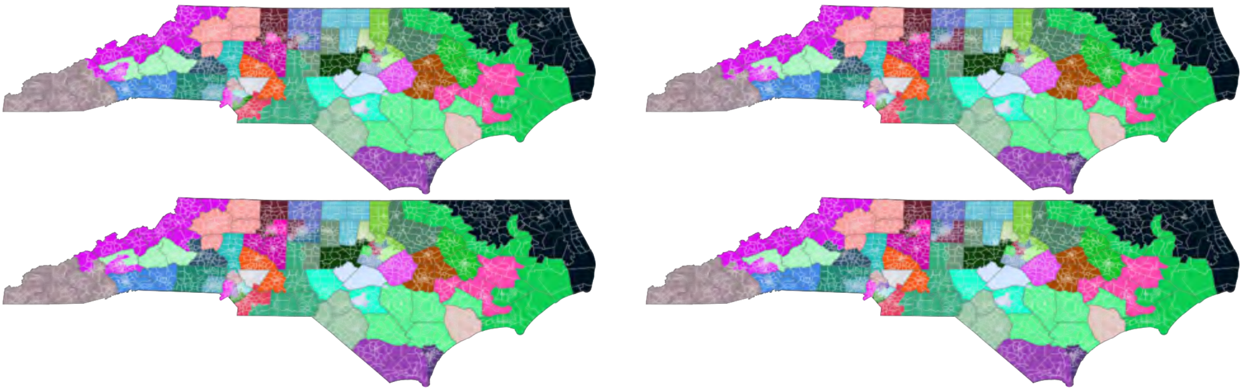
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999936%	5	99.9999996%	9	99.9999998%	13	99.999999973%
2	99.999999949%	6	99.9999974%	10	99.9999987%	14	99.9999985%
3	99.99999978%	7	99.9999999929%	11	99.9999998%	15	99.999999961%
4	99.9999989%	8	99.9999999969%	12	99.999999973%	16	99.9999985%



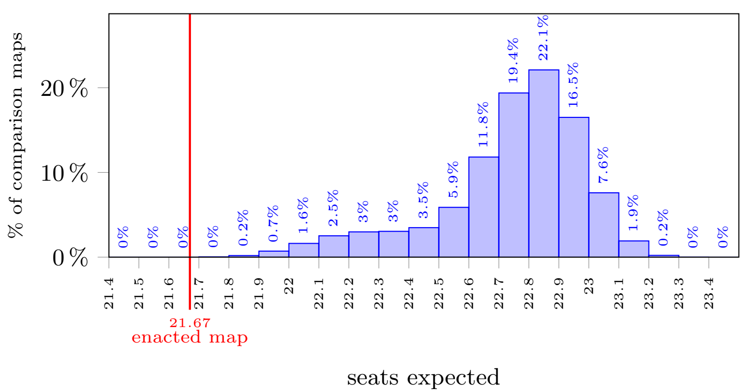
D.2 Robustness to incumbency protection

Here I show results when my analysis of the Senate map is repeated without ensuring the protection of incumbents.

D.2.1 Comparison map examples



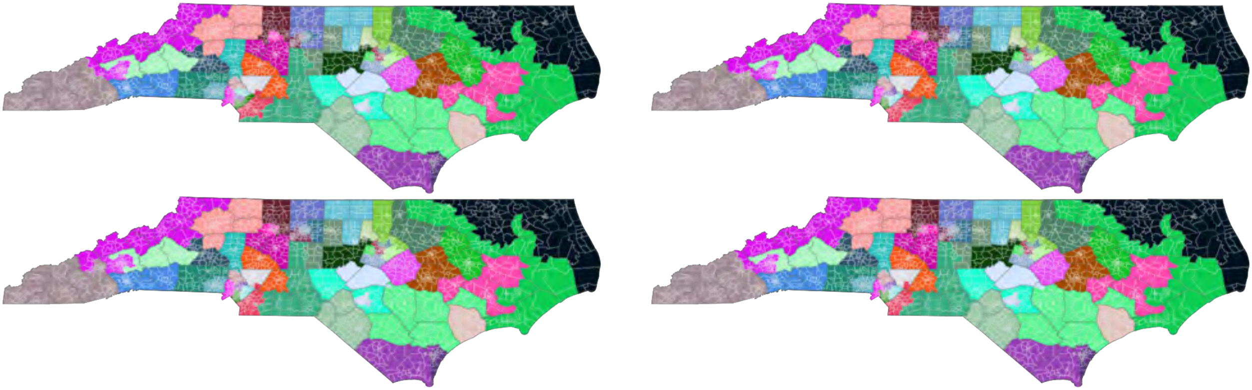
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9998%	5	99.9993%	9	99.99989%	13	99.99906%
2	99.99988%	6	99.99985%	10	99.99968%	14	99.9987%
3	99.99971%	7	99.999907%	11	99.9998%	15	99.99928%
4	99.99922%	8	99.9985%	12	99.99976%	16	99.9943%



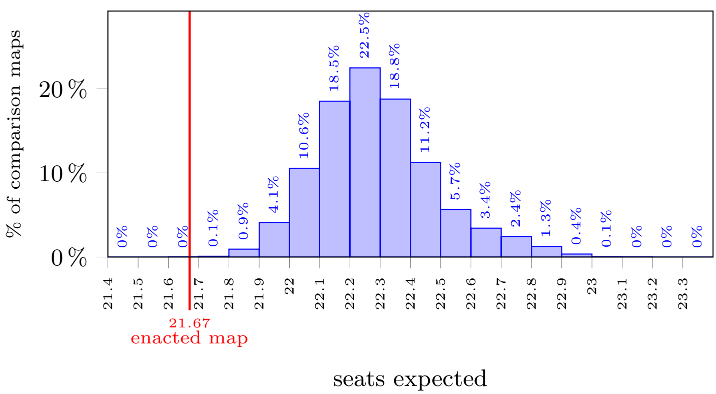
D.3 Compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the Senate map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

D.3.1 Comparison map examples



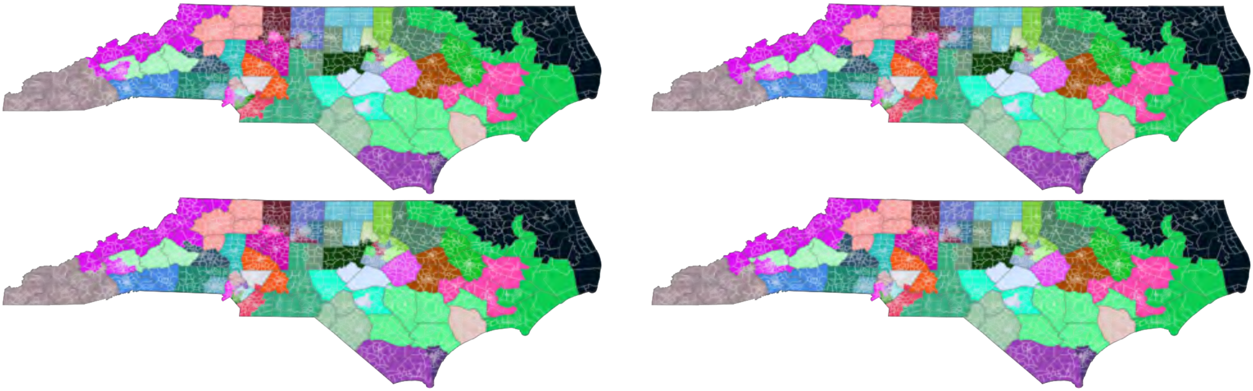
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9979%	5	99.9978%	9	99.995%	13	99.9986%
2	99.99909%	6	99.9968%	10	99.9982%	14	99.9989%
3	99.9968%	7	99.99933%	11	99.9987%	15	99.9973%
4	99.99927%	8	99.9979%	12	99.99923%	16	99.9976%



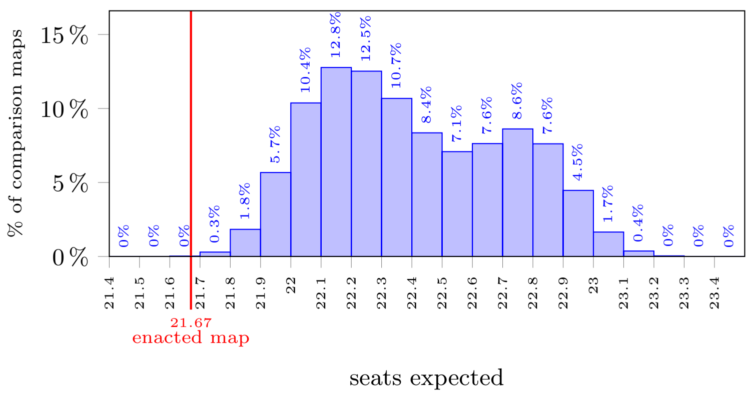
D.4 Compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the Senate map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

D.4.1 Comparison map examples



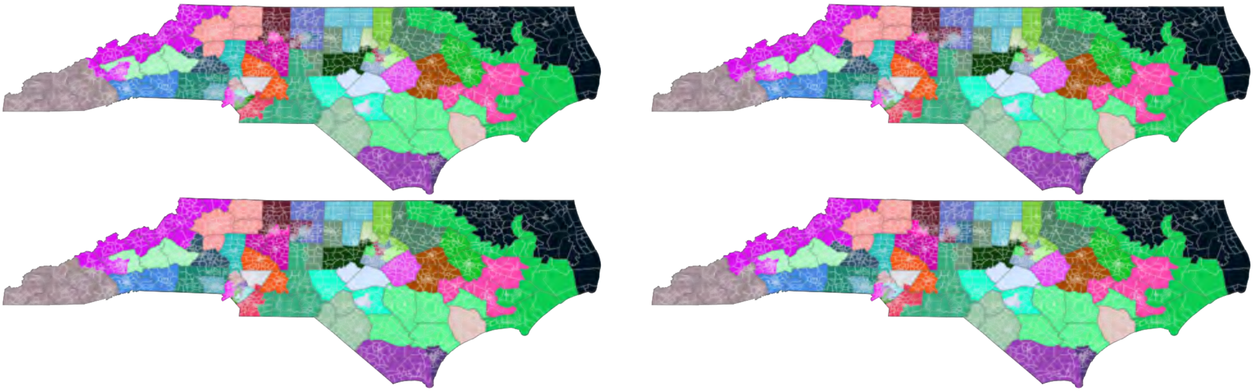
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9963%	5	99.992%	9	99.971%	13	99.98%
2	99.9928%	6	99.986%	10	99.985%	14	99.9917%
3	99.988%	7	99.993%	11	99.9924%	15	99.978%
4	99.987%	8	99.9957%	12	99.9908%	16	99.9969%



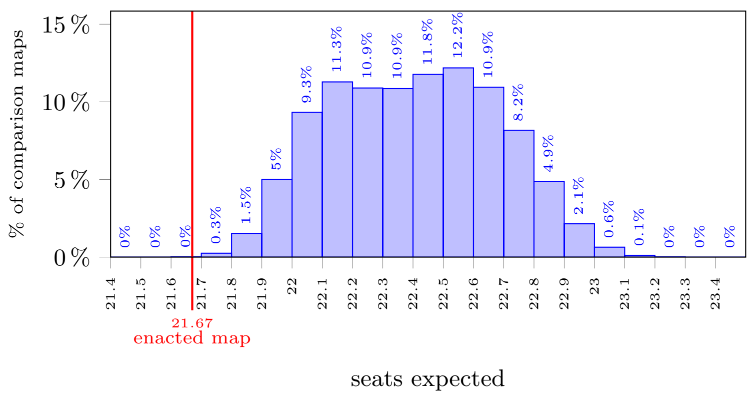
D.5 Compactness 5% Perimeter compactness

Here I show results when my analysis of the Senate map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

D.5.1 Comparison map examples

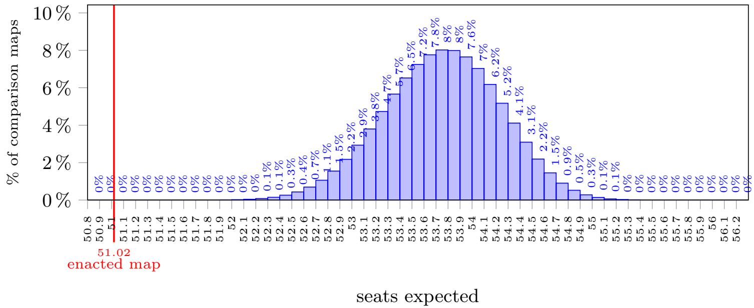


Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9913%	5	99.985%	9	99.988%	13	99.9907%
2	99.9907%	6	99.989%	10	99.988%	14	99.982%
3	99.9949%	7	99.9929%	11	99.986%	15	99.981%
4	99.989%	8	99.989%	12	99.987%	16	99.9919%



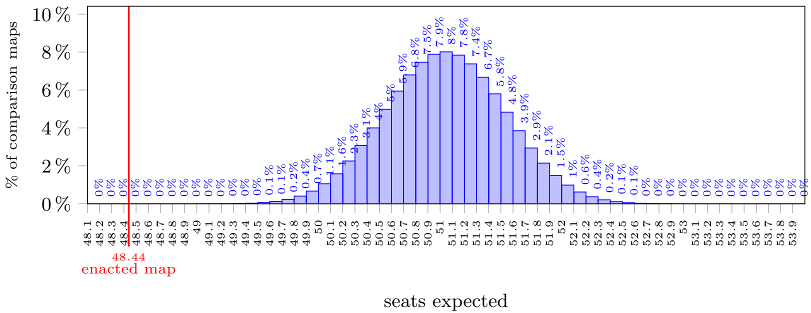
E.1.1 Results with 2020 Presidential election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999999985%	5	99.99999945%	9	99.9999986%	13	99.9999986%
2	99.999999981%	6	99.99999948%	10	99.99999912%	14	99.999999976%
3	99.99999997%	7	99.999999963%	11	99.99999986%	15	99.9999984%
4	99.9999969%	8	99.9999981%	12	99.9999985%	16	99.9999989%



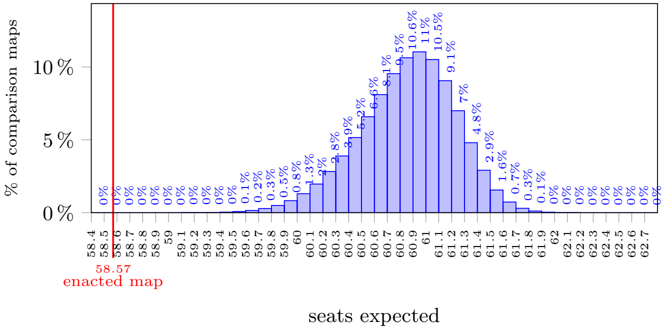
E.1.2 Results with 2020 Lieutenant Governor election

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999988%	5	99.9999983%	9	99.999997%	13	99.9999957%
2	99.999981%	6	99.9999926%	10	99.9999979%	14	99.9999905%
3	99.99999907%	7	99.9999927%	11	99.9999974%	15	99.99999914%
4	99.9999969%	8	99.999993%	12	99.9999981%	16	99.99999924%



E.1.3 Results with 2020 Governor election

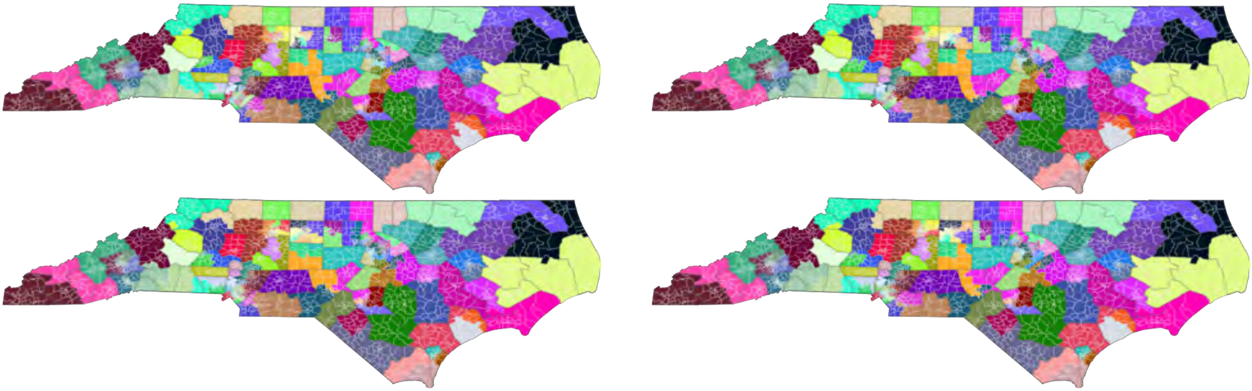
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999985%	5	99.999999931%	9	99.999999975%	13	99.99999986%
2	99.999999984%	6	99.9999994%	10	99.9999986%	14	99.99999988%
3	99.99999997%	7	99.999999986%	11	99.9999998%	15	99.99999948%
4	99.9999985%	8	99.99999985%	12	99.99999914%	16	99.99999989%



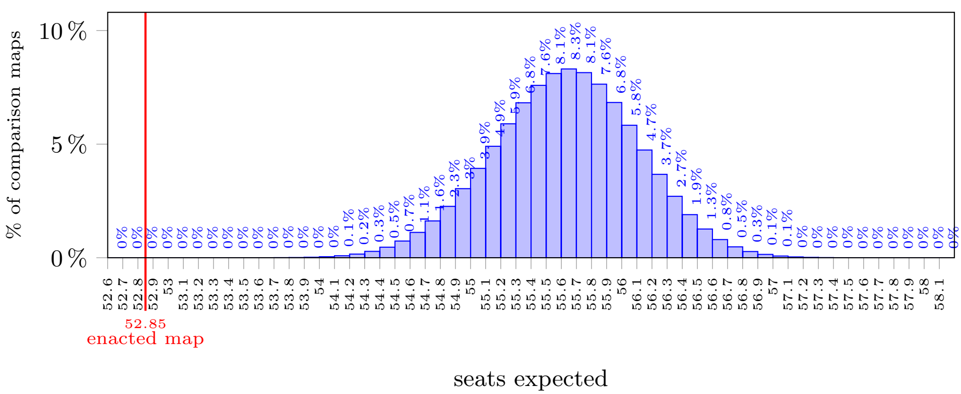
E.2 Robustness to incumbency protection

Here I show results when my analysis of the House map is repeated without ensuring the protection of incumbents.

E.2.1 Comparison map examples



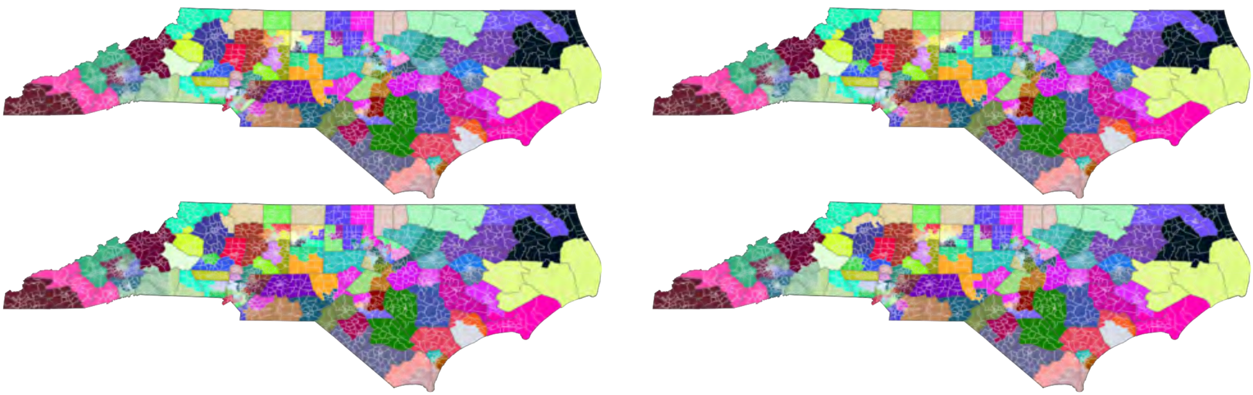
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99999987%	5	99.9999933%	9	99.99999967%	13	99.99999989%
2	99.999999981%	6	99.9999962%	10	99.99999944%	14	99.99999981%
3	99.99999997%	7	99.9999968%	11	99.9999944%	15	99.99999%
4	99.999999908%	8	99.99999961%	12	99.999999963%	16	99.99999947%



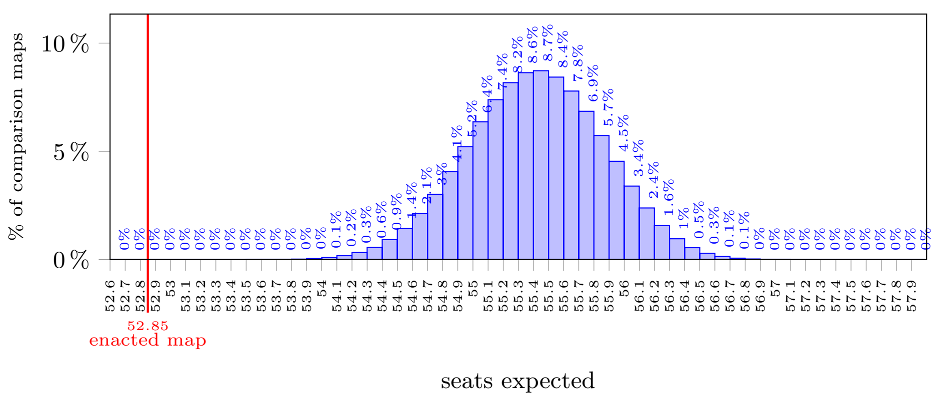
E.3 Compactness: 0% Polsby-Popper threshold

Here I show results when my analysis of the House map is repeated with a 0% threshold for compactness in place of the 5% error I allow in my primary analysis.

E.3.1 Comparison map examples



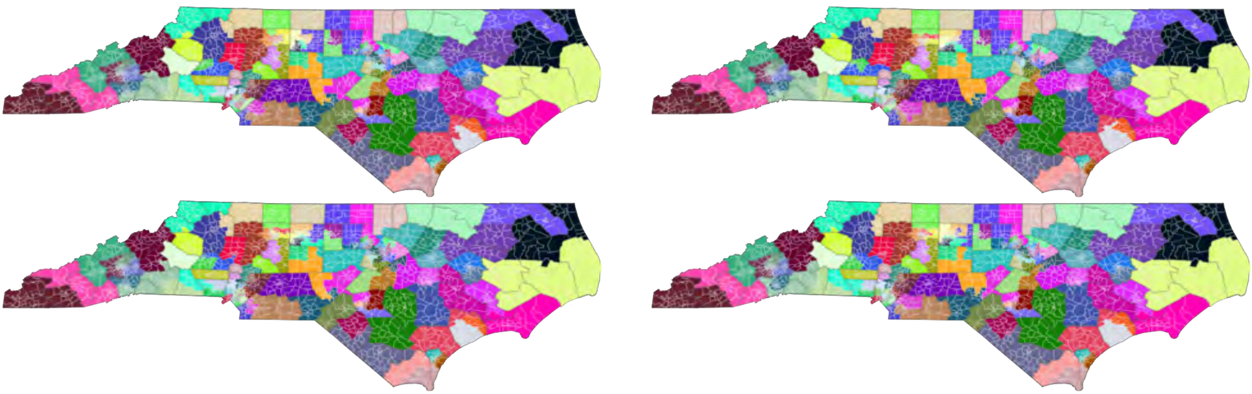
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999996%	5	99.99999927%	9	99.9999987%	13	99.9999978%
2	99.99999982%	6	99.999999941%	10	99.9999966%	14	99.9999986%
3	99.999987%	7	99.9999971%	11	99.9999963%	15	99.99999975%
4	99.9999912%	8	99.9999988%	12	99.99999928%	16	99.9999968%



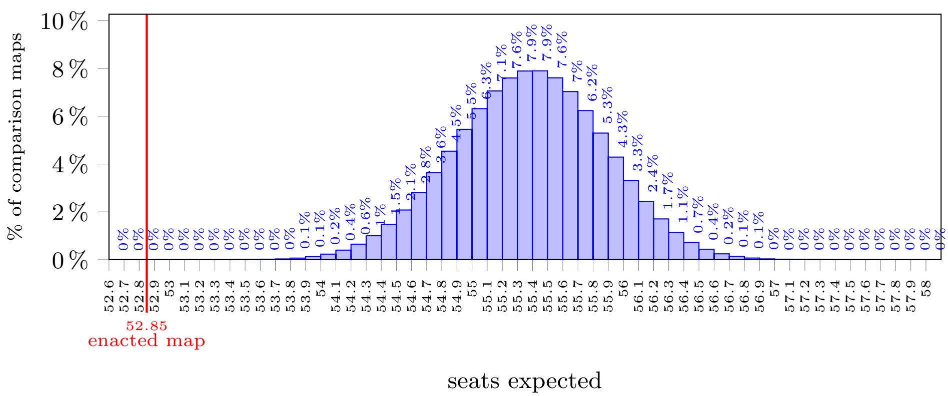
E.4 Compactness: 10% Polsby-Popper threshold

Here I show results when my analysis of the House map is repeated with a 10% threshold for compactness in place of the 5% error I allow in my primary analysis.

E.4.1 Comparison map examples



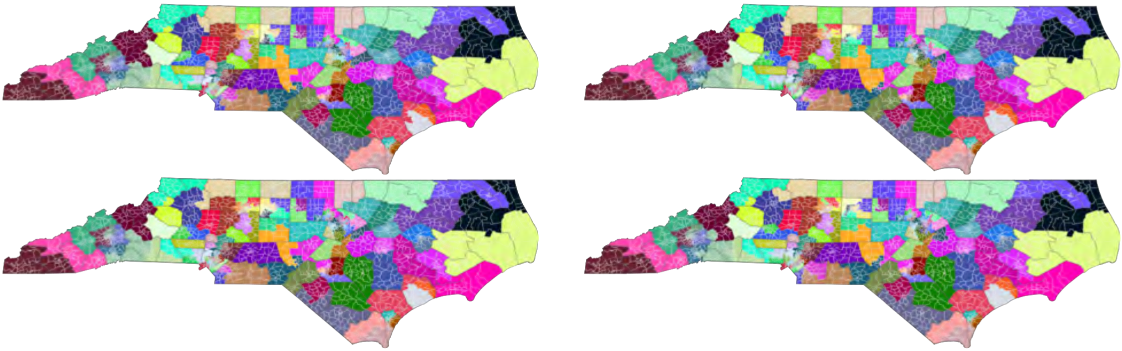
Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.9999904%	5	99.9999989%	9	99.999999917%	13	99.9999983%
2	99.999999957%	6	99.9999971%	10	99.9999983%	14	99.99999989%
3	99.9999948%	7	99.9999999916%	11	99.999988%	15	99.99999962%
4	99.9999987%	8	99.9999955%	12	99.9999922%	16	99.9999974%



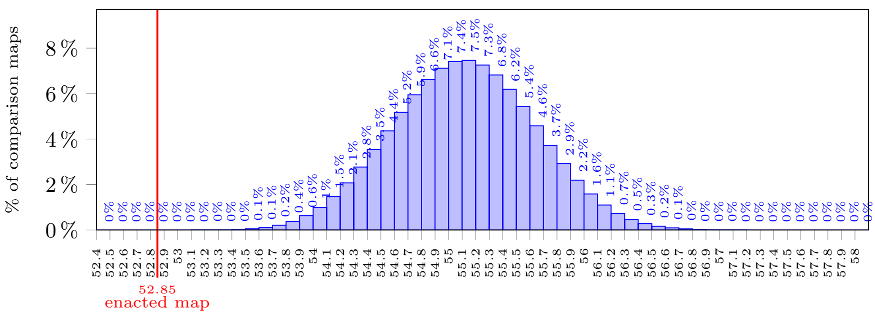
E.5 Compactness 5% Perimeter compactness

Here I show results when my analysis of the House map is repeated with a completely different compactness score, based just on the total perimeter of all districts in the districting.

E.5.1 Comparison map examples

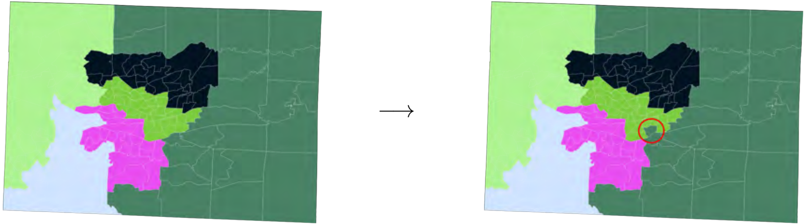


Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.99985%	5	99.999957%	9	99.999988%	13	99.999953%
2	99.999977%	6	99.999976%	10	99.999978%	14	99.99991%
3	99.99988%	7	99.9999904%	11	99.999968%	15	99.999981%
4	99.999978%	8	99.999951%	12	99.999925%	16	99.99995%



5 Random Changes

As described earlier, my method involves making small random changes to a map. For example, depicted here is a small random change made to the enacted House districting within the Guilford county cluster:



The geographical units used for these small random changes in this district are voting tabulation districts—VTDs. In particular, at each step of the sequence of random changes for the house districting within Guilford county, I move a randomly VTD that is at the boundary of two districts from one of those districts to the other (unless it would violate the constraints laid out in Section 4.3.1

Plaintiffs' Exhibit 597

(.mp4 video file produced to Court in original format)

Plaintiffs' Exhibit 598

(.mp4 video file produced to Court in original format)

Rebuttal to report of Michael Barber

Wesley Pegden

December 28, 2021

1 Introduction

In his report, Michael Barber presents the results of simulated district plans as part of an analysis which purports to elicit whether the enacted House and Senate maps of North Carolina are "partisan outliers". Barber makes choices in his analysis that reduce its ability to detect gerrymandering North Carolina clusters; for example, he discusses the partisan bias of the enacted House and Senate maps through the lens of the whole number of "Democratric-lean" districts in one hypothetical election, a lens through which even the effects of extreme gerrymandering in NC county clusters—each with a small number of districts—are made to appear less dramatic.

Nevertheless, his primary analyses (Tables 2 and 32) still find the whole-state House and Senate plans to be partisan outliers compared to his simulated maps, according to the definition he lays out in his report; in particular, he reports the middle-50% of simulated maps to have 46-51 total "Democratic-lean" districts across the House clusters he analyzes, and reports that the enacted map contains 45 such districts. For the Senate he reports a middle-50% range of 19-19 total Democratic-lean districts in his simulations, and that the enacted map contains 16 such districts.

In fact, Barber incorrectly calculated the distribution of Democrat-leaning seats for the whole-state outcomes of his simulation analysis, incorrectly reporting the sums of lower- and upper-quartile seat counts in individual clusters as the lower- and upper-quartile for total statewide seats. When the distribution of "lean Democrat district" counts at the whole-state level are calculated correctly for Barber's simulations (still using the partisan index he defines), one finds that the middle-50% range for Barber's simulated maps in the House is actually 48-50 Democratic-lean districts, not 46-51 as Barber shows, and that **the enacted North Carolina House map lies in the most Republican-biased 00.18% of whole state maps composed of Barber's simulations, and the enacted North Carolina Senate map lies in the most Republican-based 00.39% of whole state maps composed of Barber's simulations.** This computation can be carried out entirely with the figures provided in Barber's report, and uses Barber's simulated maps and Barber's metric of partisan bias (number of lean-Democrat districts), calculated with Barber's own partisan voting index.

Finally, when re-analyzing Barber's simulated maps (as provided in his backup data) to compare their expected performance over a range of electoral outcomes rather than comparing the crude number of "lean Democratic districts" for a fixed election average, the differences between the enacted map and Barber's ensemble of simulated comparison maps becomes more dramatic at the cluster level as well. Through this lens, every cluster which my original analysis found to be optimized for partisanship would qualify as a partisan outlier according to Barber's "middle 50%" criterion, and many are extreme outliers, among the most Republican biased 10%, 1%, or 0.1% of maps, even in clusters where Barber reported that the enacted map was not be a partisan outlier.

2 Barber finds the enacted House and Senate maps to be outliers according to his own definition

On page 29 of his report, in the section on House clusters, Barber writes that he considers a districting plan of North Carolina to be a partisan outlier if it lies outside of the "middle 50%" of simulation results; in Barber's report, the middle 50% are the maps that lie between the 25th and 75th percentiles according to the number of lean-Democrat districts, as measured with the partisan index Barber obtains by averaging election results. He calls this a "conservative definition" of an outlier, noting that "in the social sciences, medicine, and other disciplines it is traditional to consider something an outlier if it falls outside the middle 95% or 90% of the comparison distribution."

In both of his whole-state analysis tables (Table 2 and 32), Barber's own findings report the whole map as falling outside the middle 50% of simulated outcomes for the House and Senate. For example, in the last row, labeled "Total", of Table 2 on page 31, he reports that in the 26 clusters he analyzed, the enacted map contained 45 statewide "lean-Democrat" districts according to his partisan index, while the middle 50% range of the simulated maps for the total number of seats was $46 - 51$. Similarly, in Table 32 for the Senate, he reports the enacted map scored as having a total of 16 lean-Democrat seats in the 12 clusters used by the enacted map he analyzed, while the middle 50% range for his middle 50% range for the total number of seats in his simulated maps was 19-19. By the definition he chose to offer of a partisan outlier, Barber finds the enacted House and Senate plans are partisan outliers.

3 Barber reports incorrect quartiles for totals across clusters

Recall that in his Table 2, in the last column, Barber reports the range of the "middle 50%" for the number of lean-Democratic districts for his simulations in each cluster, and, at the bottom of the column, for the total across clusters (he reports the range for this total as 46-51). Recall that the bottom of the middle-50% range is the lower quartile of the data, and the top of the range is the upper quartile.

For example, in the House:

- for the Buncombe cluster in the House map, Barber reports in Figure 45 that 28% of his simulated maps contained 2 lean-Democrat districts, while 72% contained 3.
- for the Cumberland cluster in the House map, Barber reports in Figure 55 that 82% of his simulated maps contained 3 districts, while 18% contained 4.

I summarize this information in my Table 1, below:

Cluster	0	1	2	3	4
Buncombe			28%	72%	
Cumberland				82%	18%

Table 1: Fraction of maps with various lean-Democrat-district counts, as reported by Barber for Buncombe and Cumberland county districtings.

In his Table 2, Barber correctly summarizes the middle 50% ranges for the data in each of these clusters as 2-3 and 3-3, respectively; in each case, the lower end of the range is the smallest value below which 25% of his simulated maps lie, and the upper end is the smallest value below which 75% lie.

Suppose though, just as an example, that we wished to calculate the distribution of the total number of lean-Democrat districts across just these two clusters according the Barber's simulations; this will also enable us to calculate the middle-50% of outcomes for the total lean-Democrat districts across these two clusters.

Note that for maps of these two clusters composed of maps from Barber'simulations, a total of 5, 6, or 7 lean-Democrat districts are possible. For example, 5 lean-Democrat districts can arise only by having 2 such districts in Buncombe and 3 in Cumberland, and fewer are not possible.

According to Barber's simulations, as summarized in Table 1, 28% of the maps of these two clusters would have 2 lean-Democrat districts in Buncombe, while 82% would have 3 lean-Democrat districts in Cumberland. As the districtings in each cluster can be chosen independently of each other, a total of

$$28\% \times 82\% = 22.96\%$$

of districtings of these two counties would have a total of 5 lean-Democrat districts. (Note that having fewer than 5 lean-Democrat seats happens 0% of the time, according to Barber's simulations.)

6 lean-Democrat districts can arise from having 2 lean-Democrat districts in Buncombe and 4 in Cumberland, or having 3 lean-Democrat districts in Buncombe and 3 in Cumberland. Thus according to Barber's simulation results the frequency of this outcome would be

$$28\% \times 18\% + 72\% \times 82\% = 64.08\%.$$

Finally, the likelihood of 7 lean-Democrat seats, which arise just when there are 3 lean-Democrat districts in Buncombe and 4 lean-Democrat districts in Cumberland, would be

$$72\% \times 18\% = 12.96\%,$$

(Note that altogether, 22.96%+64.08%+12.96%=100%.)

Evidently, the middle-50% range for the total of lean-Democrat seats across these two counties would be 6-6; the 6-lean-Democrat-district maps include the middle-50% of simulated maps. (6 is both the 25th percentile and the 75th percentile of the number of Democratic-lean seats in the simulated maps.)

Under Barber's incorrect approach, he would have simply added the bottom and top of the middle-50% ranges for Buncombe and Cumberland (2-3 and 3-3, respectively) to arrive at a middle-50% range for the total number of lean-Democrat-districts across these two counties; that procedure would produce a range of 5-6, which is wider than the true middle-50% range of the total number of districts across the two counties (namely 6-6), as correctly calculated above.

In general, the magnitude of this error grows larger and larger the more independent cluster-specific results are aggregated by incorrectly summing the lower and upper quartiles as a substitute for a correct calculation of the distribution of total statewide lean-Democrat districts. In Barber's report, he aggregrates across 26 clusters in this way. As we will see in the next section, this has the effect of inflating the true middle-50% range of 48-50 to an incorrectly reported range of 46-51.

Technical Remark. Probability generating functions can be used to allow larger calculations of the same type as the one above to be performed using publicly web-based computer algebra systems instead of by programming or using statistical software. Note that precisely the same three calculations above would have been performed if expanding the algebraic expression

$$\begin{aligned}(.28x^2 + .72x^3)(.82x^3 + .18x^4) &= (.28 \times .82)x^5 + (.28 \times .18 + .72 \times .82)x^6 + (.72 \times .18)x^7 \\ &= .2296x^5 + .6408x^6 + .1296x^7.\end{aligned}$$

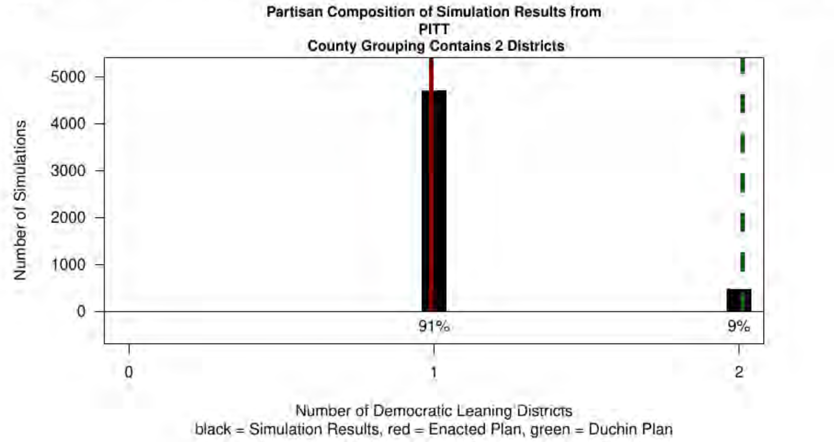
Observe that the polynomial $.28x^2 + .72x^3$ here can be seen as representing the fact that two seats occur in 28% of the maps for Buncombe, while 3 seats occur in 72% of the maps. (Similarly, then, for Cumberland and the polynomial $.82x^3 + .18x^4$.) The same answers that we found above for the fraction of simulated plans with a total of 5, 6, and 7 lean-Democrat districts, respectively, can be read off as the coefficients of x^5, x^6, and x^7, in the resulting expansion.

In the technical remark in the next section, I will point out a similar polynomial expansion which can verify the next section's calculations using public web applications, making the main findings of this rebuttal report easy to independently verify.

4 Correcting Barber's calculations

In my Table 2 on page 13 of this rebuttal report, I report the results of Barber's Figures 11, 14, 17, 20, 25, 28, 31, 34, 37, 45, 48, 51, 55, 58, 61, 64, 67, 70, 73, 76, 79, 82, 85, and 88. Each of these figures reports, for one of the clusters Barber analyzes, the fraction of his simulated maps which achieve different numbers of "lean Democrat" districts according to the partisan index he uses. For example, in Figure 14 on page 44, Barber reports that 91% of his simulated maps had one lean-Democrat district, while the remaining 9% had 2, as seen in this reproduction below:

Figure 14: **Distribution of Partisan Districts from Simulations in Pitt House County Cluster**



This information is then reproduced in my Table 2 on page 13, as the following row:

Cluster	0	1	2	3	4	5	6	7	8	9	10	11	12
Pitt		91%	9%										

In particular, everything in my Table 2 (and the corresponding Table 3 for the Senate) is taken directly from Barber's report itself.

The data in Table 2 can then be used to calculate the distribution of the total number of lean-Democrat seats based on Barber's simulations across the 26 clusters, exactly in the same way as we did above for just 2 clusters from the data in Table 1. The result of the same calculation is the histogram shown in Figure 1. In particular, according to Barber's own simulated map set, and using his own measure of the number of lean-Democrat districts under his own partisan index, **the enacted House map exhibits more Republican bias than 99.82% of maps** composed of Barber's simulations, over the clusters Barber analyzes.

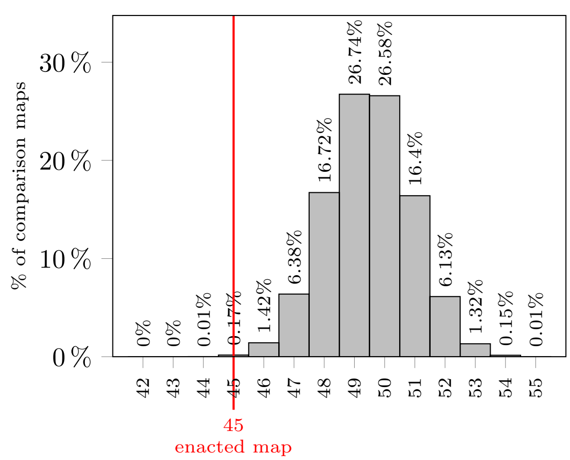


Figure 1: **Total lean-Democrat districts across Barber's House simulations**. This histogram shows the performance of Barber's simulated map set across the total set of House clusters Barber analyzes. It uses Barber's set of simulated maps, Barber's chosen metric (number of lean Democratic seats), calculated using the partisan metric Barber himself calculates in his report. The range 49-50 contains 50% of the simulated maps, the range 48-51 contains 86% of the simulated maps, and the range 47-52 contains more than 98% of the simulated maps. With 45 lean-Democratic districts across these clusters, the enacted map is in the most Republican-biased 0.18% of Barber's simulated maps.

In Table 3 I show Barber's Senate data analogous to the House data I show in Table 2. And in Figure 2, I plot the histogram showing the total of Barber's metric of Democratic-leaning districts across Barber's simulated map set, produced in the same way as I produce Figure 1 for the House. In particular, according to Barber's own simulated map set, and using his own measure of the number of lean-Democrat districts under his own partisan index, **the enacted Senate map exhibits more Republican bias than 99.61% of maps** over the clusters Barber analyzes.

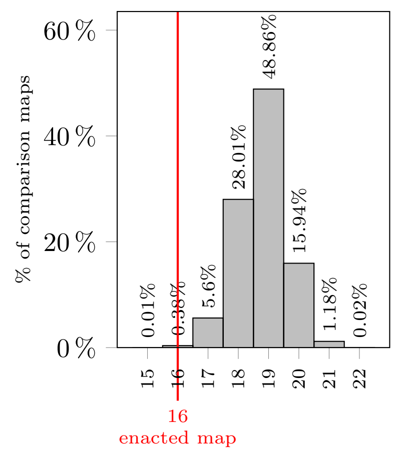


Figure 2: **Total lean-Democrat districts across Barber's Senate simulations**. This histogram shows the performance of Barber's simulated map set across the total set of Senate clusters Barber analyzes. It uses Barber's set of simulated maps, Barber's chosen metric (number of lean Democratic seats), calculated using the partisan metric Barber himself calculates in his report. The range 18-20 contains 93% of the simulated maps, and the range 17-21 contains more than 99% of the simulated maps. With 16 lean-Democrat districts, the enacted map is among the most Republican 00.39% of maps.

Technical Remark. As noted in the earlier Technical Remark, calculating the results of a histogram like Figure 1 is equivalent to expanding a certain polynomial expression. Based on the data in Table 2, (rows with only zero seats possible can be ignored), the polynomial to be expanded is

$$(.91x+.09x^2)(.44+.56x)(x^2)(x^2)(x)(.28x^2+.72x^3)(.82x^3+.18x^4)(x^4)(x)(.33x^2+.5x^3+.17x^4)(.99+.01x^1)$$
$$\cdots(.18+.82x)(.01x^4+.79x^5+.21x^6)(.01x^{10}+.56x^{11}+.44x^{12})(.02x^{10}+.32x^{11}+.66x^{12})$$

and publicly available tools such as `wolframalpha.com` can be used to verify that this polynomial expands to

$$5.55283\times10^{-7}x^{56}+0.0000685893x^{55}+0.00147488x^{54}+0.0131615x^{53}$$
$$+0.0612515x^{52}+0.163979x^{51}+0.265839x^{50}+0.267369x^{49}+0.167218x^{48}+0.0637935x^{47}+0.0141775x^{46}$$
$$+0.00167669x^{45}+0.000089375x^{44}+1.74341\times10^{-6}x^{43}+1.08123\times10^{-8}x^{42}$$

The histogram in Figure 1 can be read off the coefficients in this polynomial. For example, the fact that the coefficient of x^{49} is .267369 corresponds to the fact that Figure 1 reports the fraction of simulated maps with a total of 49 Democrat-leaning districts across the clusters Barber analyzes as 26.74% (rounded to two decimal places).

For the senate, from Table 3, the probability generating function is

$$(.77x+.23x^2)(x^2)(.23+.77x)(.93x^2+.06x^3)(.01x^4+.24x^5+.75x^6)(.05x^4+.95x^5)x(.97x+.03x^2),$$

which expands to

$$0.000227131x^{22}+0.0118152x^{21}+0.159415x^{20}+0.488577x^{19}$$
$$+0.280141x^{18}+0.0559707x^{17}+0.00377389x^{16}+0.0000807399x^{15} \quad (1)$$

giving the results shown in Figure 2.

5 A more sensitive cluster-by-cluster analysis of Barber's maps

In the previous section, I showed that even against Barber's simulated maps, using the partisan index Barber calculates, and using Barber's preferred metric for partisan bias (the number of lean-Democrat districts using that partisan index), both the enacted House and Senate plans are extreme partisan outliers.

This is true despite the fact that using the number of whole lean-Democrat districts with only a single proxy for partisanship is unlikely to capture the effects even of extreme gerrymandering in North Carolina county clusters, where a small number of seats are at stake in each, and the effects of extreme gerrymandering can be to put one or two seats into play (or take them out of contention), even in cases where districts do not change columns in a single hypothetical election.

In other words, I take Barber's single partisan index (which has a two-party statewide Democratic vote-share of 49%), and analyze what would happen under his simulations, on average, if you swung the election results so that Democrats did better or worse by a normally-distributed swing matched to past statewide North Carolina elections. This is the same metric I used in my initial report.

In this section, I re-analyze Barber's results, still using his simulated maps, and still using his partisan index, but comparing maps in each cluster using the seats-expected metric (calculated with respect to that index), which evaluates how a map would be expected to perform under a range of conditions rather than one fixed hypothetical election.

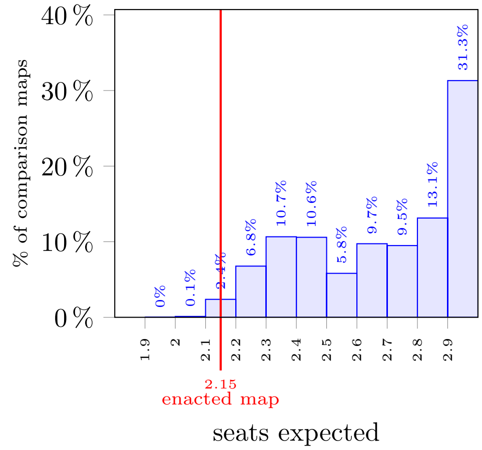
Below, I conduct this analysis for every county cluster I analyzed in my original expert report. In every cluster for which my analysis found the enacted map to be among the most optimized-for-partisanship possible maps (the first six House analyzed in the subsections below, and every Senate cluster analyzed below), Barber finds the map to be a partisan outlier according to the "middle-50%" definition he uses in his report. I summarize the outlier status of these 6+5 House and Senate clusters according to Barber's simulations in the following table:

Cluster	Enacted map among most Republican-biased…
House: Buncombe	00.797%
House: Forsyth-Stokes	00.0805%
House: Guilford	00.00646%
House: Mecklenburg	04.43%
House: Wake	05.78%
House: Pitt	24.2%
Senate: Cumberland-Moore	00.0024%
Senate: Forsyth-Stokes	00.01%
Senate: Granville-Wake	00.035%
Senate: Guilford-Rockingham	00.25%
Senate: Iredell-Mecklenburg	00.1%
	… against Barber's simulations.

Among the four remaining clusters in my report, there are two where the enacted maps are nevertheless extreme outliers against Barber's simulation sets. I summarize the results for these four clusters in the following table:

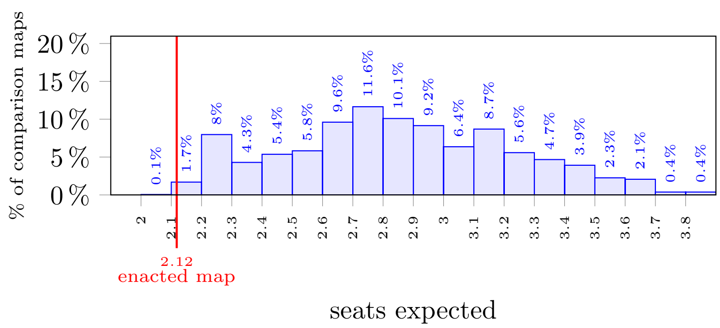
Cluster	Enacted map among most Republican-biased…
House: Alamance	39.4%
House: Brunswick-New Hanover	73.9%
House: Durham-Person	00.00265%
House: Cabarrus-Davie-Rowan-Yadkin	00.352%
	… against Barber's simulations.

5.1 House: Buncombe



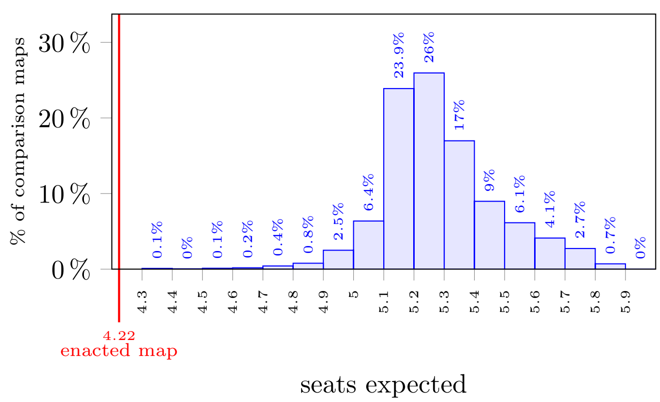
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.797% of maps.

5.2 House: Forsyth-Stokes



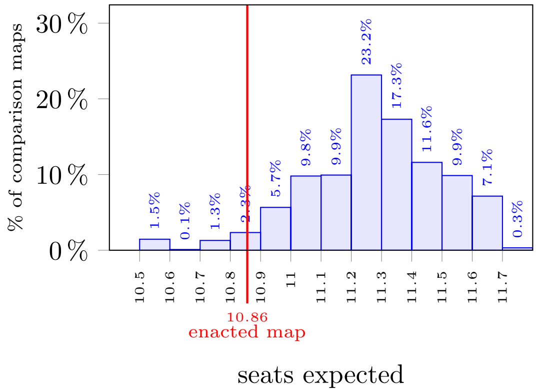
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0805% of maps.

5.3 House: Guilford



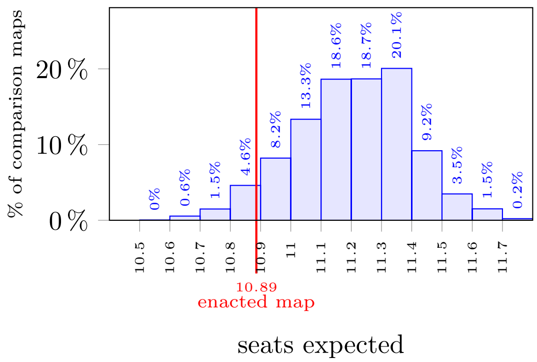
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.00646% of maps.

5.4 House: Mecklenburg



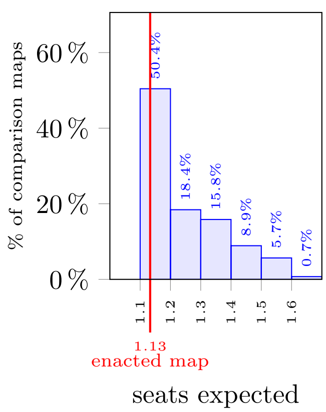
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 4.43% of maps.

5.5 House: Wake



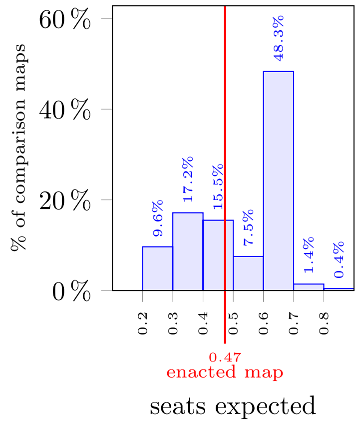
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 5.78% of maps.

5.6 House: Pitt



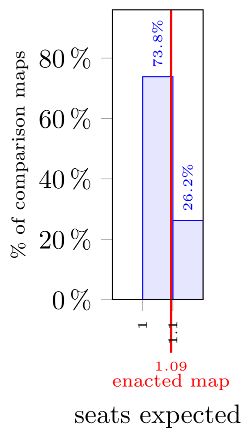
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 24.2% of maps.

5.7 House: Alamance



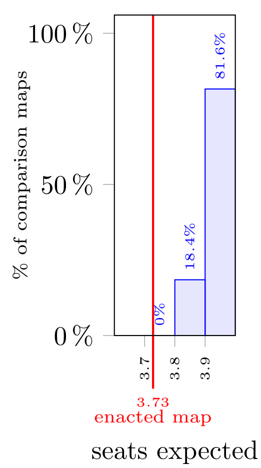
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map is not an outlier.

5.8 House: Brunswick-New Hanover



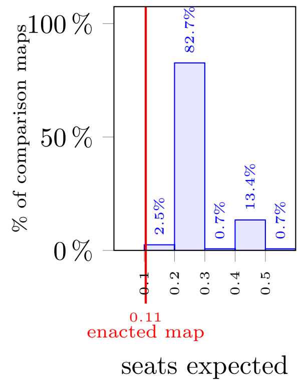
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map is not an outlier.

5.9 House: Durham-Person



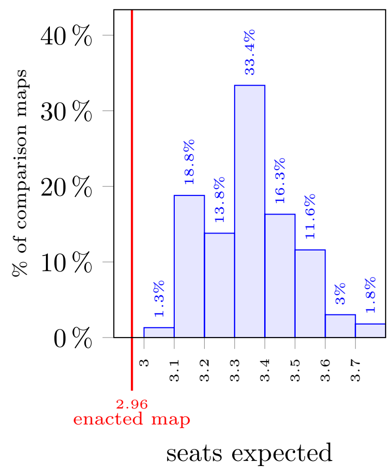
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.00265% of maps.

5.10 House: Cabarrus-Davie-Rowan-Yadkin



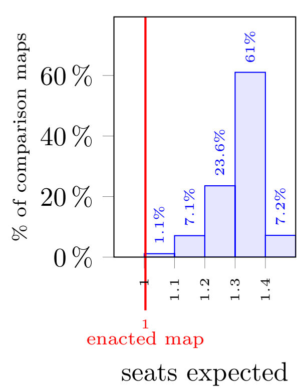
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.352% of maps.

5.11 House: Cumberland



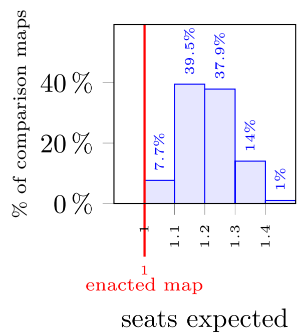
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0095% of maps.

5.12 Senate: Cumberland-Moore



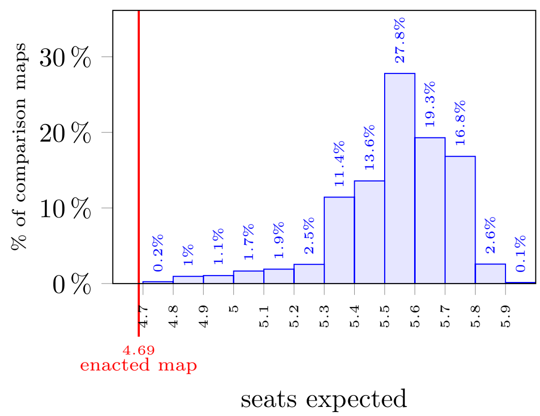
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.00235% of maps.

5.13 Senate: Forsyth-Stokes



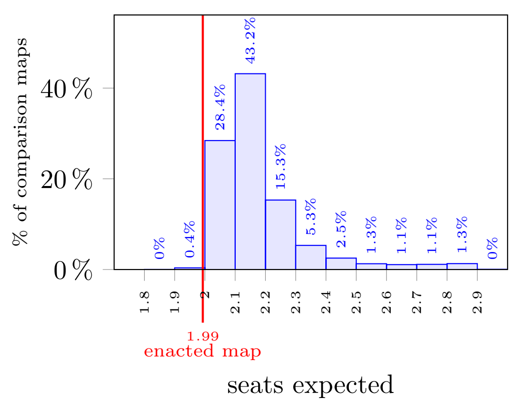
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0104% of maps.

5.14 Senate: Granville-Wake



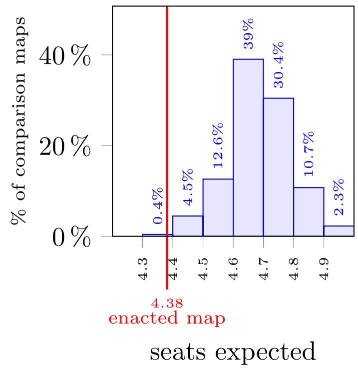
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0353% of maps.

5.15 Senate: Guilford-Rockingham



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.251% of maps.

5.16 Senate: Iredell-Mecklenburg



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.104% of maps.

Cluster	0	1	2	3	4	5	6	7	8	9	10	11	12
Davidson	100%												
Pitt		91%	9%										
Alamance	44%	56%											
Columbus-Robeson	100%												
Carteret-Craven													
Duplin-Wayne	100%												
Nash-Wilson			100%										
Caswell-Orange			100%										
Alexander-Surry-Wilkes	100%												
Franklin-Granville-Vance		100%											
Alleghany-*etc*	100%												
Beaufort-*etc*	100%												
Buncombe			28%	72%									
Anson-Union	100%												
Onslow-Pender	100%												
Cumberland				82%	18%								
Harnett-Johnston	100%												
Catawba-Iredell	100%												
Durham-Person					100%								
Brunswick-New Hanover		100%											
Forsyth-Stokes			33%	50%	17%								
Cabarrus-*etc*	99%	1%											
Chatham-*etc*	18%	82%											
Guilford					1%	79%	21%						
Avery-*etc*	100%												
Mecklenburg											1%	56%	44%
Wake											2%	32%	66%

Table 2: This table collects in one place the fraction of maps in Barber's House simulation sets realizing each number of lean-Democratic seats, as reported by Barber in his Figures 11, 14, 17, 20, 25, 28, 31, 34, 37, 45, 48, 51, 55, 58, 61, 64, 67, 70, 73, 76, 79, 82, 85, and 88. He does not present figures for the clusters in Alleghany-Ashe-Caldwell-Watauga and Beaufort-Chowan-Currituck-Dare-Hyde-Pamlico-Perquimans-Tyrrell-Washington clusters because his 0-Democratic-district results for those clusters are based on a very small number of maps. For Carteret-Craven his method does not produce any maps.

Cluster	0	1	2	3	4	5	6
Cumberland-Moore		77%	23%				
Chatham-Durham			100%				
Alleghany-*etc*	100%						
Brunswick-Columbus-New Hanover	23%	77%					
Bladen-*etc*	100%						
Guilford-Rockingham			94%	6%			
Alamance-*etc*	100%						
Granville-Wake					1%	24%	75%
Iredell-Mecklenburg					5%	95%	
Buncombe-Burke-McDowell		100%					
Cleveland-Gaston-Lincoln	100%						
Forsyth-Stokes		97%	3%				

Table 3: This table collects in one place the fraction of maps in Barber's Senate simulation sets realizing each number of lean-Democratic seats, as reported by Barber in his Figures 95, 98, 103, 106, 110, 113, 117, 120, 123, 128. He does not present figures for the Bladen-Duplin-Harnett-Jones-Lee-Pender-Sampson and Cleveland-Gaston-Lincoln clusters because his 0-district results for these clusters are based on a small number of maps.

I hereby certify that the foregoing statements are true and correct to the best of my knowledge, information, and belief.

Wesley Pegden
12/28/2021

of lean-Democratic districts for his simulations in each cluster; and, at the bottom of the column, for the total across clusters (he reports the range for this total as 46-51). Recall that the bottom of the middle-50% range is the lower quartile of the data, and the top of the range is the upper quartile.

For example, in the House:

- for the Buncombe cluster in the House map, Barber reports in Figure 45 that 28% of his simulated maps contained 2 lean-Democrat districts, while 72% contained 3.
- for the Cumberland cluster in the House map, Barber reports in Figure 55 that 82% of his simulated maps contained 3 districts, while 18% contained 4.

I summarize this information in my Table 1, below:

Cluster	0	1	2	3	4
Buncombe			28%	72%	
Cumberland				82%	18%

Table 1: Fraction of maps with various lean-Democrat-district counts, as reported by Barber for Buncombe and Cumberland county districtings.

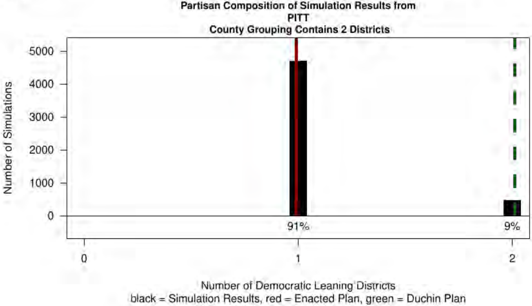
In his Table 2, Barber correctly summarizes the middle 50% ranges for the data in each of these clusters as 2-3 and 3-3, respectively; in each case, the lower end of the range is the smallest value below which 25% of his simulated maps lie, and the upper end is the smallest value below which 75% lie.

Suppose though, just as an example, that we wished to calculate the distribution of the total number of lean-Democrat districts across just these two clusters according the Barber's simulations; this will also enable us to calculate the middle-50% of outcomes for the total lean-Democrat districts across these two clusters.

Note that for maps of these two clusters composed of maps from Barber'simulations, a total of 5, 6, or 7 lean-Democrat districts are possible. For example, 5 lean-Democrat districts can arise only by having 2 such districts in Buncombe and 3 in Cumberland, and fewer are not possible.

– Ex. 4677 –

Figure 14: **Distribution of Partisan Districts from Simulations in Pitt House County Cluster**



This information is then reproduced in my Table 2 on page 13, as the following row:

Cluster	0	1	2	3	4	5	6	7	8	9	10	11	12
Pitt		91%	9%										

particular, according to Barber's own simulated map set, and using his own measure of the number of lean-Democrat districts under his own partisan index, **the enacted House map exhibits more Republican bias than 99.82% of maps** composed of Barber's simulations, over the clusters Barber analyzes.

– Ex. 4678 –

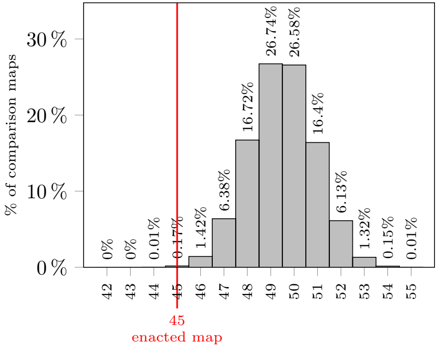


Figure 1: **Total lean-Democrat districts across Barber's House simulations**. This histogram shows the performance of Barber's simulated map set across the total set of House clusters Barber analyzes. It uses Barber's set of simulated maps, Barber's chosen metric (number of lean Democratic seats), calculated using the partisan metric Barber himself calculates in his report. The range 49-50 contains 50% of the simulated maps, the range 48-51 contains 86% of the simulated maps, and the range 47-52 contains more than 98% of the simulated maps. With 45 lean-Democratic districts across these clusters, the enacted map is in the most Republican-biased 0.18% of Barber's simulated maps.

In Table 3 I show Barber's Senate data analogous to the House data I show in Table 2. And in Figure 2, I plot the histogram showing the total of Barber's metric of Democratic-leaning districts across Barber's to Barber's own simulated map set, and using his own measure of the number of lean-Democrat districts under his own partisan index, **the enacted Senate map exhibits more Republican bias than 99.61% of maps** over the clusters Barber analyzes.

– Ex. 4679 –

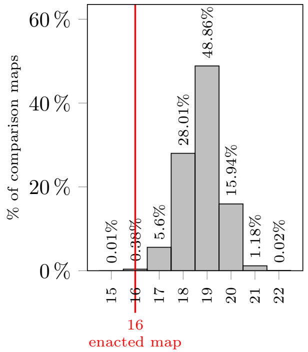


Figure 2: **Total lean-Democrat districts across Barber's Senate simulations**. This histogram shows the performance of Barber's simulated map set across the total set of Senate clusters Barber analyzes. It uses Barber's set of simulated maps, Barber's chosen metric (number of lean Democratic seats), calculated using the partisan metric Barber himself calculates in his report. The range 18-20 contains 93% of the simulated maps, and the range 17-21 contains more than 99% of the simulated maps. With 16 lean-Democrat districts, the enacted map is among the most Republican 00.39% of maps.

Technical Remark. As noted in the earlier Technical Remark, calculating the results of a histogram like Figure 1 is equivalent to expanding a certain polynomial expression. Based on the data in Table 2, (rows with only zero seats possible can be ignored), the polynomial to be expanded is possible maps (the first six House analyzed in the subsections below, and every Senate cluster analyzed below), Barber finds the map to be a partisan outlier according to the "middle-50%" definition he uses in his report. I summarize the outlier status of these 6+5 House and Senate clusters according to Barber's simulations in the following table:

Cluster	Enacted map among most Republican-biased...
House: Buncombe	00.797%
House: Forsyth-Stokes	00.0805%
House: Guilford	00.00646%
House: Mecklenburg	04.43%
House: Wake	05.78%
House: Pitt	24.2%
Senate: Cumberland-Moore	00.0024%
Senate: Forsyth-Stokes	00.01%
Senate: Granville-Wake	00.035%
Senate: Guilford-Rockingham	00.25%
Senate: Iredell-Mecklenburg	00.1%
	...against Barber's simulations.

Among the four remaining clusters in my report, there are two where the enacted maps are nevertheless extreme outliers against Barber's simulation sets. I summarize the results for these four clusters in the following table:

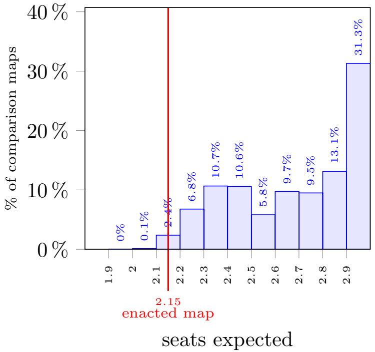
Enacted map among

	…against Barber's simulations.

Among the four remaining clusters in my report, there are two where the enacted maps are nevertheless extreme outliers against Barber's simulation sets. I summarize the results for these four clusters in the following table:

Cluster	Enacted map among most Republican-biased…
House: Alamance	39.4%
House: Brunswick-New Hanover	73.9%
House: Durham-Person	00.00265%
House: Cabarrus-Davie-Rowan-Yadkin	00.352%
	…against Barber's simulations.

5.1 House: Buncombe



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.797% of maps.

5.2 House: Forsyth-Stokes

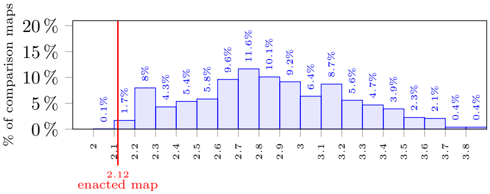
ps

seats expected

Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.797% of maps.

– Ex. 4683 –

5.2 House: Forsyth-Stokes



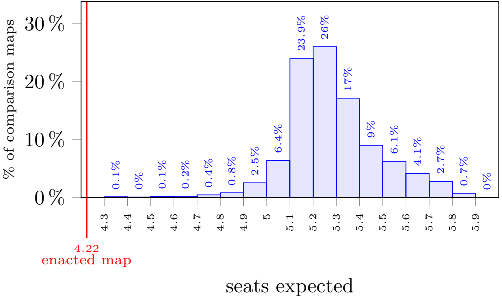
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0805% of maps.

5.3 House: Guilford

Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0805% of maps.

– Ex. 4684 –

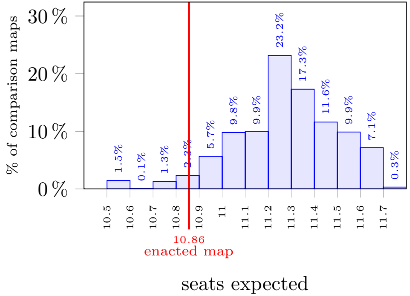
5.3 House: Guilford



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.00646% of maps.

– Ex. 4685 –

5.4 House: Mecklenburg



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 4.43% of maps.

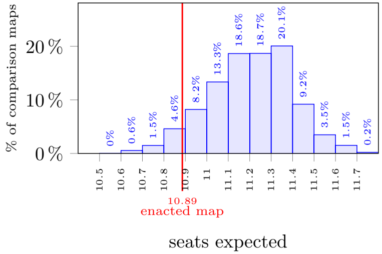
5.5 House: Wake

seats expected

Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 4.43% of maps.

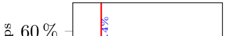
– Ex. 4686 –

5.5 House: Wake



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 5.78% of maps.

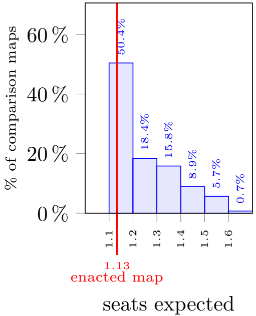
5.6 House: Pitt



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 5.78% of maps.

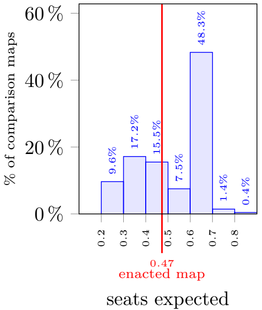
– Ex. 4687 –

5.6 House: Pitt



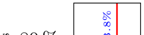
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 24.2% of maps.

5.7 House: Alamance



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map is not an outlier.

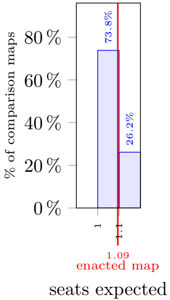
5.8 House: Brunswick-New Hanover



seats expected

Against the comparison-set of Barber's simulated maps for this cluster, the enacted map is not an outlier.

5.8 House: Brunswick-New Hanover



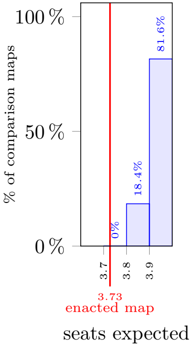
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map is not an outlier.

5.9 House: Durham-Person

enacted map
seats expected

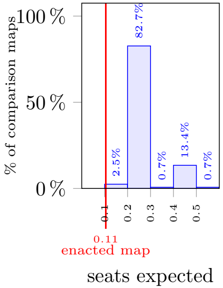
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map is not an outlier.

5.9 House: Durham-Person



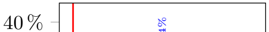
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.00265% of maps.

5.10 House: Cabarrus-Davie-Rowan-Yadkin



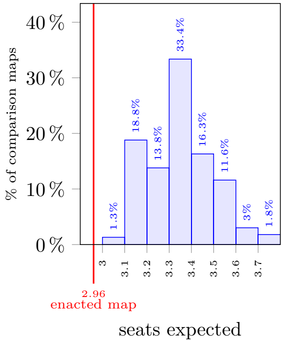
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.352% of maps.

5.11 House: Cumberland



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.352% of maps.

5.11 House: Cumberland



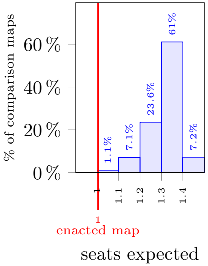
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0095% of maps.

5.12 Senate: Cumberland-Moore

seats expected

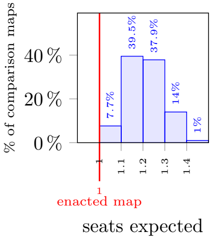
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0095% of maps.

5.12 Senate: Cumberland-Moore



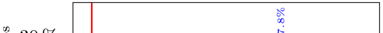
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.00235% of maps.

5.13 Senate: Forsyth-Stokes



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0104% of maps.

5.14 Senate: Granville-Wake

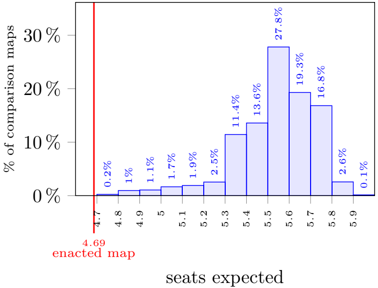


seats expected

Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0104% of maps.

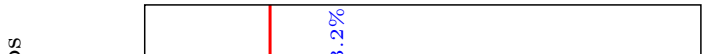
– Ex. 4695 –

5.14 Senate: Granville-Wake



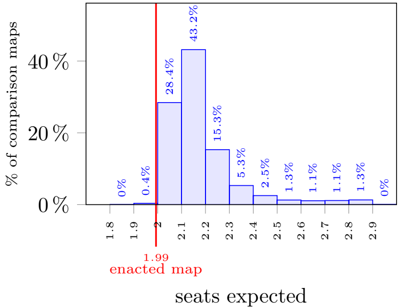
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0353% of maps.

5.15 Senate: Guilford-Rockingham



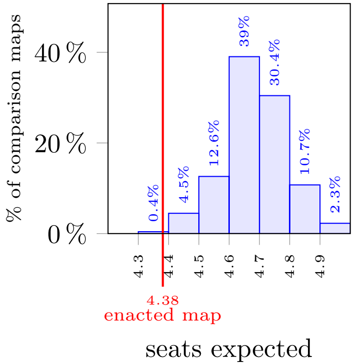
Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.0353% of maps.

5.15 Senate: Guilford-Rockingham



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.251% of maps.

5.16 Senate: Iredell-Mecklenburg



Against the comparison-set of Barber's simulated maps for this cluster, the enacted map in this cluster is among the most Republican-biased 0.104% of maps.

Cluster	0	1	2	3	4	5	6	7	8	9	10	11	12
Davidson	100%												
Pitt		91%	9%										
Alamance	44%	56%											
Columbus-Robeson	100%												
Carteret-Craven													
Duplin-Wayne	100%												
Nash-Wilson			100%										
Caswell-Orange			100%										
Alexander-Surry-Wilkes	100%												
Franklin-Granville-Vance		100%											
Alleghany-*etc*	100%												
Beaufort-*etc*	100%												
Buncombe			28%	72%									
Anson-Union	100%												
Onslow-Pender	100%												
Cumberland				82%	18%								
Harnett-Johnston	100%												
Catawba-Iredell	100%												
Durham-Person					100%								
Brunswick-New Hanover		100%											
Forsyth-Stokes			33%	50%	17%								
Cabarrus-*etc*	99%	1%											
Chatham-*etc*	18%	82%											
Guilford					1%	79%	21%						
Avery-*etc*	100%												
Mecklenburg											1%	56%	44%
Wake											2%	32%	66%

Table 2: This table collects in one place the fraction of maps in Barber's House simulation sets realizing each number of lean-Democratic seats, as reported by Barber in his Figures 11, 14, 17, 20, 25, 28, 31, 34, 37, 45, 48, 51, 55, 58, 61, 64, 67, 70, 73, 76, 79, 82, 85, and 88. He does not present figures for the clusters in Alleghany-Ashe-Caldwell-Watauga and Beaufort-Chowan-Currituck-Dare-Hyde-Pamlico-Perquimans-Tyrrell-Washington clusters because his 0-Democratic-district results for those clusters are based on a very small number of maps. For Carteret-Craven his method does not produce any maps.

Cluster	0	1	2	3	4	5	6
Cumberland-Moore		77%	23%				
Chatham-Durham			100%				
Alleghany-*etc*	100%						
Brunswick-Columbus-New Hanover	23%	77%					
Bladen-*etc*	100%						
Guilford-Rockingham			94%	6%			
Alamance-*etc*	100%						
Granville-Wake					1%	24%	75%
Iredell-Mecklenburg					5%	95%	
Buncombe-Burke-McDowell		100%					
Cleveland-Gaston-Lincoln	100%						
Forsyth-Stokes		97%	3%				

Table 3: This table collects in one place the fraction of maps in Barber's Senate simulation sets realizing each number of lean-Democratic seats, as reported by Barber in his Figures 95, 98, 103, 106, 110, 113, 117, 120, 123, 128. He does not present figures for the Bladen-Duplin-Harnett-Jones-Lee-Pender-Sampson and Cleveland-Gaston-Lincoln clusters because his 0-district results for these clusters are based on a small number of maps.

– Ex. 4700 –

Taylor & Francis
Taylor & Francis Group

Statistics and Public Policy

ISSN: (Print) 2330-443X (Online) Journal homepage: https://www.tandfonline.com/loi/uspp20

Separating Effect From Significance in Markov Chain Tests

Maria Chikina, Alan Frieze, Jonathan C. Mattingly & Wesley Pegden

To cite this article: Maria Chikina, Alan Frieze, Jonathan C. Mattingly & Wesley Pegden (2020) Separating Effect From Significance in Markov Chain Tests, Statistics and Public Policy, 7:1, 101-114, DOI: 10.1080/2330443X.2020.1806763

To link to this article: https://doi.org/10.1080/2330443X.2020.1806763

Published online: 05 Oct 2020.

STATISTICS AND PUBLIC POLICY
2020, VOL. 7, NO. 1, 101–114
https://doi.org/10.1080/2330443X.2020.1806763

Taylor & Francis
Taylor & Francis Group

OPEN ACCESS

Check for updates

Separating Effect From Significance in Markov Chain Tests

Maria Chikina[a], Alan Frieze[b], Jonathan C. Mattingly[c], and Wesley Pegden[b]

[a]Department of Computational and Systems Biology, University of Pittsburgh, Pittsburgh, PA; [b]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA; [c]Department of Mathematics, Duke University, Durham, NC

ABSTRACT

We give qualitative and quantitative improvements to theorems which enable significance testing in Markov chains, with a particular eye toward the goal of enabling strong, interpretable, and statistically rigorous claims of political gerrymandering. Our results can be used to demonstrate at a desired significance level that a given Markov chain state (e.g., a districting) is extremely unusual (rather than just atypical) with respect to the fragility of its characteristics in the chain. We also provide theorems specialized to leverage quantitative improvements when there is a product structure in the underlying probability space, as can occur due to geographical constraints on districtings.

ARTICLE HISTORY
Received September 2019
Accepted June 2020

KEYWORDS
Gerrymandering; Markov chain; Outlier

This note discusses improvements on a number of theorems for significance testing in Markov chains. The improvements to the theorem statements are both qualitative and quantitative to enable strong, easily interpretable statistical claims, and include extensions to settings where more structural assumptions lead to dramatic improvements in the bounds. This class of theorems is of particular interest because they do not assume that the chain has converged to equilibrium, which is of practical importance since the mixing time of Markov chains used in applications is frequently unknown.

The development of this class of algorithms and these particular extensions have been directly motivated by a question of significant contemporary interest; detecting and quantifying gerrymandering. The definiteness and correctness provided by these theorems affords decision makers (e.g., in a legal setting) with an uncommonly rigorous approach to understanding whether a political districting is carefully crafted for partisan advantage. These methods (and theorems) have been used successfully by one of the authors in Gerrymandering court cases in Pennsylvania and North Carolina.

The first technical part of this article gives new results along these lines, extending the work in Chikina, Frieze, and Pegden (2017) to allow separation of effect size from the quantification of statistical significance. The second part, in Section 8, develops versions of some of these results in a special setting with a particular structure on the probability space motivated by recent legal proceedings. In particular, in balancing the federal one-person-one-vote mandate with the "keep counties whole" prevision of the North Carolina Constitution, the North Carolina courts ruled in *Stephenson v. Bartlett* that a particular algorithm should be used to "cluster" the counties into independent county groups which are districted separately. This gives a product structure to the underlying probability space which can be exploited in theorems designed to take advantage of it.

In Section 1, we consider the technical background and past results necessary to frame the new results of this article; in Section 2, we discuss the practical challenges which motivate the new theoretical framework we develop in this article. The new results are stated and discussed in Section 3. Later sections prove the new theorems.

1. Background and Previous Results

Consider a reversible Markov chain $\mathcal{M}$ whose state-space Σ is endowed with some labeling $\omega : \Sigma \to \mathbb{R}$, and for which π is a stationary distribution. $\mathcal{M}, \pi, \omega$, and a fixed integer k determine a vector

$$p_0^k, p_1^k, \dots, p_k^k,$$

where for each i, p_i^k is the probability that for a k-step π-stationary trajectory $X_0, \dots, X_k$, the minimum ω value occurs at X_i. In other words, p_i^k is the probability that if we choose X_0 randomly from the stationary distribution π and take k steps in $\mathcal{M}$ to obtain the trajectory $X_0, X_1, \dots, X_k$, that we observe that $\omega(X_i)$ is the minimum among $\omega(X_0), \dots, \omega(X_k)$. Note that if we adopted the convention that we break ties among the values $\omega(X_0), \dots, \omega(X_k)$ randomly, we would have that $p_0^k + \dots + p_k^k = 1$, for any $\mathcal{M}, \pi$, and k.

In the context of districting, $\mathcal{M}$ is generally a random walk on the space of possible districtings of a state; for example, from one districting, we can randomly choose a voting precinct on the boundary of two districts, and switch its district membership if doing so does not violate constraints on contiguity, population deviation, etc. (see Figure 1 for an example of such a

CONTACT Wesley Pegden wes@math.cmu.edu Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213.

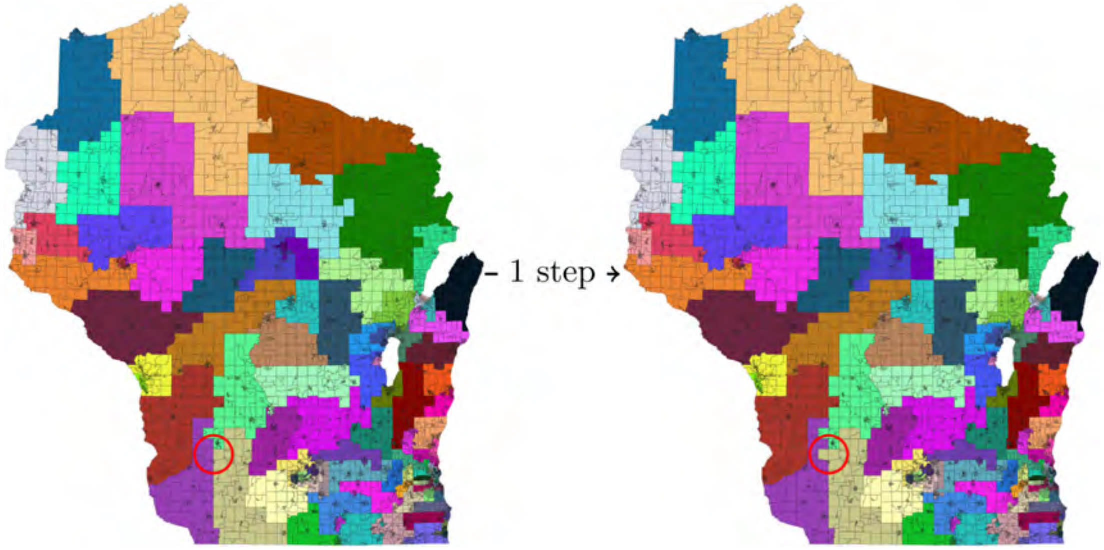


Figure 1. An example of a Markov chain transition for the legislative districting of Wisconsin.

move). This transition rule defines a reversible Markov chain on the state space consisting of all valid districtings of the state. The label function ω could be, for example, the partisanship of any given districting—which could be defined, for example, from simulated election results using historical voting data. In this context, p_0^k is the probability that if began from a random districting of a state, and carried out a sequence of k random transitions, that the initial districting in the sequence would be the most partisan districting observed in the sequence.

At first glance, it might be natural to assume that we must have something like $p_i^k \approx \frac{1}{k+1}$ for all $0 \leq i \leq k$. This would mean, for example, that if we generated a sequence of 10,000 districtings, and the initial districting was the most partisan, that it would be valid to conclude that the initial districting was not randomly chosen from the stationary distribution, at a p-value of $p = 0.0001$. But this is actually quite far from the truth; Chikina, Frieze, and Pegden (2017) showed that for some $\mathcal{M}, \pi, k$, we can have p_0^k as large as essentially $\frac{1}{\sqrt{2\pi k}}$.

As shown in Chikina, Frieze, and Pegden (2017), this is essentially the worst possible behavior for p_0^k. In particular, we can generalize the vector $\{p_i^k\}$ defined above as possible: let us define, given $\mathcal{M}, \pi, k$, and $0 \leq \varepsilon \leq 1$, the vector

$$p_{0,\varepsilon}^k, p_{1,\varepsilon}^k, \ldots, p_{k,\varepsilon}^k,$$

where each $p_{i,\varepsilon}^k$ is the probability that $\omega(X_i)$ is among the smallest ε fraction of values in the list $\omega(X_0), \ldots, \omega(X_k)$. Then in Chikina, Frieze, and Pegden (2017) we proved:

Theorem 1.1. Given a reversible Markov chain $\mathcal{M}$ with stationary distribution π, an $\varepsilon > 0$, $k \geq 0$, and with $p_{i,\varepsilon}^k$ defined as above, we have that

$$p_{0,\varepsilon}^k \leq \sqrt{2\varepsilon}.$$

In particular, in the case of the first districting which is most partisan on a sequence of 10,000, this supports a p-value of $p = \sqrt{2}/100 \approx 0.014$. Note that the example from Chikina, Frieze, and Pegden (2017) realizing $p_0^k \approx \frac{1}{\sqrt{2\pi k}}$ shows that this theorem is best possible, up to constant factors.

As indicated by the example of making a sequence of random changes to a districting, one important application of Theorem 1.1 is that it characterizes the statistical significance associated to the result of the following natural test for gerrymandering of political districtings:

Local outlier test

1. Beginning from the districting being evaluated,
2. Make a sequence of random changes to the districting, while preserving some set of constraints imposed on the districtings.
3. Evaluate the partisan properties of each districting encountered (e.g., by simulating elections using past voting data).
4. Call the original districting "carefully crafted" or "gerrymandered" if the overwhelming majority of districtings produced by making small random changes are less partisan than the original districting.

Naturally, the test described above can be implemented so that it precisely satisfies the hypotheses of Theorem 1.1. For this purpose, a (very large) set of comparison districtings are defined, to which the districting being evaluated belongs. For example, the comparison districtings may be the districtings built out of Census blocks (or some other unit) which are contiguous, equal in population up to some specified deviation, or include other constraints. A Markov chain $\mathcal{M}$ is defined on this set of districtings, where transitions in the chain correspond to changes in districtings. (E.g., a transition may correspond to randomly changing the district assignment of a randomly chosen Census block which currently borders more than one district, subject to the constraints imposed on the comparison set.) The "random changes" from Step 2 will then be precisely governed by the transition probabilities of the Markov chain $\mathcal{M}$. By designing $\mathcal{M}$ so that the uniform distribution π on the set of comparison districtings Σ is a stationary distribution for $\mathcal{M}$, Theorem 1.1 gives an upper bound on the false-positive rate (in other words, global statistical significance) for the "gerrymandered" declaration when it is made in Step 4.

Remark 1.1. Note that while a local outlier test can be used to give a statistical significant finding of gerrymandering, it can*not* be used alone to demonstrate rigorously that a districting is *not* gerrymandered. In particular, when a districting does not seem gerrymandered to a local outlier test, it is quite possible that the chain is simply not mixing well enough to explore the space of alternatives. The theorems discussed in this article give one-way guarantees: for example, observing outlier status confers a statistically significant finding of gerrymandering, but failing to observe it is inconclusive, absent other evidence.

Apart from its application to gerrymandering, Theorem 1.1 has a simple informal interpretation for the general behavior of reversible Markov chains, namely: *typical (i.e., stationary) states are unlikely to change in a consistent way under a sequence of chain transitions*, with a best-possible quantification of this fact (up to constant factors).

Also, in the general setting of a reversible Markov chain, the theorem leads to a simple quantitative procedure for asserting rigorously that σ_0 is atypical with respect to π without knowing the mixing time of $\mathcal{M}$: simply observe a random trajectory $\sigma_0 = X_0, X_1, X_2, \dots, X_k$ from σ_0 for any fixed k. If $\omega(\sigma_0)$ is an ε-outlier among $\omega(X_0), \dots, \omega(X_k)$, then this is statistically significant at $\sqrt{2\varepsilon}$ against the null hypothesis that $\sigma_0 \sim \pi$.

This quantitative test is potentially useful because $\sqrt{2\varepsilon}$ converges quickly enough to 0 as $\varepsilon \to 0$; in particular, it is possible to obtain good statistical significance from observations which can be made with reasonable computational resources. Of course, faster convergence to 0 would be even better, but, as already noted, $p \approx \sqrt{\varepsilon}$ is roughly a best possible upper bound.

On the other hand, it *is* possible to achieve better dependence on ε by changing the parameters of the test. For example, we will prove the following theorem as a stepping-stone to the main results of the present manuscript:

Theorem 1.2. Consider two independent trajectories $Y_0, \dots, Y_k$ and $Z_0, \dots, Z_k$ in the reversible Markov chain $\mathcal{M}$ (whose states have real-valued labels) from a common starting point $Y_0 = Z_0 = \sigma_0$. If we choose σ_0 from a stationary distribution π of $\mathcal{M}$, then for any k we have that

$$\Pr\big(\omega(\sigma_0) \text{ is an } \varepsilon\text{-outlier among } \omega(\sigma_0), \omega(Y_1), \dots, \omega(Y_k), \omega(Z_1), \dots, \omega(Z_k)\big) < 2\varepsilon.$$

Note that due to the reversibility of the chain, Theorem 1.2 is equivalent to the statement that the probabilities $p^k_{i,\varepsilon}$ always satisfy

$$p^{2k}_{k,\varepsilon} < 2\varepsilon. \tag{1}$$

Remark 1.2. As in the case of Theorem 1.1, it seems like an interesting question to investigate the tightness of the constant 2; we will see in Section 8 that there are settings where the impact of this constant is inflated to have outsize-importance. We point out here that at least for the case of $k = 1$, $\varepsilon = 1/3$, $\rho^2_{1,\frac{1}{3}}$ can be at least as large as $\frac{1}{2}$, showing that the constant 2 in (1) cannot be replaced by a constant less than $\frac{3}{2}$, in general. To see this, consider, for example, a bipartite complete graph $K_{n,n}$, where the labels of the vertices of one side are $1, \dots, n$ and the other are $n+1, \dots, 2n$. For the Markov chain given by the random walk on this undirected graph, we have that $\rho^2_{1,\frac{1}{3}} = \frac{1}{2}$. Note that for this example, it is still the case that $\rho^{2k}_{k,\varepsilon} \to \varepsilon$ as $k \to \infty$, leaving open the possibility that the 2 in (1) can be replaced with an expression asymptotically equivalent to 1.

The informal interpretation of Theorem 1.2 is thus: *typical (i.e., stationary) states are unlikely to change in a consistent way under* two *sequences of chain transitions.*

Unknown to the authors at the time of the publication of Chikina, Frieze, and Pegden (2017), Besag and Clifford (1989) described a test related to that based on Theorem 1.2, which has essentially a one-line proof, which we discuss in Section 5:

Theorem 1.3 (Besag and Clifford serial test). Fix any number k and suppose that σ_0 is chosen from a stationary distribution π, and that ξ is chosen uniformly in $\{0, \dots, k\}$. Consider two independent trajectories $Y_0, Y_1, \dots$ and $Z_0, Z_1, \dots$ in the reversible Markov chain $\mathcal{M}$ (whose states have real-valued labels) from $Y_0 = Z_0 = \sigma_0$. If we choose σ_0 from a stationary distribution π of $\mathcal{M}$, then for any k we have that

$$\Pr\big(\omega(\sigma_0) \text{ is an } \varepsilon\text{-outlier among } \omega(\sigma_0), \omega(Y_1), \dots, \omega(Y_\xi), \omega(Z_1), \dots, \omega(Z_{k-\xi})\big) \le \varepsilon.$$

Here, a real number a_0 is an ε*-outlier* among $a_0, \dots, a_k$ if

$$\#\{i \in \{0, \dots, k\} \mid a_i \le a_0\} \le \varepsilon(k+1).$$

In particular, the striking thing about Theorem 1.3 is that it achieves a best-possible dependence on the parameter ε. (Notice that ε would be the correct value of the probability if, for example, the Markov chain is simply a collection of independent random samples.) The sacrifice is in Theorem 1.3's slightly more complicated intuitive interpretation, which would be: *typical (i.e., stationary) states are unlikely to change in a consistent way under two sequences of chain transitions of random complementary lengths.* Note that the pattern in these theorems is that the simplicity of the intuitive interpretation of the theorem is sacrificed for the quantitative bounds offered; one goal of the present article is prove theorems which avoid this trade off.

2. The Need for New Statistical Approaches

In late 2018, the nonprofit Common Cause filed a lawsuit in North Carolina superior court challenging the state-level districting plans (for the North Carolina state House and Senate districts). The third and fourth authors of this article served as expert witnesses in this case, which ultimately found the challenged districtings to be unconstitutional partisan gerrymanders. In this section, we use some of the findings from the fourth author's expert report Pegden (2019) in that case as motivation for the need for the new theoretical results we advance in this article.

One interesting feature of districting in North Carolina is the requirement (from the state supreme court case *Stephenson v. Bartlett*) that districtings of that state must respect particular

Table 1. Results of the analysis from Pegden (2019) for the House districting of North Carolina.

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.999994%	5	99.9999999964%	9	99.999997%	13	99.99984%
2	99.99999999963%	6	99.9999999988%	10	99.99999999909%	14	99.9999999986%
3	99.99999999981%	7	99.99999945%	11	99.9999999949%	15	99.9999948%
4	99.9999984%	8	99.999998%	12	99.9999982%	16	99.9999999973%

Figure 2. First: The challenged State House districting of North Carolina, and three examples of maps encountered by a sequence of random changes.

groupings—*county clusters*—determined, essentially deterministically, by a prescribed set of rules.

A complete districting of the state of North Carolina is thus composed of completely separate, noninteracting districting problems in each county cluster. One practical question of interest in *Common Cause v. Lewis*—separate from the question of whether the challenged districtings were partisan gerrymanders—was the question of which specific county clusters were gerrymandered in each (House and Senate) districting.

The basic approach of the analysis used in Pegden (2019) begins with an experiment in which random changes are made to the actual, enacted districting of the state being evaluated. For example, Table 1 (taken from Pegden (2019)) shows the results of 16 such experiments run on the North Carolina House districting:

Each number shows, as a percentage, the fraction of districtings encountered in the sequence of random changes which were more Republican-favorable than the enacted plan. The particular choices of partisan metrics and voting data can be found in Pegden (2019). (Figure 2 shows the initial map as well as three examples of maps encountered in the first experiment.)

The percentages recorded in Table 1 show that the enacted House plan is an extreme outlier among the plans generated by making small random changes to the enacted plan. This already gives strong intuitive evidence that the enacted plan is extremely carefully drawn to maximize partisan advantage: as soon as the lines of the map are subjected to small random changes, the overwhelming fraction of encountered maps are less advantageous to Republicans than the enacted plan.

But the next step of the analysis in Pegden (2019) is the application of rigorous theorems to establish statistical significance for the findings in Table 1. This is slightly subtle, since, as we will see when setting up the technical details later, we do *not* make the heuristic assumption that the randomly generated plans are uniform samples from some set of possible maps; instead, we wish to make rigorous statistical claims with respect to the actual experiment, a Markovian process of making a sequence of random changes to the initial map.

The previous paper (Chikina, Frieze, and Pegden 2017) also took this approach in an analysis of the Congressional districting of Pennsylvania (which later formed the basis of expert testimony in the *League of Women Voters v. Pennsylvania* case which challenged the Congressional districting there). The theorem from Chikina, Frieze, and Pegden (2017) allows one to establish statistical significance of $p = \sqrt{2\varepsilon}$ when one finds that a $1 - \varepsilon$ fraction of maps are less advantageous to (say) the Republicans than the enacted map. For example, for Run 1 in Table 1, $\varepsilon = 0.00000006$ and so $p = 0.0003$ is quite statistically significant, against the null hypothesis of a randomly chosen districting.

But an important question in the *Common Cause v. Lewis* lawsuit was not simply whether unconstitutional gerrymandering took place in the drawing of the North Carolina plans, but in which county clusters it took place (so that districtings in those clusters could be ordered redrawn). As an example we will consider the county cluster consisting of Forsyth and Yadkin counties; the challenged map and 3 examples of maps encountered by the sequence of random changes are shown in Figure 3, while the table of results for this cluster are shown in Table 2. While Table 2 shows that the overwhelming fraction of maps encountered by making random changes were less favorable to Republicans, ε values of around 0.002 in this case are not small enough that $\sqrt{2\varepsilon}$ would be statistically significant. In particular, even if we applied a test like that given by Theorem 1.2 or Theorem 1.3, our level of statistical confidence would be limited by the degree to which the districting in this county cluster is an outlier.

Instead, we wish to decouple the statistical significance of our finding from the level of outlier status we can report for the

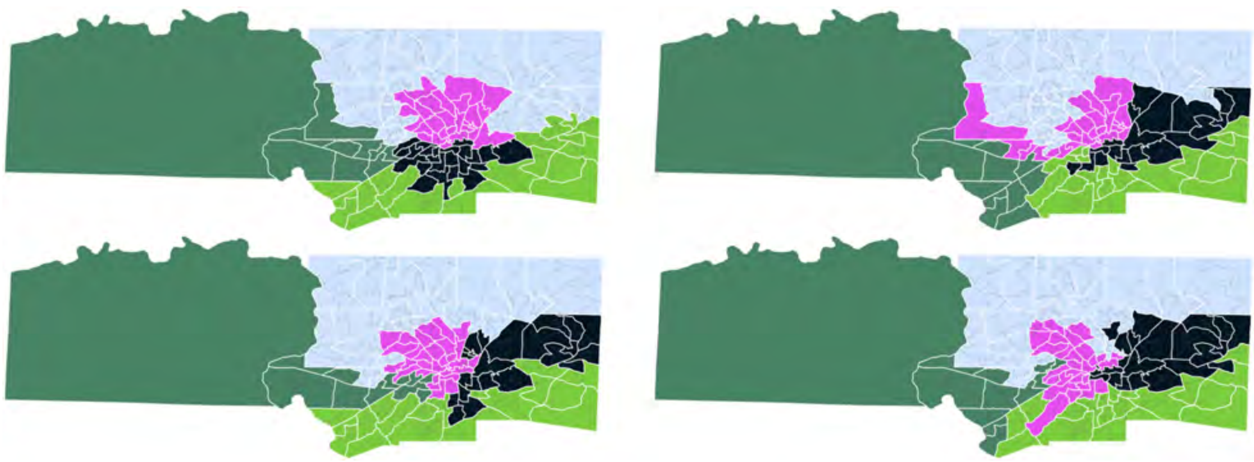


Figure 3. First: The challenged State House districting within the Forsyth-Yadkin county cluster, and three examples of maps encountered by a sequence of random changes to this districting.

Table 2. Results of the analysis from Pegden (2019) for the House districting of North Carolina in the Forsyth-Yadkin cluster.

Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan	Run	Percentage of comparison maps less partisan than enacted plan
1	99.945%	9	99.905%	17	99.87%	25	99.92%
2	99.76%	10	99.925%	18	99.84%	26	99.84%
3	99.87%	11	99.8%	19	99.929%	27	99.81%
4	99.86%	12	99.911%	20	99.73%	28	99.85%
5	99.77%	13	99.8%	21	99.88%	29	99.83%
6	99.89%	14	99.927%	22	99.906%	30	99.77%
7	99.91%	15	99.82%	23	99.8%	31	99.947%
8	99.79%	16	99.88%	24	99.88%	32	99.937%

districting in this cluster; this is a major goal of the approach we take in this note.

3. New Results

One common feature of the tests based on Theorems 1.1 and 1.3 is the use of randomness. In particular, the probability space at play in these theorems includes both the random choice of σ_0 assumed by the null hypothesis and the random steps taken by the Markov chain from σ_0. Thus, the measures of "how (globally) unusual" σ_0 is with respect to its performance in the local outlier test and "how sure" we are that σ_0 is unusual in this respect are intertwined in the final p-value. In particular, the effect size and the statistical significance are not explicitly separated.

To further the goal of simplifying the interpretation of the results of these tests, our approach in this note will also show that tests like these can be efficiently used in a way which separates the measure of statistical significance from the question of the magnitude of the effect. For example, our new results would allow one to capture the outlier status of a cluster like Forsyth-Yadkin discussed in the previous section, at a p-value which can be made arbitrarily small, independent of the observed ε values.

To begin, let us recall the probabilities $p^k_{0,\varepsilon}, \dots, p^k_{k,\varepsilon}$ defined previously, let us define the ε-failure probability $p^k_{0,\varepsilon}(\sigma_0)$ to be the probability that on a trajectory $\sigma_0 = X_0, X_1, \dots, X_k$, $\omega(\sigma_0)$ is among the smallest ε fraction of the list $\omega(X_0), \dots, \omega(X_k)$. Now we make the following definition:

Definition 3.1. With respect to k, the state σ_0 is an *(ε, α)-outlier* in $\mathcal{M}$ if, among all states in $\mathcal{M}$, $p^k_{0,\varepsilon}(\sigma_0)$ is in the largest α fraction of the values of $p^k_{0,\varepsilon}(\sigma)$ over *all* states $\sigma \in \mathcal{M}$, weighted according to π.

In particular, being an (ε, α)-outlier measures the likelihood of σ_0 to fail the local outlier test, ranked against *all other states* $\sigma \sim \pi$ of the chain $\mathcal{M}$. For example, fix $k = 10^9$. If σ_0 is a $(10^{-6}, 10^{-5})$-outlier in $\mathcal{M}$ and π is the uniform distribution, this means that among *all* states $\sigma \in \mathcal{M}$, σ_0 is more likely than all but a 10^{-5} fraction of states to have an ω-value in the bottom 10^{-6} values $\omega(X_0), \omega(X_1), \dots, \omega(X_{10^9})$. Note that the probability space underlying the "more likely" claim here just concerns the choice of the random trajectory $X_1, \dots, X_{10^9}$ from $\mathcal{M}$.

Note that whether σ_0 is a (ε, α)-outlier is a deterministic question about the properties of $\sigma_0, \mathcal{M}$, and ω. Thus, it is a deterministic measure (defined in terms of certain probabilities) of the extent to which σ_0 is unusual (globally, in all of $\mathcal{M}$) with respect to its local fragility in the chain.

The following theorem enables one to assert statistical significance for the property of being an (ε, α)-outlier. In particular, while tests based on Theorems 1.1 and 1.3 take as their null hypothesis that $\sigma_0 \sim \pi$, the following theorem takes as its null hypothesis merely that σ_0 is not an (ε, α)-outlier.

Theorem 3.1. Consider m independent trajectories

$$\mathcal{T}^1 = (X_0^1, X_1^1, \ldots, X_k^1),$$
$$\vdots$$
$$\mathcal{T}^m = (X_0^m, X_1^m, \ldots, X_k^m)$$

of length k in the reversible Markov chain $\mathcal{M}$ (whose states have real-valued labels) from a common starting point $X_0^1 = \cdots = X_0^m = \sigma_0$. Define the random variable ρ to be the number of trajectories $\mathcal{T}^i$ on which σ_0 is an ε-outlier.

If σ_0 is not an (ε, α)-outlier, then

$$\Pr\left(\rho \geq m\sqrt{\frac{2\varepsilon}{\alpha}} + r\right) \leq e^{-\min(r^2\sqrt{\alpha/2\varepsilon}/3m, r/3)}. \tag{2}$$

In particular, apart from separating measures of statistical significance from the quantification of a local outlier, Theorem 3.1 connects the intuitive local outlier test tied to Theorem 1.1 (which motivates the definition of a (ε, α)-outlier) to the better quantitative dependence on ε in Theorem 1.3.

To compare the quantitative performance of Theorem 3.1 to Theorems 1.1 and 1.3, consider the case of a state σ_0 for which a random trajectory $\sigma_0 = X_0, X_1, \ldots, X_k$ is likely (say with some constant probability p') to find σ_0 an ε'-outlier. For Theorem 1.1, significance at $p \approx \sqrt{2\varepsilon}$ would be obtained[1], while using Theorem 1.3, one would hope to obtain significance of $\approx \varepsilon'$. Applying Theorem 3.1, we would expect to see ρ around $m \cdot p'$. In particular, we could demonstrate that σ_0 is an (ε', α) outlier for $\alpha = \frac{3\varepsilon}{(p')^2}$ (a linear dependence on ε) at a p-value which can be made arbitrarily small (at an exponential rate) as we increase the number of observed trajectories m. As we will see in Section 6, the exponential tail in (2) can be replaced by a binomial tail. In particular, the following special case applies:

Theorem 3.2. With $\mathcal{T}^1, \ldots, \mathcal{T}^m$ as in Theorem 3.1, we have that if σ_0 is not an (ε, α) outlier, then

$$\Pr\left(\sigma_0 \text{ an } \varepsilon\text{-outlier on all of } \mathcal{T}^1, \ldots, \mathcal{T}^m\right) \leq \left(\frac{2\varepsilon}{\alpha}\right)^{m/2}.$$

Theorem 3.2 also has advantages from the standpoint of avoiding the need to correct for multiple hypothesis testing, as we discuss in Section 4.

Example 3.1. Based on the output in Table 2, we can apply Theorem 3.2 to report that the enacted districting of the Forsyth-Yadkin cluster is an (α, ε)-outlier for $\alpha = 0.9\%$ and $\varepsilon = 0.3\%$, at a statistical significance of $p = 0.002$. In other words, among *all* possible districtings defined by the constraints imposed (not just those encountered in the 32 runs), the enacted plan has a greater ε-failure probability than 99.1% of districtings of the cluster, a finding we are confident in at a statistical significance of $p = 0.002$.

[1]Multiple tests have limited utility here or with Theorem 1.3 since there is no independence (the null hypothesis $\sigma_0 \sim \pi$ is not being resampled). In particular, multiple runs might be done merely until a trajectory is seen on which σ_0 is indeed an ε' outlier (requiring $1/p'$ runs, on average), in conjunction with multiple hypothesis testing.

In the article where Theorem 1.3 is proved, Besag and Clifford also describe a parallel test, which we will discuss in Section 7. In particular, in Section 7, we will describe a test which generalizes Besag and Clifford's serial and parallel tests in a way which could be useful in certain parallel regimes.

Finally, we consider an interesting case in the analysis of districtings that arises when the districting problem can be decomposed into several noninteracting districting problems, as is the case in North Carolina because of the county clusters. In this case, the probability space of random districtings is really a product space, and this structure can be exploited in a strong way for the statistical tests developed in this manuscript. We develop results for this setting in Section 8.

The remaining sections of the article are devoted to the proofs of the new results.

4. Multiple Hypothesis Considerations

When applying Theorem 3.1 directly, one cannot simply run m trajectories, observe the list $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ where each ε_i is the minimum ε_i for which σ_0 is an ε_i-outlier on $\mathcal{T}^i$, and then, post-hoc, freely choose the parameters α and ε in Theorem 3.1 to achieve some desired trade-off between α and the significance p.

The problem, of course, is that in this case one is testing multiple hypotheses (infinitely many in fact; one for each possible pair ε and α) which would require a multiple hypothesis correction.

One way to avoid this problem is to essentially do a form of cross-validation, were a few trajectories are run for the purposes of selecting suitable ε and α, and then discarded from the set of trajectories from which we obtain significance.

A simpler approach, however, is to simply set the parameter $\varepsilon = \varepsilon_{(t)}$ as the tth-smallest element of the list $\varepsilon_1, \ldots, \varepsilon_m$ for some fixed value t. The case $t = m$, for example, corresponds to taking ε as the maximum value, leading to the application of Theorem 3.2.

The reasons this avoids the need for a multiple hypothesis correction is that we can order our hypothesis events by containment. In particular, when we apply this test with some value of t, we will always have $\rho = t$. Thus, the significance obtained will depend just on the parameter $\varepsilon_{(t)}$ returned by taking the tth smallest ε_i and on our choice of α (as opposed to say, the particular values of the other ε_i's which are not the t-th smallest). In particular, regardless of how we wish to trade-off the values of α and p we can assert from our test, our optimum choice of α (for our fixed choice of t) will depend just on the value $\varepsilon_{(t)}$. In particular, we can view α as a function $\alpha(\varepsilon_{(t)})$, so that we when applying Theorem 3.1 with $\varepsilon = \varepsilon_{(t)}$, we are evaluating the single-parameter infinite family of hypotheses $H_{\varepsilon_{(t)},\alpha(\varepsilon_{(t)})}$, and we do not require multiple hypothesis correction since the hypotheses are nested; that is, since

$$\varepsilon_{(t)} \leq \varepsilon'_{(t)} \implies H_{\varepsilon_{(t)},\alpha(\varepsilon_{(t)})} \subseteq H_{\varepsilon'_{(t)},\alpha(\varepsilon_{(t)})}. \tag{3}$$

Indeed, (3) implies that

$$\Pr\left(\bigcup_{\varepsilon_{(t)} \leq \beta} H_{\varepsilon_{(t)},\alpha(\varepsilon_{(t)})}\right) = \Pr(H_{\beta,\alpha(\beta)}),$$

which ensures that when applying Theorem 3.1 in this scenario, the probability of returning a p-value $\leq p_0$ for any fixed value p_0 will indeed be at most p_0.

5. Proof Background

We begin this section by giving the proof of Theorem 1.2. In doing so we will introduce some notation that will be useful throughout the rest of this note. To make things as accessible as possible, we give every detail of the proof.

In this article, a Markov chain $\mathcal{M}$ on Σ is specified by the transition probabilities $\{\pi_{\sigma_1,\sigma_2} \mid \sigma_1, \sigma_2 \in \Sigma\}$ of a chain. A *trajectory* of $\mathcal{M}$ is a sequence of random variables $X_0, X_1, \ldots$ required to have the property that for each i and $\sigma_0, \ldots, \sigma_i$, we have

$$\begin{aligned}\Pr\left(X_i = \sigma_i \mid X_{i-1} = \sigma_{i-1}, X_{i-2} = \sigma_{i-2} \ldots, X_0 = \sigma_0\right) \\ = \pi_{\sigma_i,\sigma_{i-1}}. \qquad (4)\end{aligned}$$

In particular, the Markov property of the trajectory is that the conditioning on $X_{i-2}, X_{i-3}, \ldots$ is irrelevant once we condition on the value of X_{i-1}. Recall that π is a stationary distribution if $X_0 \sim \pi$ implies that $X_1 \sim \pi$ and thus also that $X_i \sim \pi$ for all $i \geq 0$; in this case we that the trajectory $X_0, X_1, \ldots$ is π-*stationary*. The Markov chain $\mathcal{M}$ is *reversible* if any π-stationary trajectory $X_0, \ldots, X_k$ is equivalent in distribution to its reverse $X_k, \ldots, X_0$.

We say that a_j is ℓ-*small* among $a_0, \ldots, a_s$ if there are at most ℓ indices $i \neq j$ among $0, \ldots, s$ such that $a_i \leq a_j$. The following simple definition is at the heart of the proofs of Theorems 1.1–1.3.

Definition 5.1. Given a Markov chain $\mathcal{M}$ with labels $\omega : \Sigma \to \mathbb{R}$ and stationary distribution π, we define for each $\ell, j \leq k$ a real number $\rho^k_{j,\ell}$, which is the probability that for a π-stationary trajectory $X_0, X_1, \ldots, X_k$, we have that $\omega(X_j)$ is ℓ-small among $\omega(X_0), \ldots, \omega(X_k)$.

Observe that (4) implies that all π-stationary trajectories of a fixed length are all identical in distribution, and in particular, that the $\rho^k_{j,\ell}$'s are well-defined.

Next observe that if the sequence of random variables $X_0, X_1, \ldots$ is a π-stationary trajectory for $\mathcal{M}$, then so is any interval of it. For example,

$$(X_{k-j}, \ldots, X_k, \ldots, X_{2k-j})$$

is another stationary trajectory, and thus the probability that $\omega(X_k)$ is ℓ-small among $\omega(X_{k-j}), \ldots, \omega(X_{2k-j})$ is equal to $\rho^k_{j,\ell}$. In particular, since

$$(\omega(X_k) \text{ is } \ell\text{-small among } \omega(X_{k-j}), \ldots, \omega(X_{2k-j}))$$

follows from

$$(\omega(X_k) \text{ is } \ell\text{-small among } \omega(X_0), \ldots, \omega(X_{2k}))$$

for all $j = 0, \ldots, k$, we have that

$$\rho^{2k}_{k,\ell} \leq \rho^k_{j,\ell}. \qquad (5)$$

We also have that $\sum_{j=0}^{k} \rho^k_{j,\ell} \leq \ell + 1$. Indeed, by linearity of expectation, this sum is the expected number of indices $j \in 0, \ldots, k$ such that $\omega(X_j)$ is ℓ-small among $\omega(X_0), \ldots, \omega(X_k)$. Thus, averaging the left and right sides of (5) over j from 0 to k, we obtain

$$\rho^{2k}_{k,\ell} \leq \frac{\ell+1}{k+1} < 2 \cdot \frac{\ell+1}{2k+1}. \qquad (6)$$

Line (6) already gives the theorem, once we make the following trivial observation:

Observation 5.1. Under the hypotheses of Theorem 1.2, we have that

$$Y_k, Y_{k-1}, \ldots, Y_1, \sigma_0, Z_1, Z_2, \ldots, Z_k$$

is a π-stationary trajectory.

This is an elementary consequence of the definitions, but since we will generalize this statement in Section 7, we give all the details here:

Proof of Observation 5.1. Our hypothesis is that $Y_1, Y_2, \ldots, Y_k$ and $Z_1, Z_2, \ldots, Z_k$ are independent trajectories from a common state $Y_0 = Z_0 = \sigma_0$ chosen from the stationary distribution π. Stationarity implies that

$$(Z_0, Z_1, \ldots, Z_k) \sim (X_k, X_{k+1}, \ldots, X_{2k}).$$

Similarly, stationarity and reversibility imply that

$$(Y_k, Y_{k-1}, \ldots, Y_0) \sim (X_0, X_1, \ldots, X_k).$$

Finally, our assumption that $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ are independent trajectories from σ_0 is equivalent to the condition that, for any $s_0, y_1, z_1, y_2, z_2, \ldots, y_k, z_k \in \Sigma$, we have for all $j \geq 0$ that

$$\begin{aligned}\Pr\big(Z_j = z_j \mid Z_{j-1} = z_{j-1}, \ldots, Z_1 = z_1, Z_0 = Y_0 = s_0, \\ Y_1 = y_1, \ldots, Y_k = y_k\big) \\ = \Pr\big(Z_j = z_j \mid Z_{j-1} = z_{j-1}, \ldots, Z_1 = z_1, Z_0 = s_0\big). \qquad (7)\end{aligned}$$

Of course, since $\mathcal{M}$ is a Markov chain, this second probability is simply

$$\Pr(Z_j = z_j \mid Z_{j-1} = z_{j-1}) = \Pr(X_{k+j} = z_j \mid X_{k+j-1} = z_{j-1}).$$

In particular, by induction on $j \geq 1$,

$$\begin{aligned}(Y_k, Y_{k-1}, \ldots, Y_0 = Z_0, Z_1, \ldots, Z_j) \\ \sim (X_0, X_1, \ldots, X_k, X_{k+1}, \ldots, X_{k+j}),\end{aligned}$$

and in particular

$$(Y_k, \ldots, \sigma_0, \ldots, Z_k) \sim (X_0, \ldots, X_k, \ldots, X_{2k}). \qquad (8)$$

□

Pared down to its bare minimum, this proof of Theorem 1.2 works by using that $\rho^{2k}_{k,\ell}$ is a lower bound on each $\rho^k_{j,\ell}$, and then applying the simple inequality

$$\sum_{j=0}^{k} \rho^k_{j,\ell} \leq \ell + 1. \tag{9}$$

The proof of Theorem 1.3 of Besag and Clifford is in some sense even simpler, using only (9), despite the fact that Theorem 1.3 has better dependence on ε (on the other hand, it is not directly applicable to (ε, α)-outliers in the way that we will use Theorem 1.2). Recall from Definition 5.1 that the $\rho^k_{j,\ell}$'s are fixed real numbers associated to a stationary Markov chain. If ℓ, k are fixed and ξ is chosen randomly from 0 to k, then the resulting $\rho^k_{\xi,\ell}$ is a random variable uniformly distributed on the set of real numbers $\{\rho^k_{0,\ell}, \rho^k_{1,\ell}, \ldots, \rho^k_{k,\ell}\}$. In particular, Theorem 1.3 is proved by writing that the probability that $\omega(\sigma_0)$ is ℓ-small among $\omega(\sigma_0), \omega(Y_1), \ldots, \omega(Y_\xi), \omega(Z_1), \ldots, \omega(Z_{k-\xi})$ is given by

$$\frac{1}{k+1}\left(\rho^k_{0,\ell} + \rho^k_{1,\ell} + \cdots + \rho^k_{k,\ell}\right) \leq \frac{\ell+1}{k+1},$$

where the inequality is from (9). Note that we are using an analog of Observation 5.1 to know that for any j, $Y_j, \ldots, Y_1, \sigma_0, Z_1, Z_{k-j}$ is a π-stationary trajectory.

6. Global Significance for Local Outliers

We now prove Theorem 3.1 from Theorem 1.2.

Proof of Theorem 3.1. For a π-stationary trajectory $X_0, \ldots, X_k$, let us define $p^k_{j,\varepsilon}(\sigma)$ to be the probability that $\omega(X_j)$ is in the bottom ε fraction of the values $\omega(X_0), \ldots, \omega(X_k)$, *conditioned* on the event that $X_j = \sigma$.

In particular, to prove Theorem 3.1, we will prove the following claim:

Claim: If σ_0 is not an (ε, α)-outlier, then

$$p^k_{0,\varepsilon}(\sigma_0) \leq \sqrt{\frac{2\varepsilon}{\alpha}}. \tag{10}$$

Let us first see why the claim implies the theorem. Recall the random variable ρ is the number of trajectories $\mathcal{T}^i$ from σ_0 on which σ_0 is observed to be an ε-outlier with respect to the labeling ω. The random variable ρ is thus a sum of m independent Bernoulli random variables, which each take value 1 with probability $\leq \sqrt{\frac{2\varepsilon}{\alpha}}$ by the claim. In particular, by Chernoff's bound, we have

$$\Pr\left(\rho \geq (1+\delta)m\sqrt{\tfrac{2\varepsilon}{\alpha}}\right) \leq e^{-\min(\delta,\delta^2)m\sqrt{\frac{2\varepsilon}{\alpha}}/3}, \tag{11}$$

giving the theorem. (Note the key point of the claim is that α is *inside* the square root in (10), while a straightforward application of Theorem 1.1 would give an expression with α outside the square root.)

To prove (10), consider a π-stationary trajectory $X_0, \ldots, X_k, \ldots, X_{2k}$ and condition on the event that $X_k = \sigma$ for some arbitrary $\sigma \in \Sigma$. Since $\mathcal{M}$ is reversible, we can view this trajectory as two independent trajectories $X_{k+1}, \ldots, X_{2k}$ and $X_{k-1}, X_{k-2}, \ldots, X_0$ both beginning from σ. In particular, letting A and B be the events that $\omega(X_k)$ is an ε-outlier among the lists $\omega(X_0), \ldots, \omega(X_k)$ and $\omega(X_k), \ldots, \omega(X_{2k})$, respectively, we have that

$$p^k_{0,\varepsilon}(\sigma)^2 = \Pr(A \cap B) \leq p^{2k}_{k,\varepsilon}(\sigma). \tag{12}$$

Now, the assumption that the given $\sigma_0 \in \Sigma$ is not an (ε, α)-outlier gives that for a random $\sigma \sim \pi$, we have that

$$\Pr\left(p^k_{0,\varepsilon}(\sigma) \geq p^k_{0,\varepsilon}(\sigma_0)\right) \geq \alpha. \tag{13}$$

Line (12) gives that $p^k_{0,\varepsilon}(\sigma)^2 \leq p^{2k}_{k,\varepsilon}(\sigma)$, and Theorem 1.2 gives that $p^{2k}_{k,\varepsilon} \leq 2\varepsilon$. Thus taking expectations with respect to a random $\sigma \sim \pi$, we obtain that

$$\mathbf{E}_{\sigma\sim\pi}\left(p^k_{0,\varepsilon}(\sigma)^2\right) \leq \mathbf{E}_{\sigma\sim\pi}\left(p^{2k}_{k,\varepsilon}(\sigma)\right) = p^{2k}_{k,\varepsilon} \leq 2\varepsilon.$$

On the other hand, we can use (13) to write

$$\mathbf{E}_{\sigma\sim\pi}\left(p^k_{0,\varepsilon}(\sigma)^2\right) \geq \alpha \cdot p^k_{0,\varepsilon}(\sigma_0)^2,$$

so that we have

$$p^k_{0,\varepsilon}(\sigma_0)^2 \leq \frac{2\varepsilon}{\alpha}.$$

□

The following theorem is the analog of Theorem 3.1 obtained when one uses an analog of Besag and Clifford's Theorem 1.3 in place of 1.2 in the proof. This version pays the price of using a random k instead of a fixed k for the notion of an (ε, α)-outlier, but has the advantage that the constant 2 is eliminated from the bound. (Note that as in Theorem 3.1, the notion of (ε, α)-outlier used here is still just defined with respect to a single path, although Theorem 1.3 depends on using two independent trajectories.)

Theorem 6.1. Consider m independent trajectories

$$\mathcal{T}^1 = (X^1_0, X^1_1, \ldots, X^1_{k_1}),$$
$$\vdots$$
$$\mathcal{T}^m = (X^m_0, X^m_1, \ldots, X^m_{k_m})$$

in the reversible Markov chain $\mathcal{M}$ (whose states have real-valued labels) from a common starting point $X^1_0 = \cdots = X^m_0 = \sigma_0$, where each of the lengths k_i are independently drawn random numbers from a geometric distribution. Define the random variable ρ to be the number of trajectories $\mathcal{T}^i$ on which σ_0 is an ε-outlier.

If σ_0 is not an (ε, α)-outlier with respect to k drawn from the geometric distribution, then

$$\Pr\left(\rho \geq m\sqrt{\tfrac{\varepsilon}{\alpha}} + r\right) \leq e^{-\min(r^2\sqrt{\alpha/\varepsilon}/3m,\, r/3)}. \tag{14}$$

Again, there is an analogous version to Theorem 3.2, where 2ε is replaced by ε.

Proof of Theorem 6.1. For a π-stationary trajectory $X_0, \ldots, X_k$ and a real number μ, let us define $p^{\mu}_{0,\varepsilon}(\sigma)$ to be the probability that $\omega(X_j)$ is in the bottom ε fraction of the values $\omega(X_0), \ldots, \omega(X_k)$, *conditioned* on the event that $X_0 = \sigma$, where the length k is chosen from a geometric distribution with mean μ supported on 0,1,2,...; that is, $k = t$ with probability $\frac{1}{\mu+1}(1 - \frac{1}{\mu+1})^t$.

To prove Theorem 6.1, it suffices to prove that if σ_0 is not an (ε, α)-outlier with respect to k drawn from the geometric distribution with mean μ, then

$$p^{\mu}_{0,\varepsilon}(\sigma_0) \leq \sqrt{\frac{\varepsilon}{\alpha}}. \tag{15}$$

To prove (15), suppose that k_1 and k_2 are independent random variables which are geometrically distributed with mean μ, and consider a π-stationary trajectory

$$X_0, \ldots, X_{k_1}, \ldots, X_{k_1+k_2}$$

of random length $k_1 + k_2$, and condition on the event that $X_{k_1} = \sigma$ for some arbitrary $\sigma \in \Sigma$. Since $\mathcal{M}$ is reversible, we can view this trajectory as two independent trajectories $X_{k_1}, X_{k_1+1}, \ldots, X_{k_1+k_2}$ and $X_{k_1}, X_{k_1-1}, X_{k_1-2}, \ldots, X_0$ both beginning from $X_{k_1} = \sigma$, of random lengths k_2 and k_1, respectively. In particular, letting A and B be the events that $\omega(X_{k_1})$ is an ε-outlier among the lists $\omega(X_0), \ldots, \omega(X_{k_1})$ and $\omega(X_{k_1}), \ldots, \omega(X_{k_1+k_2})$, respectively, we have that

$$\begin{aligned} p^{\mu}_{0,\varepsilon}(\sigma)^2 = \Pr(A \cap B) \leq \Pr\big(&\omega(X_{k_1}) \text{ is an } \varepsilon\text{-outlier among} \\ &\omega(X_0), \ldots, \omega(X_{k_1+k_2}) \mid X_{k_1} = \sigma\big), \end{aligned} \tag{16}$$

where, in this last expression, k_1 and k_2 are random variables. Now, the assumption that the given $\sigma_0 \in \Sigma$ is not an (ε, α)-outlier gives that for a random $\sigma \sim \pi$, we have that

$$\Pr\left(p^{\mu}_{0,\varepsilon}(\sigma) \geq p^{\mu}_{0,\varepsilon}(\sigma_0)\right) \geq \alpha. \tag{17}$$

Thus, we write

$$\begin{aligned} &\alpha \cdot p^{\mu}_{0,\varepsilon}(\sigma_0)^2 \leq \mathbf{E}_{\sigma\sim\pi}\left(p^{\mu}_{0,\varepsilon}(\sigma)^2\right) \\ &\leq \Pr\left(\omega(X_{k_1}) \text{ is an } \varepsilon\text{-outlier among } \omega(X_0), \ldots, \omega(X_{k_1+k_2})\right), \end{aligned} \tag{18}$$

where the last inequality follows from line (16).

On the other hand, considering the right-hand side of Line (18), we have that conditioning on any value for the length $\ell = k_1 + k_2$ of the trajectory, k_1 is uniformly distributed in the range $\{0, \ldots, \ell\}$. This is ensured by the geometric distribution, simply because for any ℓ and any $x \in (0, \ldots, \ell)$, we have that the probability

$$\begin{aligned} &\Pr\left(k_1 = x \text{ AND } k_2 = \ell - x\right) \\ &= \tfrac{1}{\mu+1}(1 - \tfrac{1}{\mu+1})^x \tfrac{1}{\mu+1}(1 - \tfrac{1}{\mu+1})^{\ell-x} \\ &= \left(\tfrac{1}{\mu+1}\right)^2 \left(1 - \tfrac{1}{\mu+1}\right)^{\ell} \end{aligned}$$

is independent of x. In particular, conditioning on any particular value for the length $\ell = k_1 + k_2$, we have that the probability that $\omega(X_{k_1})$ is an ε-outlier on the trajectory is at most ε, since X_{k_1} is a uniformly randomly chosen element of the trajectory $X_0, \ldots, X_{k_1+k_2}$; note that this part of the proof is exactly the same as the proof of Theorem 1.3. In particular, for the righthand-side of line (18), we are writing

$$\begin{aligned} \alpha \cdot p^{\mu}_{0,\varepsilon}(\sigma_0)^2 &\leq \Pr\big(\omega(X_{k_1}) \text{ is an } \varepsilon\text{-outlier among} \\ &\qquad \omega(X_0), \ldots, \omega(X_{k_1+k_2})\big) \\ &\leq \max_{\ell} \Pr\big(\omega(X_{k_1}) \text{ is an } \varepsilon\text{-outlier among} \\ &\qquad \omega(X_0), \ldots, \omega(X_{k_1+k_2}) \big| k_1 + k_2 = \ell\big) \leq \varepsilon. \end{aligned} \tag{19}$$

This gives line (15) and completes the proof. □

We close this section by noting that in implementations where m is not enormous, it may be sensible to use the exact binomial tail in place of the Chernoff bound in (11). In particular, this gives the following versions:

Theorem 6.2. With ρ as in Theorem 3.1, we have that if σ_0 is not an (ε, α) outlier, then

$$\Pr(\rho \geq K) \leq \sum_{k=K}^{m} \binom{m}{k} \left(\frac{2\varepsilon}{\alpha}\right)^{k/2} \left(1 - \sqrt{\frac{2\varepsilon}{\alpha}}\right)^{m-k}. \tag{20}$$

Theorem 6.3. With ρ as in Theorem 6.1, we have that if σ_0 is not an (ε, α) outlier, then

$$\Pr(\rho \geq K) \leq \sum_{k=K}^{m} \binom{m}{k} \left(\frac{\varepsilon}{\alpha}\right)^{k/2} \left(1 - \sqrt{\frac{\varepsilon}{\alpha}}\right)^{m-k}. \tag{21}$$

7. Generalizing the Besag and Clifford Tests

Theorem 3.1 is attractive because it succeeds at separating statistical significance from effect size, and at demonstrating statistical significance for an intuitively-interpretable deterministic property of state in the Markov chain. This is especially important when public-policy decisions must be made by nonexperts on the basis of such tests.

In some cases, however, these may not be important goals. In particular, one may simply desire a statistical test which is as effective as possible at disproving the null hypothesis $\sigma \sim \pi$. This is a task at which Besag and Clifford's Theorem 1.3 excels.

In their paper, Besag and Clifford also proved the following result, to enable a test designed to take efficient advantage of parallelism:

Theorem 7.1 (Besag and Clifford parallel test). Fix numbers k and m. Suppose that σ_0 is chosen from a stationary distribution π of the reversible Markov chain $\mathcal{M}$, and suppose we sample a trajectory $X_1, X_2, \ldots, X_k$ from $X_0 = \sigma_0$, and then branch to sample $m - 1$ trajectories $Z^s_1, Z^s_2, \ldots, Z^s_k$ $(2 \leq s \leq m)$ all from the state $Z^s_0 = X_k$. Then we have that

$$\begin{aligned} \Pr\big(&\omega(\sigma_0) \text{ is an } \varepsilon\text{-outlier among} \\ &\omega(\sigma_0), \omega(Z^2_k), \omega(Z^3_k), \ldots, \omega(Z^m_k)\big) \leq \varepsilon. \end{aligned}$$

Proof. For this theorem, it suffices to observe that $\sigma_0, Z^2_k, \ldots, Z^m_k$ are *exchangeable* random variables—that is, all permutations of the sequence $\sigma_0, Z^2_k, \ldots, Z^m_k$ are identical in distribution. This is because if σ_0 is chosen from π and then the Z^i_k's are chosen as above, the result is equivalent in distribution to the case where X_k is chosen from π and then each Z_k^i is chosen (independently) as the end of a trajectory $X_k, Z_1^i, \dots, Z_k^i$, and $\sigma_0 = Y_k$ is chosen (independently) as the end of a trajectory $X_k, Y_1, \dots, Y_k$. Here, we are using that reversibility implies that $(X_k, Y_1, \dots, Y_k)$ is identical in distribution to $(\sigma_0, X_1, \dots, X_k)$. □

With an eye toward finding a common generalization of Besag and Clifford's serial and parallel tests, we define a *Markov outlier test* as a significance test with the following general features:

- The test begins from a state σ_0 of the Markov chain which, under the null hypothesis, is assumed to be stationary;
- random steps in the Markov chain are sampled from the initial state and/or from subsequent states exposed by the test;
- the ranking of the initial state's label is compared among the labels of some (possibly all) of the visited states; it is an ε-outlier if it's label is among the bottom ε of the comparison labels. Some function $\rho(\varepsilon)$ assigns valid statistical significance to the test results, as in the above theorems.

In particular, such a test may consist of single or multiple trajectories, may branch once or multiple times, etc. In this section, we prove the validity of a parallelizable Markov outlier test with best possible function $\rho(\varepsilon) = \varepsilon$, but for which it is natural to expect the ε-*power* of the test—that is, its tendency to return small values of ε when σ_0 truly is an outlier—surpasses that of Theorems 1.3 and 7.1. In particular, we prove the following theorem:

Theorem 7.2 (Star-split test). Fix numbers m and k. Suppose that σ_0 is chosen from a stationary distribution π of the reversible Markov chain $\mathcal{M}$, and suppose that ξ is chosen randomly in $\{1, \dots, k\}$. Now sample trajectories $X_1, \dots, X_\xi$ and $Y_1, \dots, Y_{k-\xi}$ from σ_0, and then branch and sample $m - 1$ trajectories $Z_1^s, Z_2^s, \dots, Z_k^s$ $(2 \le s \le m)$ all from the state $Z_0^s = X_\xi$. Then we have that

$$\begin{aligned}\Pr\Big(&\omega(\sigma_0) \text{ is an } \varepsilon\text{-outlier among } \omega(\sigma_0),\\ &\omega(X_1) \dots, \omega(X_{\xi-1}),\\ &\omega(Y_1), \dots, (Y_{k-\xi}),\\ &\omega(Z_1^2), \dots, \omega(Z_k^2)\\ &\vdots\\ &\omega(Z_k^m), \dots, \omega(Z_k^m)\Big) \le \varepsilon.\end{aligned}$$

In particular, note that the set of comparison random variables used consists of all random variables exposed by the test *except* X_ξ.

To compare Theorem 7.2 with Theorems 7.1 and 1.3, let us note that it is natural to expect the ε-power of a Markov chain significance test to depend on:

1. How many comparisons are generated by the test, and
2. how far typical comparison states are from the state being tested, where we measure distance to a comparison state by the number of Markov chain transitions which the test used to generate the comparison.

If unlimited parallelism is available, then the Besag/Clifford parallel test is essentially optimal from these parameters, as it draws an unlimited number of samples, whose distance from the initial state is whatever serial running time is used. Conversely, in a purely serial setting, the Besag/Clifford test is essentially optimal with respect to these parameters.

But it is natural to expect that even when parallelism is available, the number n of samples we desire will often be significantly greater than the parallelism factor ℓ available. In this case, the Besag/Clifford parallel test will use n comparisons at distance $d \approx \ell t/n$, where t is the serial time used by the test. In particular, the typical distance to a comparison can be considerably less than t when ℓ compares unfavorably with n.

On the other hand, Besag/Clifford serial test generates comparisons whose typical distance is roughly $t/2$, but cannot make use of parallelism beyond $\ell = 2$. For an apples-to-apples comparison, it is natural to consider the case of carrying out their serial test using only every dth state encountered as a comparison state for some d. This is equivalent to applying the test to the dth-power of the Markov chain, instead of applying it directly. (In practical applications, this is a sensible choice when comparing the labels of states is expensive relative to the time required to carry out transitions of the chain.) Now if ℓ is a small constant, we see that with $t \cdot d$ steps, the BC parallel test can generate roughly n comparisons all at distance d from the state being tested, the serial test could generate comparisons at distances $d, 2d, 3d, \dots, kd$ (measured in terms of transitions in $\mathcal{M}$), where these distances occur with multiplicity at most 2, and $k = \max(\xi, n - \xi) \ge n/2$. In particular, the serial test generates a similar number of comparisons in this way but at much greater distances from the state we are evaluating, making it more likely that we are able to detect that the input state is an outlier.

Consider now the star-split test. Again, to facilitate comparison, we suppose the test is being applied to the dth power of $\mathcal{M}$. If serial time $t \approx sd$ is to be used, then we will branch into $\ell - 1$ trajectories after $\xi \cdot \mathcal{M}^d$ chain, where ξ is randomly chosen from $\{0, \frac{s}{2}\}$. Thus, comparisons used lie at a set of distances $d, 2d, \dots, (\xi + \frac{s}{2})d$ similar to the case of the Besag/Clifford serial test above. But now the distances $d, 2d, \dots, (\xi d - 1)d$ will have multiplicities at most 2 in the set of comparison distances, while the distances $(\xi + 1)d, (\xi + 2)d, \dots, (\xi + \frac{s}{2})d$ all have multiplicity at least $\ell - 1$. In particular, the test allows us to make more comparisons to more distance states, essentially by a factor of the parallelism factor being used. In particular, it is natural to expect performance to improve as ℓ increases. Moreover, the star-split test is equivalent to the Besag/Clifford serial test for $\ell \le 2$, and essentially equivalent to their parallel test in the large ℓ limit. (To make this latter correspondence exact, once can apply Theorem 7.2 to the dth power of a Markov chain $\mathcal{M}$, and take $k = 1$.)

We now turn to the task of proving Theorem 7.2. Unlike Theorems 1.1–1.3, the comparison states used in Theorems 7.1 and 7.2 cannot be viewed as a single trajectory in $\mathcal{M}$. This motivates the natural generalization of the notion of a π-stationary trajectory as follows:

Definition 7.1. Given a reversible Markov chain $\mathcal{M}$ with stationary distribution π and an undirected tree T, a *π-stationary T-projection* is a collection of random variables $\{X_v\}_{v\in T}$ such that:

1. for all $v \in T$, $X_v \sim \pi$;
2. for any edge $\{u, v\}$ in T, if we let T_u denote the vertex-set of the connected component of u in $T \setminus \{u, v\}$ and $\{\sigma_w\}_{w\in T}$ is an arbitrary collection of states, then

$$\Pr\left(X_v = \sigma_v \middle| \bigwedge_{w\in T_u} X_w = \sigma_w\right) = \pi_{\sigma_u,\sigma_v}.$$

In analogy to the case of π-stationary trajectories, Definition 7.1 easily gives the following, by induction:

Observation 7.1. For fixed π and T, if $\{X_w\}_{w\in T}$ and $\{Y_w\}_{w\in T}$ are both π-stationary T-projections, then the two collections $\{X_w\}_{w\in T}$ and $\{Y_w\}_{w\in T}$ are equivalent in distribution. □

This enables the following natural analog of Definition 5.1:

Definition 7.2. Given a Markov chain $\mathcal{M}$ with labels $\omega : \Sigma \to \mathbb{R}$ and stationary distribution π, we define for each ℓ, each undirected tree T, each vertex subset $S \subset T$ and each vertex $v \in S$ a real number $\rho^{T,S}_{v,\ell}$, which is the probability that for a π-stationary T-projection $\{X_w\}_{w\in T}$, we have that $\omega(X_v)$ is ℓ-small among $\{\omega(X_w)\}_{w\in S}$.

Observe that as in (9) we have for any tree T and any vertex subset S of T, we have that

$$\sum_{w\in S} \rho^{T,S}_{w,\ell} \le \ell + 1. \tag{22}$$

The following observation, applied recursively, gives the natural analog of Observation 5.1. Again the proof is an easy exercise in the definitions.

Observation 7.2. Suppose that T is an undirected tree, v is a leaf of T, $T' = T \setminus v$, and $\{X_w\}_{w\in T'}$ is a π-stationary T'-projection. Suppose further that X_v is a random variable such that for all $\{\sigma_w\}_{w\in T}$ we have that

$$\Pr\left(X_v = \sigma_v \middle| \bigwedge_{w\in T'} (X_w = \sigma_w)\right) = \pi_{\sigma_u,\sigma_v}, \tag{23}$$

where u is the neighbor of v in T. Then $\{X_w\}_{w\in T}$ is a π-stationary T-projection. □

We can rephrase the proof of Theorem 7.1 in this language. Let T be the tree consisting of m paths of length k sharing a common endpoint and no other vertices, and let S be the leaves of T. By symmetry, we have that $\rho^{T,S}_{w,\ell}$ is constant over $w \in S$. On the other hand, Observation 7.2 gives that under the hypotheses of Theorem 7.1, $\sigma_0, X_1, \ldots, X_k$, and the Z^s_i's are a π-stationary T-projection, with obvious assignments (e.g., σ_0 corresponds to a leaf of T; X_k corresponds to the center). In particular, (22) implies that $\rho^{T,S}_{w,\ell} \le \frac{\ell+1}{n}$, which gives the theorem.

On the other hand, the definitions make the following proof easy as well, using the same simple idea as Besag and Clifford's Theorem 1.3.

Proof of Theorem 7.2. Define T to be the undirected tree with vertex set $\{v_0\} \cup \{v^s_j \mid 1 \le s \le m, 1 \le j \le k\}$, with edges $\{v_0, v^s_1\}$ for each $1 \le s \le m$ and $\{v^s_j, v^s_{j+1}\}$ for each $1 \le s \le m$, $1 \le j \le k-1$. Now we let S consist of all vertices of T except the center v_0, and let S_j denote the set of m vertices in S at distance j from v_0. By symmetry, we have that $\rho^{T,S}_{v,\ell}$ is constant in each S_j; in particular, we have that

$$\rho^{T,S}_{v^1_j,\ell} = \frac{1}{n}\sum_{s=1}^{m} \rho^{T,S}_{v^s_j,\ell}$$

and together with (22) this gives that

$$\sum_{j=1}^{k} \rho^{T,S}_{v^1_j,\ell} \le \frac{\ell+1}{n}. \tag{24}$$

Now if we let

$$W_{v_0} = X_k,$$

$$W_{v^s_j} = \begin{cases} X_{\xi-j} & s = 1, 1 \le j < \xi \\ \sigma_0 & s = 1, j = \xi \\ Y_{j-\xi} & s = 1, j > \xi \\ Z^s_j & 2 \le s \le m, 1 \le j \le k, \end{cases}$$

then $\{W_w\}_{w\in T}$ is a π-stationary T-projection under the hypotheses of Theorem 7.2, by recursively applying Observation 7.2. Moreover, as ξ is chosen randomly among $\{1, \ldots, k\}$, the probability that $\omega(\sigma_0) = \omega(W_{v^1_\xi})$ is ℓ-small among $\{\omega(W_w)\}_{w\in S}$ is given by

$$\frac{1}{k}\left(\rho^{T,S}_{v^1_1,\ell} + \cdots + \rho^{T,S}_{v^1_k,\ell}\right) \le \frac{\ell+1}{kn},$$

where the inequality is from (24), giving the theorem. □

8. The Product Space Setting

The appeal of the theorems developed thus far in this article is that they can be applied to any reversible Markov chain without any knowledge of its structure. However, there are some important cases where additional information about the structure of the stationary distribution of a chain *is* available, and can be exploited to enable more powerful statistical claims.

In this section, we consider the problem of evaluating claims of gerrymandering with a Markov chain where the probability distribution on districtings is known to have a product structure imposed by geographical constraints. For example, the North Carolina Supreme Court has ruled in *Stephenson v. Bartlett* that districtings of that state must respect groupings of counties determined by a prescribed algorithm. In particular, a set of explicit rules (nearly) determine a partition of the counties of North Carolina into county groupings whose populations are each close to an integer multiple of an ideal district size (see Carter et al. 2020 for recent results on these rules), and then the districting of the state is comprised of independent districtings of each of the county groupings.

In this way, the probability space of uniformly random districtings is a product space, with a random districting of the whole state equivalent to collection of random independent districtings of each of the separate county groupings. We wish to exploit this structure for greater statistical power. In particular, running trajectories of length k in each of d clusters generates a total of k^d comparison maps with only $k \cdot d$ total Markov chain steps. To take advantage of the potential power of this enormous comparison set, we need theorems which allow us to compare a given map not just to a trajectory of maps in a Markov chain (since the k^d maps do not form a trajectory) but to the product of trajectories. This is what we show in this section.

Formally, in the product space setting, we have a collection $\mathcal{M}^{[d]}$ of d Markov chains $\mathcal{M}_1, \ldots, \mathcal{M}_d$, each $\mathcal{M}_i$ on state space Σ_i (each corresponding to one county grouping in North Carolina, for example). We are given a label function $\omega : \Sigma^{[d]} \to \mathbb{R}$, where here $\Sigma^{[d]} = \Sigma_1 \times \cdots \times \Sigma_d$. In the first theorem in this section, which is a direct analog of the Besag and Clifford test, we consider a $\boldsymbol{\sigma}_0 \in \Sigma^{[d]}$ distributed as $\boldsymbol{\sigma}_0 \sim \pi^{[d]}$, where here $\pi^{[d]}$ indicates the product space of stationary distributions π_i of the $\mathcal{M}_i$. (In the gerrymandering case, $\pi^{[d]}$ is a random map chosen by randomly selecting a map for each separate county cluster.) In the tests discussed earlier in this article, a state $\sigma_0 \sim \mathcal{M}$ is evaluated by comparing a state σ_0 to other states on a trajectory containing σ_0. In the product setting, we compare $\boldsymbol{\sigma}_0$ against a product of one trajectory from each $\mathcal{M}_i$.

In particular, given the collection $\mathcal{M}^{[d]}$, a state $\boldsymbol{\sigma}_0 = (\sigma_0^1, \ldots, \sigma_0^d) \in \Sigma^{[d]}$, and $\mathbf{j} = (j_1, \ldots j_d)$, $\mathbf{k} = (k_1, \ldots, k_d)$, we define the *trajectory product* $\mathbf{X}_{\boldsymbol{\sigma}_0,\mathbf{j},\mathbf{k}}$ which is obtained by considering, for each i, a trajectory $X_0^i, \ldots, X_{k^i}^i$ in $\mathcal{M}_i$ conditioned on $X_{j_i}^i = \sigma_0^i$. $\mathbf{X}_{\boldsymbol{\sigma}_0,\mathbf{j},\mathbf{k}}$ is simply the set of all d-tuples consisting of one element from each such trajectory.

We define the *stationary trajectory product* $\mathbf{X}_{\pi^{[d]},\mathbf{k}}$, analogously, except that the trajectories used are all stationary, instead of conditioning on $X_{j_i}^i = \sigma_0^i$.

Theorem 8.1. Given reversible Markov chains $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_d$, fix any number k and suppose that $\sigma_0^1, \ldots, \sigma_0^d$ are chosen from stationary distributions $\pi_1, \ldots, \pi_d$ of $\mathcal{M}_1, \ldots, \mathcal{M}_d$, and that $\xi_1, \ldots, \xi_d$ are chosen uniformly and independently in $\{0, \ldots, k\}$. For each $s = 1, \ldots, d$, consider two independent trajectories $Y_0^s, Y_1^s, \ldots$ and $Z_0^s, Z_1^s, \ldots$ in the reversible Markov chain $\mathcal{M}_s$ from $Y_0^s = Z_0^s = \sigma_0^s$. Let $\omega : \mathcal{M}_1 \times \cdots \times \mathcal{M}_d \to \mathbb{R}$ be a label function on the product space, write $\boldsymbol{\sigma}_0 = (\sigma_0^1, \ldots, \sigma_0^d)$, and denote by $\mathbf{Z}_{\boldsymbol{\sigma}_0,k}$ the (random) set of all vectors $(a_1, \ldots, a_d)$ such that for each i, $a_i \in \left(\sigma_0^i, Y_1^i, \ldots, Y_{\xi_i}^i, Z_1^i, \ldots, Z_{k-\xi_i}^i\right)$. Then we have that

$$\Pr\left(\omega(\boldsymbol{\sigma}_0) \text{ is an } \varepsilon\text{-outlier among } \omega(\mathbf{x}), \mathbf{x} \in \mathbf{Z}_{\sigma_0,k}\right) \leq \varepsilon. \quad (25)$$

Proof. Like the proof of Theorem 1.3, this proof is very simple; it is just a matter of digesting notation. First observe that $\mathbf{Z}_{\boldsymbol{\sigma}_0,k}$ is simply a trajectory product $\mathbf{X}_{\boldsymbol{\sigma}_0,\xi,\mathbf{k}}$, where $\mathbf{k} = (k, \ldots, k)$ and ξ is the random variable $(\xi_1, \ldots, \xi_d)$.

In particular, under the hypothesis that $\sigma_0^i \sim \pi_i$ for all i, $\mathbf{Z}_{\boldsymbol{\sigma}_0,k}$ is in fact a stationary trajectory product $\mathbf{X}_{\pi^{[d]},\mathbf{k}}$. In particular, by the random, independent choice of the ξ_i's, the probability in (25) is equivalent to the probability that the label of a random element of the a stationary trajectory product is among ε smallest labels in the stationary trajectory product; this probability is at most ε. □

The following is an analog of Theorem 1.2 for the product space setting.

Theorem 8.2. Given reversible Markov chains $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_d$, fix any number k and suppose that $\sigma_0^1, \ldots, \sigma_0^d$ are chosen from stationary distributions $\pi_1, \ldots, \pi_d$ of $\mathcal{M}^1, \ldots, \mathcal{M}^d$. For each $s = 1, \ldots, d$, consider two independent trajectories $Y_0^s, Y_1^s, \ldots$ and $Z_0^s, Z_1^s, \ldots$ in the reversible Markov chain $\mathcal{M}^s$ from $Y_0^s = Z_0^s = \sigma_0^s$. Let $\omega : \mathcal{M}_1 \times \cdots \times \mathcal{M}_d \to \mathbb{R}$ be a label function on the product space, write $\boldsymbol{\sigma}_0 = (\sigma_0^1, \ldots, \sigma_0^d)$, and denote by $\mathbf{Z}_{\boldsymbol{\sigma}_0,k}$ the (random) set of all vectors $(a_1, \ldots, a_d)$ such that for each i, $a_i \in \left(\sigma_0^i, Y_1^i, \ldots, Y_k^i, Z_1^i, \ldots, Z_k^i\right)$. Then we have that

$$\Pr\left(\omega(\boldsymbol{\sigma}_0) \text{ is an } \varepsilon\text{-outlier among } \omega(\mathbf{x}), \mathbf{x} \in \mathbf{Z}_{\boldsymbol{\sigma}_0,k}\right) \leq 2^d \cdot \varepsilon. \quad (26)$$

Proof. First consider d independent stationary trajectories $X_0^i, X_1^i, X_2^i, \ldots$ for each $i = 1, \ldots, d$, and define $\mathbf{X}_{\pi,k}$ to be the collection of all $(k+1)^d$ d-tuples $(a_1, \ldots, a_d)$ where, for each i, $a_i \in \{X_0^i, \ldots, X_k^i\}$.

In analogy to Definition 5.1, we define $\rho_{\mathbf{j},\ell}^k$ for $\mathbf{j} = (j_1, j_2, \ldots, j_k)$ to be the probability that for $X_{\mathbf{j}} = (X_{j_1}^1, \ldots, X_{j_d}^d) \in \mathbf{X}_{\pi,k}$, we have that $\omega(X_{\mathbf{j}})$ is ℓ-small among the ω-labels of all elements of $\mathbf{X}_{\pi,k}$.

Observe that for $\mathbf{k} = (k, \ldots, k)$, we have in analogy to Equation (5) that

$$\rho_{\mathbf{k},\ell}^{2k} \leq \rho_{\mathbf{j},\ell}^k \quad (27)$$

for any $j = (j_1, \ldots, j_d)$. And of course we have that

$$\sum_{\mathbf{j}} \rho_{\mathbf{j},\ell}^k \leq \ell + 1.$$

Thus, averaging both sides of (27) gives that

$$\rho_{\mathbf{k},\ell}^{2k} \leq \frac{\ell+1}{(k+1)^d} \leq 2^d \frac{\ell+1}{(2k+1)^d}. \quad (28)$$

Now observe that the statement that

$$\omega(\boldsymbol{\sigma}_0) \text{ is an } \varepsilon\text{-outlier among } \omega(\mathbf{x}), \mathbf{x} \in \mathbf{Z}_{\boldsymbol{\sigma}_0,k}$$

equivalent to the statement that

$$\omega(\boldsymbol{\sigma}_0) \text{ is an } \ell\text{-small among } \omega(\mathbf{x}), \mathbf{x} \in \mathbf{Z}_{\boldsymbol{\sigma}_0,k}$$

for $\ell = \varepsilon \cdot (2k+1)^d - 1$; thus (28) gives the theorem, since $\rho_{\mathbf{k},\ell}^{2k}$ is precisely the probability that this second statement holds. □

The presence of the 2^d in (26) is now potentially more annoying than the constant 2 in (1.2), and it is natural to ask whether it can be avoided. However, using the example from Remark 1.2, it is easy to see that an exponential factor $\left(\frac{3}{2}\right)^d$ may really be necessary, at least if $k = 1$. Whether such a factor can be avoided for larger values of k is an interesting question. However, as we discuss below, this seemingly large exponential penalty is actually likely dwarfed by the quantitative benefits of the product setting, in many real-world cases.

8.1. Illustrative Product Examples

The fact the estimate in Theorem 8.1 looks like original Theorem 1.3, hides the power in the product version. More misleading is the fact that Theorem 8.2 has a 2^d which seems to make the theorem degrade with increasing d.

Let us begin by considering the simplest example we are looking for the single extreme outlier across the entire product space. Let us further assume that this global extreme is obtained by choosing each of the extreme element in each part of the product space. An example of this comes for the Gerrymandering application where one is naturally interested in the seat count. Each of the product coordinates represents the seats from a particular geographic region. In some states such as North Carolina judicial rulings break the problem up into the product measure required by Theorems 8.1 and 8.2 by stipulating that particular geographic regions must be redistricted independently.

For illustrative purposes, let assume that there are L different outcomes in each of the d different factors of the product space. Hence, the chance of getting the minimum in any of the d different components is $1/L$. However, getting the minimum in the whole product space requires getting the minimum in each of the components and so is $1/L^d$. Hence is this setting one can take $\varepsilon = 1/L^d$ in Theorems 8.1 and 8.2. Thus even in Theorem 8.2 as long as $L > 2$, one has a significant improvement as d grows.

Now let's consider a second slightly more complicated example which builds on the proceeding one. Let us equip each $\mathcal{M}_i$ with a function ω_i and decide that we are interested in the event

$$\mathcal{E}(\delta) = \left\{ \{\xi_i\}_1^d \colon \sum_{i=1}^{d} \omega_i(\xi_i) \leq \delta \right\}. \tag{29}$$

Then one can take

$$\varepsilon = \frac{|\mathcal{E}(\delta)|}{L^d}$$

in Theorem 8.2 and 2^d times this in Theorem 8.2, where $|\mathcal{E}(\delta)|$ is simply the number of elements in the set $\mathcal{E}(\delta)$. This can lead to a significant improvement in the power of the test in the product case over the general case when $|\mathcal{E}(\delta)|$ grows slower than L^d.

There remains the task of calculating $|\mathcal{E}(\delta)|$. In the gerrymandering examples we have in mind, this can be done efficiently. When counting seat counts, the map ω_i is a many-to-one map with a range consisting of a few discrete values. This means that one can tabulate exactly the number of samples which produce a given value of ω_i. Since we are typically interested extreme values of

$$\omega(\xi) = \sum_{i=1}^{d} \omega_i(\xi_i)\,,$$

there are often only a few partitions of each value of ω made from possible values of ω_i. When this true, the size of $\mathcal{E}$ can be calculated exactly efficiently.

For example, let us assume there are d geographical regions which each needs to be divided into 4 districts. Furthermore each party always wins at least one seat in each geographical region; hence, the only possible outcomes are 1, 2, or 3 seats in each region for a given party. If ω_i counts the number of seats for the party of interest in geographic region i, let us suppose for concreteness that we want are interested in $\delta = 2d$. To calculate $|\mathcal{E}(\delta)|$, we need to only keep track of the number of times 1, 2, or 3 seats is produced in each geographic region. We can then combine these numbers by summing over all of the ways the numbers 1, 2, and 3 can add numbers between d and $2d$. (The smallest $\omega(\xi)$ can be given our assumptions is d.) This is a straightforward calculation for which there exist fast algorithms which leverage the hierarchical structure. Namely, group each region with another and calculate the combined possible seat counts and their frequencies. Continuing up the tree recursively one can calculate $|\mathcal{E}(\delta)|$ in only logarithmically many levels.

It is worth remarking, that not all statistics of interest fall as neatly into this framework which enables simple and efficient computation. For instance, calculating the ranked marginals used in Herschlag et al. (2018) requires choosing some representation of the histogram, such as a fixed binning, and would yield only approximate results.

8.2. Toward an (ε,α)-Outlier Theorem for Product Spaces

In general, the cost of making a straightforward translation of Theorems 3.1 or 6.1 to the product-space setting are surprisingly large: in both cases, the square root is replaced by a 2^dth root, according to the natural generalization of the proofs of those theorems.

Accordingly, in this section, we point out simply that by using a more complicated definition of (ε,α)-outliers for the product space setting, an analog of Theorem 6.1 is then easy. In particular, let us define

$$p^{\mathbf{k}}_{\mathbf{U},\varepsilon}(\boldsymbol{\sigma}_0) := \Pr\left(\omega(\boldsymbol{\sigma}_0) \text{ an } \varepsilon\text{-outlier in } \mathbf{X}_{\boldsymbol{\sigma}_0,\mathbf{j},\mathbf{k}}\right), \tag{30}$$

where $\mathbf{j} = (j_1, \ldots, j_d)$ is chosen randomly with respect to the uniform distributions $j_i \sim \mathrm{Unif}[0, k_i]$ (here $\mathbf{k} = (k_1, \ldots, k_d)$).

Now we define a state $\boldsymbol{\sigma}_0$ to be an (ε,α)-outlier with respect to a distribution $\mathbf{k}$ if among all states in $\Sigma^{[d]}$, we have that $p^{\mathbf{k}}_{\mathbf{U},\varepsilon}(\boldsymbol{\sigma}_0)$ is in the largest α fraction of the values of $p^{\mathbf{k}}_{\mathbf{U},\varepsilon}(\boldsymbol{\sigma})$ over *all* states $\boldsymbol{\sigma} \in \mathcal{M}^{[d]}$, weighted according to π.

Theorem 8.3. We are given Markov chains $\mathcal{M}_1, \ldots, \mathcal{M}_d$. Suppose that σ_0 is not an (ε,α)-outlier with respect to $\mathbf{k}$. Then

$$p^{\mathbf{k}}_{\mathbf{U},\varepsilon}(\sigma_0) \leq \frac{\varepsilon}{\alpha}.$$

Proof. This follows immediately from the definitions. From the definition of (ε,α)-outlier given above for the product setting, we have that if σ_0 is not an (ε,α)-outlier, then for a random $\sigma \sim \pi$,

$$\Pr\left(p^{\mathbf{k}}_{\mathbf{U},\varepsilon}(\sigma) \geq p^{\mathbf{k}}_{\mathbf{U},\varepsilon}(\sigma_0)\right) \geq \alpha.$$

Thus, we can write

$$\mathbf{E}_{\sigma\sim\pi}\, p^{\mathbf{k}}_{\mathbf{U},\varepsilon}(\sigma) \geq \alpha \cdot p^{\mathbf{k}}_{\mathbf{U},\varepsilon}(\sigma_0).$$

And of course this expectation is just the probability that a random element of $\mathbf{X}_{\pi,\mathbf{k}}$ is an ε-outlier on $X_{\pi,\mathbf{k}}$, which is at most ε. □

Of course this kind of trivial proof would be possible in the general non-product space setting also, but the sacrifice is that (ε, α)-outliers cannot be defined with respect to the endpoints of trajectories, which appears most natural. Whether analogous to Theorems 3.1 and 6.1 are possible in the product space setting without an explosive dependence on the dimension d seems like a very interesting question.

Funding

Research supported in part by NSF grant DMS-1362785. Research supported in part by NSF grant DMS-1363136 and the Sloan Foundation.

References

Besag, J., and Clifford, P. (1989), "Generalized Monte Carlo Significance Tests," *Biometrika*, 76, 633–642. [103]

Carter, D., Hunter, Z., Teague, D., Herschlag, G., and Mattingly, J. (2020), "Optimal Legislative County Clustering in North Carolina," *Statistics and Public Policy*, 7, 19–29. [111]

Chikina, M., Frieze, A., and Pegden, W. (2017), "Assessing Significance in a Markov Chain Without Mixing," *Proceedings of the National Academy of Sciences of the United States of America*, 114, 2860–2864. [101,102,103,104]

Herschlag, G., Kang, H. S., Luo, J., Graves, C. V., Bangia, S., Ravier, R., and Mattingly, J. C. (2018), "Quantifying Gerrymandering in North Carolina," arXiv no. 1801.03783. [113]

Pegden, W. (2019), "Expert Report," *Common Cause v. Lewis*. [103,104,105]

Assessing significance in a Markov chain without mixing

Maria Chikina[a], Alan Frieze[b], and Wesley Pegden[b,1]

[a]Department of Computational and Systems Biology, University of Pittsburgh, Pittsburgh, PA 15213; and [b]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213

Edited by Kenneth W. Wachter, University of California, Berkeley, CA, and approved January 24, 2017 (received for review October 21, 2016)

We present a statistical test to detect that a presented state of a reversible Markov chain was not chosen from a stationary distribution. In particular, given a value function for the states of the Markov chain, we would like to show rigorously that the presented state is an outlier with respect to the values, by establishing a *p* value under the null hypothesis that it was chosen from a stationary distribution of the chain. A simple heuristic used in practice is to sample ranks of states from long random trajectories on the Markov chain and compare these with the rank of the presented state; if the presented state is a 0.1% outlier compared with the sampled ranks (its rank is in the bottom 0.1% of sampled ranks), then this observation should correspond to a *p* value of 0.001. This significance is not rigorous, however, without good bounds on the mixing time of the Markov chain. Our test is the following: Given the presented state in the Markov chain, take a random walk from the presented state for any number of steps. We prove that observing that the presented state is an ε-outlier on the walk is significant at $p=\sqrt{2\varepsilon}$ under the null hypothesis that the state was chosen from a stationary distribution. We assume nothing about the Markov chain beyond reversibility and show that significance at $p\approx\sqrt{\varepsilon}$ is best possible in general. We illustrate the use of our test with a potential application to the rigorous detection of gerrymandering in Congressional districting.

Markov chain | mixing time | gerrymandering | outlier | *p* value

The essential problem in statistics is to bound the probability of a surprising observation under a null hypothesis that observations are being drawn from some unbiased probability distribution. This calculation can fail to be straightforward for a number of reasons. On the one hand, defining the way in which the outcome is surprising requires care; for example, intricate techniques have been developed to allow sophisticated analysis of cases where multiple hypotheses are being tested. On the other hand, the correct choice of the unbiased distribution implied by the null hypothesis is often not immediately clear; classical tools like the t test are often applied by making simplifying assumptions about the distribution in such cases. If the distribution is well-defined but is not be amenable to mathematical analysis, a p value can still be calculated using bootstrapping if test samples can be drawn from the distribution.

A third way for p value calculations to be nontrivial occurs when the observation is surprising in a simple way and the null hypothesis distribution is known but where there is no simple algorithm to draw samples from this distribution. In these cases, the best candidate method to sample from the null hypothesis is often through a Markov chain, which essentially takes a long random walk on the possible values of the distribution. Under suitable conditions, theorems are available that guarantee that the chain converges to its stationary distribution, allowing a random sample to be drawn from a distribution quantifiably close to the target distribution. This principle has given rise to diverse applications of Markov chains, including to simulations of chemical reactions, Markov chain Monte Carlo statistical methods, protein folding, and statistical physics models.

A persistent problem in applications of Markov chains is the often unknown rate at which the chain converges with the stationary distribution (1, 2). It is rare to have rigorous results on the mixing time of a real-world Markov chain, which means that, in practice, sampling is performed by running a Markov chain for a "long time" and hoping that sufficient mixing has occurred. In some applications, such as in simulations of the Potts model from statistical physics, practitioners have developed modified Markov chains in the hopes of achieving faster convergence (3), but such algorithms have still been shown to have exponential mixing times in many settings (4–6).

In this article, we are concerned with the problem of assessing statistical significance in a Markov chain without requiring results on the mixing time of the chain or indeed, any special structure at all in the chain beyond reversibility. Formally, we consider a reversible Markov chain $\mathcal{M}$ on a state space Σ, which has an associated label function $\omega:\Sigma\to\Re$. (The definition of Markov chain is recalled at the end of this section.) The labels constitute auxiliary information and are not assumed to have any relationship to the transition probabilities of $\mathcal{M}$. We would like to show that a presented state σ_0 is unusual for states drawn from a stationary distribution π. If we have good bounds on the mixing time of $\mathcal{M}$, then we can simply sample from a distribution of $\omega(\pi)$ and use bootstrapping to obtain a rigorous p value for the significance of the smallness of the label of σ_0. However, such bounds are rarely available.

We propose the following simple and rigorous test to detect that σ_0 is unusual relative to states chosen randomly according to π, which does not require bounds on the mixing rate of $\mathcal{M}$.

The $\sqrt{\varepsilon}$ test. Observe a trajectory $\sigma_0,\sigma_1,\sigma_2\dots,\sigma_k$ from the state σ_0 for any fixed k. The event that $\omega(\sigma_0)$ is an ε-outlier among $\omega(\sigma_0),\dots,\omega(\sigma_k)$ is significant at $p=\sqrt{2\varepsilon}$ under the null hypothesis that $\sigma_0\sim\pi$.

Here, we say that a real number α_0 is an ε-outlier among $\alpha_0,\alpha_2,\dots,\alpha_k$ if there are, at most, $\varepsilon(k+1)$ indices i for which

Significance

Markov chains are simple mathematical objects that can be used to generate random samples from a probability space by taking a random walk on elements of the space. Unfortunately, in applications, it is often unknown how long a chain must be run to generate good samples, and in practice, the time required is often simply too long. This difficulty can preclude the possibility of using Markov chains to make rigorous statistical claims in many cases. We develop a rigorous statistical test for Markov chains which can avoid this problem, and apply it to the problem of detecting bias in Congressional districting.

Author contributions: M.C., A.F., and W.P. performed research and wrote the paper.

The authors declare no conflict of interest.

This article is a PNAS Direct Submission.

[1]To whom correspondence should be addressed. Email: wes@math.cmu.edu.

This article contains supporting information online at www.pnas.org/lookup/suppl/doi:10.1073/pnas.1617540114/-/DCSupplemental.

$\alpha_i \le \alpha_0$. In particular, note for the $\sqrt{\varepsilon}$ test that the only relevant feature of the label function is the ranking that it imposes on the elements of Σ. In *SI Text*, we consider the statistical power of the test and show that the relationship $p \approx \sqrt{\varepsilon}$ is best possible. We leave as an open question whether the constant $\sqrt{2}$ can be improved.

Roughly speaking, this kind of test is possible, because a reversible Markov chain cannot have many local outliers (Fig. 1). Rigorously, the validity of the test is a consequence of the following theorem.

Theorem 1.1. Let $\mathcal{M} = X_0, X_1, \ldots$ be a reversible Markov chain with a stationary distribution π, and suppose the states of $\mathcal{M}$ have real-valued labels. If $X_0 \sim \pi$, then for any fixed k, the probability that the label of X_0 is an ε-outlier from among the list of labels observed in the trajectory $X_0, X_1, X_2, \ldots, X_k$ is, at most, $\sqrt{2\varepsilon}$.

We emphasize that Theorem 1.1 makes no assumptions on the structure of the Markov chain beyond reversibility. In particular, it applies even if the chain is not irreducible (in other words, even if the state space is not connected), although in this case, the chain will never mix.

In Detecting Bias in Political Districting, we apply the test to Markov chains generating random political districting for which no results on rapid mixing exist. In particular, we show that, for various simple choices of constraints on what constitutes a "valid" Congressional districting (e.g., that the districts are contiguous and satisfy certain geometric constraints), the current Congressional districting of Pennsylvania is significantly biased under the null hypothesis of a districting chosen at random from the set of valid districting. (We obtain p values between $\approx 2.5 \cdot 10^{-4}$ and $\approx 8.1 \cdot 10^{-7}$ for the constraints that we considered.)

One hypothetical application of the $\sqrt{\varepsilon}$ test is the possibility of rigorously showing that a chain is not mixed. In particular, suppose that Research Group 1 has run a reversible Markov chain for n_1 steps and believes that this was sufficient to mix the chain. Research Group 2 runs the chain for another n_2 steps, producing a trajectory of total length $n_1 + n_2$, and notices that a property of interest changes in these n_2 additional steps. Heuristically, this observation suggests that n_1 steps were not sufficient to mix the chain, and the $\sqrt{\varepsilon}$ test quantifies this reasoning rigorously. For this application, however, we must allow X_0 to be distributed not exactly as the stationary distribution π but as some distribution π' with total variation distance to π that is small, as is the scenario for a "mixed" Markov chain. In *SI Text*, we give a version of Theorem 1.1, which applies in this scenario.

One area of research related to this manuscript concerns methods for perfect sampling from Markov chains. Beginning with the Coupling from the Past (CFTP) algorithm of Propp and Wilson (7, 8) and several extensions (9, 10), these techniques are designed to allow sampling of states exactly from the stationary distribution π without having rigorous bounds on the mixing time of the chain. Compared with the $\sqrt{\varepsilon}$ test, perfect sampling techniques have the disadvantages that they require the Markov chain to possess a certain structure for the method to be implementable and that the time that it takes to generate each perfect sample is unbounded. Moreover, although perfect sampling methods do not require rigorous bounds on mixing times to work, they will not run efficiently on a slowly mixing chain. The point is that for a chain that has the right structure and that actually mixes quickly (despite an absence of a rigorous bound on the mixing time), algorithms like CFTP can be used to rigorously generate perfect samples. However, the $\sqrt{\varepsilon}$ test applies to any reversible Markov chain, regardless of the structure, and has running time k chosen by the user. Importantly, it is quite possible that the test can detect bias in a sample even when k is much smaller than the mixing time of the chain, which seems to be the case in the districting example discussed in Detecting Bias in Political Districting. Of course, unlike perfect sampling methods, the $\sqrt{\varepsilon}$ test can only be used to show that a given sample is not chosen from π; it does not give a way for generating samples from π.

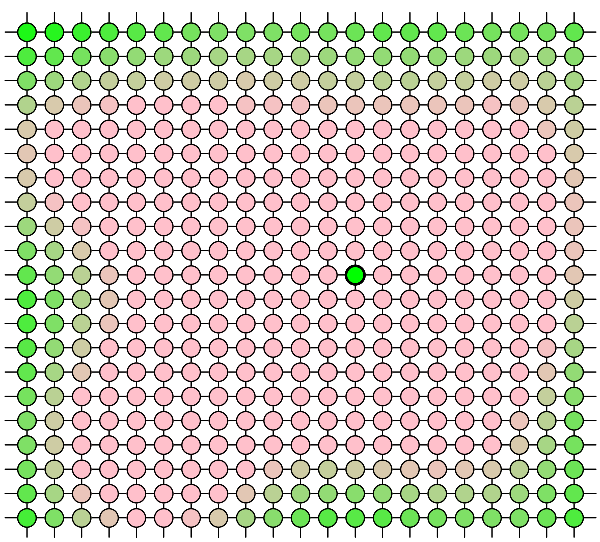


Fig. 1. This schematic illustrates a region of a potentially much larger Markov chain with a very simple structure; from each state seen here, a jump is made with equal probability to each of four neighboring states. Colors from green to pink represent labels from small to large, respectively. It is impossible to know from this local region alone whether the highlighted green state has unusually small label in this chain overall. However, to an unusual degree, this state is a local outlier. The $\sqrt{\varepsilon}$ test is based on the fact that no reversible Markov chain can have too many local outliers.

Definitions

We remind the reader that a Markov chain is a discrete time random process; at each step, the chain jumps to a new state, which only depends on the previous state. Formally, a Markov chain $\mathcal{M}$ on a state space Σ is a sequence $\mathcal{M} = X_0, X_1, X_2, \ldots$ of random variables taking values in Σ (which correspond to states that may be occupied at each step), such that, for any $\sigma, \sigma_0, \ldots, \sigma_{n-1} \in \Sigma$,

$$\begin{aligned}\mathbf{Pr}(X_n = \sigma | X_0 = \sigma_0, X_1 = \sigma_1, \ldots, X_{n-1} = \sigma_{n-1})\\ = \mathbf{Pr}(X_1 = \sigma | X_0 = \sigma_{n-1}).\end{aligned}$$

Note that a Markov chain is completely described by the distribution of X_0 and the transition probabilities $\mathbf{Pr}(X_1 = \sigma_1 | X_0 = \sigma_0)$ for all pairs $\sigma_0, \sigma_1 \in \Sigma$. Terminology is often abused, so that the Markov chain refers only to the ensemble of transition probabilities, regardless of the choice of distribution for X_0.

With this abuse of terminology, a stationary distribution for the Markov chain is a distribution π, such that $X_0 \sim \pi$ implies that $X_1 \sim \pi$ and therefore, that $X_i \sim \pi$ for all i. When the distribution of X_0 is a stationary distribution, the Markov chain $X_0, X_1, \ldots$ is said to be stationary. A stationary chain is said to be reversible if, for all i, k, the sequence of random variables $(X_i, X_{i+1}, \ldots, X_{i+k})$ is identical in distribution to the sequence $(X_{i+k}, X_{i+k-1}, \ldots, X_i)$. Finally, a chain is reducible if there is a pair of states σ_0, σ_1, such that σ_1 is inaccessible from σ_0 via legal transitions and irreducible otherwise.

A simple example of a Markov chain is a random walk on a directed graph beginning from an initial vertex X_0 chosen from some distribution. Here, Σ is the vertex set of the directed graph. If we are allowed to label the directed edges with positive reals

MATHEMATICS

POLITICAL SCIENCES

and if the probability of traveling along an arc is proportional to the label of the arc (among those leaving the present vertex), then any Markov chain has such a representation, because the transition probability $\mathbf{Pr}(X_1 = \sigma_1 | X_0 = \sigma_0)$ can be taken as the label of the arc from σ_0 to σ_1. Finally, if the graph is undirected, the corresponding Markov chain is reversible.

Detecting Bias in Political Districting

A central feature of American democracy is the selection of Congressional districts in which local elections are held to directly elect national representatives. Because a separate election is held in each district, the proportions of party affiliations of the slate of representatives elected in a state do not always match the proportions of statewide votes cast for each party. In practice, large deviations from this seemingly desirable target do occur.

Various tests have been proposed to detect "gerrymandering" of districting, in which a district is drawn in such a way as to bias the resulting slate of representatives toward one party, which can be accomplished by concentrating voters of the unfavored party in a few districts. One class of methods to detect gerrymandering concerns heuristic "smell tests," which judge whether districting seems generally reasonable in its statistical properties (11, 12). For example, such tests may frown on districting in which difference between the mean and median votes on district by district basis is unusually large (13).

The simplest statistical smell test, of course, is whether the party affiliation of the elected slate of representatives is close in proportion to the party affiliations of votes for representatives. Many states have failed this simple test spectacularly, such as in Pennsylvania; in 2012, 48.77% of votes were cast for Republican representatives and 50.20% of votes were cast for Democrat representatives in an election that resulted in a slate of 13 Republican representatives and 5 Democrat representatives.

Heuristic statistical tests such as these all suffer from lack of rigor, however, because of the fact that the statistical properties of "typical" districting are not rigorously characterized. For example, it has been shown (14) that Democrats may be at a natural disadvantage when drawing electoral maps, even when no bias is at play, because Democrat voters are often highly geographically concentrated in urban areas. Particularly problematic is that the degree of geographic clustering of partisans is highly variable from state to state: what looks like gerrymandered districting in one state may be a natural consequence of geography in another.

Some work has been done in which the properties of valid districting are defined (which may be required to have roughly equal populations among districts, districts with reasonable boundaries, etc.), so that the characteristics of a given districting can be compared with what would be typical for valid districting of the state in question, by using computers to generate random districting (15, 16); there is discussion of this in ref. 13. However, much of this work has relied on heuristic sampling procedures, which do not have the property of selecting districting with equal probability (and more generally, distributions that are not well-characterized), undermining rigorous statistical claims about the properties of typical districts.

In an attempt to establish a rigorous framework for this kind of approach, several groups (17–19) have used Markov chains to sample random valid districting for the purpose of such comparisons. Like many other applications of real-world Markov chains, however, these methods suffer from the completely unknown mixing time of the chains in question. Indeed, no work has even established that the Markov chains are irreducible (in the case of districting, irreducibility means that any valid districting can be reached from any other by a legal sequence of steps), even if valid districting was only required to consist of contiguous districts of roughly equal populations. Additionally, indeed, for very restrictive notions of what constitutes valid districting, irreducibility certainly fails.

As a straightforward application of the $\sqrt{\varepsilon}$ test, we can achieve rigorous p values in Markov models of political districting, despite the lack of bounds on mixing times of the chains. In particular, for all choices of the constraints on valid districting that we tested, the $\sqrt{\varepsilon}$ test showed that the current Congressional districting of Pennsylvania is an outlier at significance thresholds ranging from $p \approx 2.5 \cdot 10^{-4}$ to $p \approx 8.1 \cdot 10^{-7}$. Detailed results of these runs are in *SI Text*.

A key advantage of the Markov chain approach to gerrymandering is that it rests on a rigorous framework, namely comparing the actual districting of a state with typical (i.e., random) districting from a well-defined set of valid districting. The rigor of the approach thus depends on the availability of a precise definition of what constitutes valid districting; in principle and in practice, the best choice of definition is a legal question. Although some work on Markov chains for redistricting (in particular, ref. 19) has aimed to account for complex constraints on valid districting, our main goal in this manuscript is to illustrate the application of the $\sqrt{\varepsilon}$ test. In particular, we have erred on the side of using relatively simple sets of constraints on valid districting in our Markov chains, while checking that our significance results are not highly sensitive to the parameters that we use. However, our test immediately gives a way of putting the work, such as that in ref. 19, on a rigorous statistical footing.

The full description of the Markov chain that we use in this work is given in *SI Text*, but its basic structure is as follows: Pennsylvania is divided into roughly 9,000 census blocks. (These blocks can be seen on close inspection of Fig. 2.) We define a division of these blocks into 18 districts to be a valid districting of Pennsylvania if districts differ in population by less than 2%, are contiguous, are simply connected (districts do not contain holes), and are "compact" in ways that we discuss in *SI Text*; roughly, this final condition prohibits districts with extremely contorted structure. The state space of the Markov chain is the set of valid districting of the state, and one step of the Markov chain

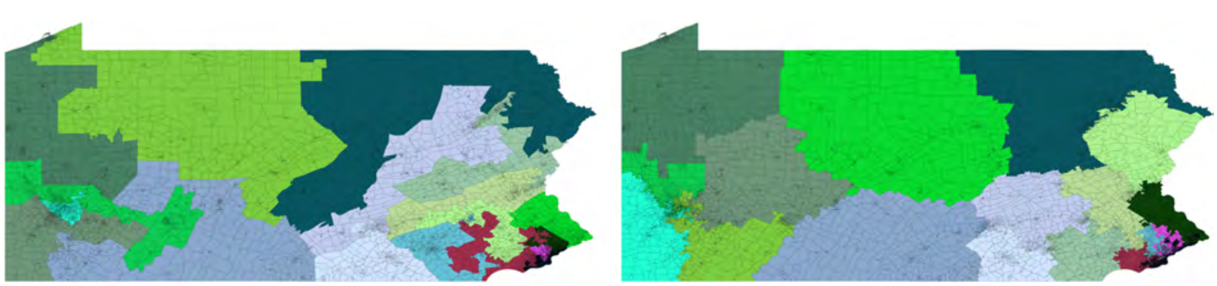


Fig. 2. (*Left*) The current districting of Pennsylvania. (*Right*) Districting produced by the Markov chain after 2^{40} steps. (Detailed parameters for this run are given in *SI Text*.)

consists of randomly swapping a precinct on the boundary of a district to a neighboring district if the result is still a valid districting. As we discuss in *SI Text*, the chain is adjusted slightly to ensure that the uniform distribution on valid districting is indeed a stationary distribution for the chain. Observe that this Markov chain has a potentially huge state space; if the only constraint on valid districting was that the districts have roughly equal population, there would be 10^{10000} or so valid districtings. Although contiguity and especially, compactness are severe restrictions that will decrease this number substantially, it seems difficult to compute effective upper bounds on the number of resulting valid districtings, and certainly, it is still enormous. Impressively, these considerations are all immaterial to our very general method.

Applying the $\sqrt{\varepsilon}$ test involves the choice of a label function $\omega(\sigma)$, which assigns a real number to each districting. We have conducted runs using two label functions: ω_{var} is the (negative) variance of the proportion of Democrats in each district of the districting (as measured by 2012 presidential votes), and ω_{MM} is the difference between the median and mean of the proportions of Democrats in each district; ω_{MM} is motivated by the fact that this metric has a long history of use in gerrymandering and is directly tied to the goals of gerrymandering, whereas the use of the variance is motivated by the fact that it can change quickly with small changes in districtings. These two choices are discussed further in *SI Text*, but an important point is that our use of these label functions is not based on an assumption that small values of ω_{var} or ω_{MM} directly imply gerrymandering. Instead, because Theorem 1.1 is valid for any fixed label function, these labels are tools used to show significance, which are chosen because they are simple and natural functions on vectors that can be quickly computed, seem likely to be different for typical versus gerrymandered districtings, and have the potential to change relatively quickly with small changes in districtings. For the various notions of valid districtings that we considered, the $\sqrt{\varepsilon}$ test showed significance at p values in the range from 10^{-4} to 10^{-5} for the ω_{MM} label function and the range from 10^{-4} to 10^{-7} for the ω_{var} label function (see Fig. S1 and Table S1).

As noted earlier, the $\sqrt{\varepsilon}$ test can easily be used with more complicated Markov chains that capture more intricate definitions of the set of valid districtings. For example, the current districting of Pennsylvania splits fewer rural counties than the districting in Fig. 2, *Right*, and the number of county splits is one of many metrics for valid districtings considered by the Markov chains developed in ref. 19. Indeed, our test will be of particular value in cases where complex notions of what constitute valid districting slow the chain to make the heuristic mixing assumption particularly questionable. Regarding mixing time, even our chain with relatively weak constraints on the districtings (and very fast running time in implementation) seems to mix too slowly to sample π, even heuristically; in Fig. 2, we see that several districts still seem to have not left their general position from the initial districting, even after 2^{40} steps.

On the same note, it should also be kept in mind that, although our result gives a method to rigorously disprove that a given districting is unbiased—e.g., to show that the districting is unusual among districtings X_0 distributed according to the stationary distribution π—it does so without giving a method to sample from the stationary distribution. In particular, our method cannot answer the question of how many seats Republicans and Democrats should have in a typical districting of Pennsylvania, because we are still not mixing the chain. Instead, Theorem 1.1 has given us a way to disprove $X_0 \sim \pi$ without sampling π.

Proof of Theorem 1.1

We let π denote any stationary distribution for $\mathcal{M}$ and suppose that the initial state X_0 is distributed as $X_0 \sim \pi$, so that in fact, $X_i \sim \pi$ for all i. We say σ_j is ℓ-small among $\sigma_0, \ldots, \sigma_k$ if there are, at most, ℓ indices $i \neq j$ among $0, \ldots, k$, such that the label of σ_i is, at most, the label of σ_j. In particular, σ_j is 0-small among $\sigma_0, \sigma_1, \ldots, \sigma_k$ when its label is the unique minimum label, and we encourage readers to focus on this $\ell = 0$ case in their first reading of the proof.

For $0 \leq j \leq k$, we define

$$\rho_{j,\ell}^k := \mathbf{Pr}\left(X_j \text{ is } \ell\text{-small among } X_0, \ldots, X_k\right)$$

$$\rho_{j,\ell}^k(\sigma) := \mathbf{Pr}(X_j \text{ is } \ell\text{-small among } X_0, \ldots, X_k \mid X_j = \sigma).$$

Observe that, because $X_s \sim \pi$ for all s, we also have that

$$\rho_{j,\ell}^k(\sigma) = \mathbf{Pr}\left(X_{s+j} \text{ is } \ell\text{-small among } X_s, \ldots, X_{s+k} \mid X_{s+j} = \sigma\right). \quad [1]$$

We begin by noting two easy facts.

Observation 4.1.

$$\rho_{j,\ell}^k(\sigma) = \rho_{k-j,\ell}^k(\sigma).$$

Proof. Because $\mathcal{M} = X_0, X_1, \ldots$ is stationary and reversible, the probability that $(X_0, \ldots, X_k) = (\sigma_0, \ldots, \sigma_k)$ is equal to the probability that $(X_0, \ldots, X_k) = (\sigma_k, \ldots, \sigma_0)$ for any fixed sequence $(\sigma_0, \ldots, \sigma_k)$. Thus, any sequence $(\sigma_0, \ldots, \sigma_k)$ for which $\sigma_j = \sigma$ and σ_j is a ℓ-small corresponds to an equiprobable sequence $(\sigma_k, \ldots, \sigma_0)$, for which $\sigma_{k-j} = \sigma$ and σ_{k-j} is ℓ-small. □

Observation 4.2.

$$\rho_{j,2\ell}^k(\sigma) \geq \rho_{j,\ell}^j(\sigma) \cdot \rho_{0,\ell}^{k-j}(\sigma).$$

Proof. Consider the events that X_j is an ℓ-small among $X_0, \ldots, X_j$ and among $X_j, \ldots, X_k$. These events are conditionally independent when conditioning on the value of $X_j = \sigma$, and $\rho_{j,\ell}^j(\sigma)$ gives the probability of the first of these events, whereas applying Eq. **1** with $s = j$ gives that $\rho_{0,\ell}^{k-j}(\sigma)$ gives the probability of the second event.

Finally, when both of these events happen, we have that X_j is 2ℓ-small among $X_0, \ldots, X_k$. □

We can now deduce that

$$\begin{aligned}\rho_{j,2\ell}^k(\sigma) &\geq \rho_{j,\ell}^j(\sigma) \cdot \rho_{0,\ell}^{k-j}(\sigma) = \rho_{0,\ell}^j(\sigma) \cdot \rho_{0,\ell}^{k-j}(\sigma) \\ &\geq \left(\rho_{0,\ell}^k(\sigma)\right)^2. \end{aligned} \quad [2]$$

Indeed, the first inequality follows from Observation 4.2, the equality follows from Observation 4.1, and the final inequality follows from the fact that $\rho_{j,\ell}^k(\sigma)$ is monotone nonincreasing in k for fixed j, ℓ, σ.

Observe now that $\rho_{j,\ell}^k = E\, \rho_{j,\ell}^k(X_j)$, where the expectation is taken over the random choice of $X_j \sim \pi$.

Thus, taking expectations in Eq. **2**, we find that

$$\begin{aligned}\rho_{j,2\ell}^k &= \mathbf{E}\rho_{j,2\ell}^k(\sigma) \geq \mathbf{E}\left(\left(\rho_{0,\ell}^k(\sigma)\right)^2\right) \\ &\geq \left(\mathbf{E}\rho_{0,\ell}^k(\sigma)\right)^2 = \left(\rho_{0,\ell}^k\right)^2, \end{aligned} \quad [3]$$

where the second of the two inequalities is the Cauchy–Schwartz inequality.

For the final step in the proof, we sum the left- and right-hand sides of Eq. **3** to obtain

$$\sum_{j=0}^k \rho_{j,2\ell}^k \geq (k+1)\left(\rho_{0,\ell}^k\right)^2.$$

MATHEMATICS

POLITICAL SCIENCES

If we let ξ_j $(0 \leq i \leq k)$ be the indicator variable that is one whenever X_j is 2ℓ-small among $X_0, \ldots, X_k$, then $\sum_{j=0}^{k} \xi_j$ is the number of 2ℓ-small terms, which is always, at most, $2\ell + 1$. Therefore, linearity of expectation gives that

$$2\ell + 1 \geq (k+1)\left(\rho_{0,\ell}^{k}\right)^2, \qquad \textbf{[4]}$$

giving that

$$\rho_{0,\ell}^{k} \leq \sqrt{\frac{2\ell+1}{k+1}}. \qquad \textbf{[5]}$$

Theorem 1.1 follows, because if X_i is an ε-outlier among $X_0, \ldots, X_k$, then X_i is necessarily ℓ-small among $X_0, \ldots, X_k$ for $\ell = \lfloor \varepsilon(k+1) - 1 \rfloor \leq \varepsilon(k+1) - 1$, and then, we have $2\ell + 1 \leq 2\varepsilon(k+1) - 1 \leq 2\varepsilon(k+1)$. □

ACKNOWLEDGMENTS. We thank John Nagle, Danny Sleator, and Dan Zuckerman for helpful conversations. M.C. was supported, in part, by NIH Grants 1R03MH10900901A1 and U545U54HG00854003. A.F. was supported, in part, by National Science Foundation (NSF) Grants DMS1362785 and CCF1522984 and Simons Foundation Grant 333329. W.P. was supported, in part, by NSF Grant DMS-1363136 and the Sloan Foundation.

1. Gelman A, Rubin DB (1992) Inference from iterative simulation using multiple sequences. *Stat Sci* 7(4):457–472.
2. Gelman A, Rubin DB (1992) A single series from the Gibbs sampler provides a false sense of security. *Bayesian Statistics 4: Proceedings of the Fourth Valencia International Meeting*, eds Bernardo JM, Berger JO, Dawid AP, Smith AFM (Clarendon, Gloucestershire, UK), pp 625–631.
3. Swendsen RH, Wang JS (1987) Nonuniversal critical dynamics in Monte Carlo simulations. *Phys Rev Lett* 58(2):86–88.
4. Borgs C, Chayes J, Tetali P (2012) Tight bounds for mixing of the Swendsen–Wang algorithm at the Potts transition point. *Probab Theory Relat Fields* 152(3-4):509–557.
5. Cooper C, Frieze A (1999) Mixing properties of the Swendsen-Wang process on classes of graphs. *Random Struct Algorithms* 15(3-4):242–261.
6. Gore VK, Jerrum MR (1999) The Swendsen–Wang process does not always mix rapidly. *J Stat Phys* 97(1-2):67–86.
7. Propp JG, Wilson DB (1996) Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Struct Algorithms* 9(1-2):223–252.
8. Propp J, Wilson D (1998) Coupling from the past: A user's guide. *Microsurveys in Discrete Probability*, eds Aldous DJ, Propp J (American Mathematical Society, Providence, RI), Vol 41, pp 181–192.
9. Fill JA (1997) An interruptible algorithm for perfect sampling via Markov chains. Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing (Association for Computing Machinery, New York), pp 688–695.
10. Huber M (2004) Perfect sampling using bounding chains. *Ann Appl Probab* 14(2): 734–753.
11. Wang SS-H (2016) Three tests for practical evaluation of partisan gerrymandering (December 28, 2015). *Stanford Law Rev* 68:1263–1321.
12. Nagle JF (2015) Measures of partisan bias for legislating fair elections. *Elect Law J* 14(4):346–360.
13. McDonald MD, Best RE (2015) Unfair partisan gerrymanders in politics and law: A diagnostic applied to six cases. *Elect Law J* 14(4):312–330.
14. Chen J, Rodden J (2013) Unintentional gerrymandering: Political geography and electoral bias in legislatures. *Quart J Polit Sci* 8(3):239–269.
15. Cirincione C, Darling TA, O'Rourke TG (2000) Assessing South Carolina's 1990s Congressional districting. *Polit Geogr* 19(2):189–211.
16. Rogerson PA, Yang Z (1999) The effects of spatial population distributions and political districting on minority representation. *Soc Sci Comput Rev* 17(1):27–39.
17. Fifield B, Higgins M, Imai K, Tarr A (2015) *A New Automated Redistricting Simulator Using Markov Chain Monte Carlo. Working Paper*. *Technical Report*. Available at imai.princeton.edu/research/files/redist.pdf. Accessed October 21, 2016.
18. Chenyun Wu L, Xiaotian Dou J, Frieze A, Miller D, Sleator D (2015) Impartial redistricting: A Markov chain approach. arXiv:1510.03247.
19. Vaughn C, Bangia S, Dou B, Guo S, Mattingly J (2016) *Quantifying Gerrymandering@Duke*. Available at https://services.math.duke.edu/projects/gerrymandering. Accessed October 21, 2016.
20. Ansolabehere S, Palmer M, Lee A (2014) *Precinct-Level Election Data, Harvard Dataverse, v1*. Available at https://dataverse.harvard.edu/dataset.xhtml?persistentId=hdl:1902.1/21919. Accessed October 21, 2016.
21. Edgeworth FY (1897) Miscellaneous applications of the calculus of probabilities. *J R Stat Soc* 60(3):681–698.
22. Levin DA, Peres Y, Wilmer EL (2009) *Markov Chains and Mixing Times* (American Mathematical Society, Providence, RI).
23. Aldous D, Fill J (2002) *Reversible Markov Chains and Random Walks on Graphs*. Available at https://www.stat.berkeley.edu/~aldous/RWG/book.html. Accessed October 21, 2016.
24. Frieze A, Karoński M (2015) *Introduction to Random Graphs* (Cambridge Univ Press, Cambridge, UK).

Expert Report on the North Carolina State Legislature and Congressional Redistricting (Corrected Version)

Jonathan C. Mattingly

December 23, 2021

Contents

1 Introduction

I am a Professor of Mathematics and Statistical Science at Duke University. My degrees are from the North Carolina School of Science and Math (High School Diploma), Yale University (B.S.), and Princeton University (Ph.D.). I grew up in Charlotte, North Carolina and currently live in Durham, North Carolina.

I lead a group at Duke University which conducts non-partisan research to understand and quantify gerrymandering. This report grows out of aspects of our group's work around the current North Carolina legislative districts which are relevant to the case being filed.

I previously submitted an expert report in Common Cause v. Rucho, No. 18-CV-1026 (M.D.N.C.), Diamond v. Torres, No. 17-CV-5054 (E.D. Pa.), Common Cause v. Lewis (N.C. Sup. Ct No. 18-cvs-014001), and Harper v. Lewis (No. 19-cv-012667) and was an expert witness for the plaintiffs in Common Cause v Rucho and Common Cause v. Lewis. I am being paid at a rate of $400/per hour for the work on this case. Much of the work derives from an independent research effort, unrelated to this lawsuit, to understand gerrymandering nationally and in North Carolina specifically. Much of the core analysis described in this report was previously released publicly as part of a non-partisan effort to inform the discussion around the redistricting process.

2 General Overview

I was asked in this case to analyze whether the enacted Congressional, state House, and state Senate redistricting plans for North Carolina were drawn intentionally for partisan advantage. In summary, to conduct our analysis, we used historic voting data to compare election results under the enacted plans with elections results under a collection of non-partisan maps generated using Markov Chain Monte Carlo methods, referred to throughout this report as an "ensemble." No partisan information is used to construct this ensemble of maps; only the generally accepted districting criteria of approximately equal population per district, contiguous and relatively compact districts, reducing traversals, and keeping counties, precincts, and possibly municipalities whole. One strength of the ensemble method is that it makes no assumptions in advance about what structure an election should have such as a relation to proportional representation or some type of symmetry considerations. Rather it shows what results would naturally occur, and the structure of those results, because of political geography of the state when non-partisan maps are used. We examine both the number of seats that would have been won under these vote counts, along with the expected margins of victory.

We see that each of the enacted plans is an extreme outlier with respect to its partisan properties in comparison to the ensemble. The Congressional, House, and Senate plans each systematically favor the Republican Party to an extent which is rarely, if ever, seen in the non-partisan collection of maps. Under many historic elections considered, each of the enacted maps elects significantly fewer Democrats than the typical number of Democrats found in the collection of maps. Specifically, the enacted Congressional plan produces 10 Republican seats and 4 Democratic seats across a wide range of historic elections, spanning roughly a 6-point differential in the statewide two-party vote share. In other words, Republicans win 10 congressional seats despite large shifts in the statewide vote fraction and across a variety of election structures. Over the statewide vote Democratic partisan vote range of 46.59% to 52.32%, the enacted map only twice changes the number of Republicans elected. The outcome of the election is largely stuck at 4 Democrats. Our non-partisan ensemble plans, by contrast, are far more responsive to changes in the election structure and the statewide vote fraction.

Under the enacted Senate and House plans, at times the Democratic Party is either denied a majority of seats or denied breaking a Republican supermajority when the overwhelming majority of maps in our ensemble would have resulted in either a Democratic majority or a simple Republican majority. In the Senate, we find instances in which the Republicans would have gained a supermajority under the enacted plan, but would have lost a supermajority in nearly every map in our collection. In the House, we find instances in which the Republicans won the supermajority of seats under the enacted plan but they would have not won the supermajority in the majority of maps in our collection.

In the House and Senate plans, the extreme statewide tilt towards the Republican Party is the result of a significant number of truly independent choices at the level of the county-clusters into which the state is divided. The chance of making so many independent choices which bias the results towards the Republican Party unintentionally, without corresponding choices favoring the Democratic party, is astronomically small.

In addition to this systematic bias towards the Republican Party which when aggregated produces highly atypical results, the enacted House and Senate plans also have highly atypical results in a number of county clusters even when viewed alone. Beyond often creating atypical results in terms of the number of seats won in a given cluster, our results also show a durability in the results in certain clusters under the enacted plans. By durable, we mean that the results remain atypically unchanged over a wide range of elections. This unresponsiveness to changes in vote counts is another problematic feature revealed by our analysis of the enacted plans.

Our analysis show that each of the three enacted plans is an extreme gerrymander over a range of voter behavior seen historically in North Carolina. The effect of these extreme gerrymanders is to prevent the Democrats from winning as many seats in Congress, the House, and the Senate as they would have had the maps been drawn in a neutral way without political considerations. In many cases, the enacted maps reduce the extent to which the results of an election respond to the changing options of the electorate as expressed at the ballot box.

3 Discussion on Interpreting The Ensemble Method

3.1 The Political Geography

In redistricting conversations, there are often discussions of the urban versus rural divide and natural packing. These points demonstrate the need for a methodology that accounts for this political geography; ensemble methods precisely capture it. The distribution on redistricting plans can distinguish between typical plans and atypical plans. This determination is fundamentally informed by the geometry of the state, its political geography, and the spatial structure of the elections used to probe the redistricting plan.

The fundamental power of the ensemble method is that it begins with a clear set of redistricting criteria as an input. It then creates a representative ensemble of redistricting plans which accounts for the geometry of the state and the geography of where people live and how they vote. Any collection of voting data can then be applied to this ensemble of restricting plans to obtain a collection of election results. The election results give a benchmark against which a particular redistricting may be compared under the same set of voting data. It is only the relative difference between the ensemble and the enacted plan which matters. Our ensemble of restricting plans naturally incorporates how nonpartisan redistricting criteria interact with the political geography and geometry of the state. It naturally adapts to natural packing in urban areas and other effects. It is capable of separating these natural effects from those of partisan gerrymandering. Because of this, this mode of analysis can separate bias that natural packing might induce from other effects.

Additionally, none of these analyses rely on any forms of partisan symmetry or ideas of proportional representation. The ensemble method does not impose any idea of fairness nor does it select for a particular seats-to-votes curve. Rather it illuminates what the result would have typically been had only the stated redistricting criteria been utilized. It is quite possible, and often happens, that the results from the ensemble method do not yield proportional representation and one party has a natural advantage relative to the statewide vote fraction. One can then use this natural advantage as a benchmark to detect when a particular plan is biased beyond the neutral standard the ensemble establishes.

3.2 Different Elections have Different Voting Patterns

Elections differ both in the statewide partisan vote fraction and the spatial patterns of voting across the state. Hence, it is not at all surprising that a given map can act differently under different voting patterns; even those that share the same statewide partisan vote fractions. For instance, a map could be designed to neutralize the effectiveness of a particular set of coalitions, and hence would only be a statistical outlier in elections when those coalitions are active.

On a number of occasions, we have seen maps that particularly show the effect of the Gerrymander when there is a danger that the majority or supermajority are lost. To better understand why this is natural, consider the following example. Let us assume that a region has three varieties of people who always vote as a block and are spatially contiguous. For definiteness, let us call them red, purple, and blue people. We will assume that red always vote for the red candidate and blue for the blue candidate. Sometimes the purple vote for the red candidate and sometimes for the blue candidate. Hence, sometimes red wins two seats, and sometimes three seats, depending on how the purple people vote. Let us assume that most redistricting plans that one would naturally draw (without knowing where the red, purple, and blue people lived) would produce 2 majority red districts, 2 majority blue districts, and one majority purple district. We will call these neutral plans. Now let us consider a plan which is carefully drawn so that the purple people are never a majority but rather the purple people are split such that there are three majority blue districts and two majority red. We will call this the gerrymandered plan.

Under the gerrymandered plan the red candidates always win two of the five seats, but never more. This is typical of elections where the purple people vote with the blue people. It is typical because the majority purple district in the neutral plans would vote for the blue candidate to elect three blue candidates. On the other hand, in elections where the purple people vote with the red people, the outcome would be highly atypical as the neutral maps would have always produced three red winners but the gerrymandered plan only produces two red winners. In summary, atypical maps may lead to a typical split of elected officials under some vote counts, but not under others. It is not unusual for gerrymandered maps to sometimes produce typical results.

3.3 Collected Seat Histograms and Uniform Swing Analysis

It is a misconception that a gerrymandered map will behave atypically under all different types of elections. Gerrymandered maps can behave atypically under some types of elections and typically under other types of elections. For example, a map may only become atypical when a party is in danger of losing the majority. We demonstrate this through a type of plot we call *Collected Seat Histograms*. The election data use can either be historical elections or data generated using a uniform swing hypothesis.[1]

In both cases, we plot the histograms tabulating the fraction of the ensemble maps which produce a particular number of Democratic seats under a particular choice of statewide votes (tabulated at the precinct level). We then collect these histograms on a single plot where they are arranged on the vertical axis according to their statewide vote fractions, with the most Republican at the bottom and the most Democratic at the top. On each of the individual histograms, we also place a mark corresponding to the number of seats the enacted map would produce using those votes. Using these plots, one can identify trends and types of elections were the enacted maps products outlier results. When considering the NC State House and Senate, we also place vertical lines on each plot to mark where the supermajorities are in effect and where the simple majority in the chamber changes hand.

In addition to using historical statewide votes to produce our Collected Seat Histograms, we also create a collection of Collected Seat Histograms built from a single historical vote which is shifted using the Uniform Swing Hypothesis to produce a collection of votes which preserve the relative voting pattern across the state while seeing the effect of shifting the partisan tilt of the election.

Both kinds of Collected Seat Histograms are effective at identifying maps that are non-responsive to changing voter opinions or under-respond to those changes. A district map that results in different representation when the number of votes for a particular party changes sufficiently is a minimal requirement of a democratic process that is responsive to the changing will of the people. The Collected Seat Histograms can be used to determine the level of responsiveness to changes in the votes one should expect of the maps that were drawn without a partisan bias. The Rank Ordered Boxplots in the next section can help illuminate the structure of the map which is responsible for any systematic bias or lack of responsiveness relative to the nonpartisan benchmark embodied in the ensemble.

[1]The uniform swing hypothesis takes a single election and then uniformly increases (or decreases) the percentage for a given party across all the predicts. This creates a new set of voting data with the same spatial structure but a different statewide partisan percentage for each party.

3.4 Structure of Maps and Rank-Ordered Marginal Boxplots and Histograms

While the partisan seat count is clearly a quantity of interest, it can be less effective at illuminating the structure of a map that also explores how the elections are won. To this end, we introduce the *Rank-Ordered Marginal Boxplots and Histograms*. These are formed by considering the partisan vote fraction for one of the political parties (say the Democrats, or equally the Republicans) in each of the districts for a given redistricting plan. These marginal vote fractions are then ordered from smallest to largest, that is to say; from most Republican district to most Democratic district. These ordered numbers are then tabulated over all of the plans in the ensemble.

The Rank-Ordered Marginal Boxplots plot the typical range of the most Republican district to most Democratic district. Ranges are represented by box-plots. In these box-plots, 50% of all plans have corresponding ranked districts that lie within the box; the median is given by the line within the box; the ticks mark the 2.5%, 10%, 90% and 97.5% quartiles; the extent of the lines outside of the boxes represent the range of results observed in the ensemble. The number of boxes is the same as the number of seats. That is 120 seats for the NC House, 50 seats for the NC Senate, and 14 seats for the NC Congressional Delegation. Any box that lies above the 50% line on the vertical axis will elect (or typically elect) a Democrat; any box that lies below the 50% line will elect (or typically elect) a Republican.

We take the enacted plan with each set of votes and plot the ordered district returns over the box plots. If the districts of an enacted plan lie either far above or far below the ensemble at a particular ranking, this can indicate that the district was either packed or cracked to provide an atypical result.

4 State Legislature

Using historic voting data, we compare election results under the enacted districting plans for the North Carolina House and North Carolina Senate with election results under a collection of non-partisan maps. One strength of this method is that it makes no assumptions in advance about what structure an election should have such as a relation to proportional representation or some type of symmetry considerations. We examine both the number of seats that would have been won under these vote counts, along with the expected margins of victory.

4.1 State Legislature: Overview of Findings

4.2 State Legislature: Overview of Method

We generate a collection of alternative restricting maps using Markov Chain Monte Carlo methods, and used this collection to characterize what would be expected if only non-partisan redistricting criteria where used. We have described this method in detail in our academic work. See [7, 3, 8, 10, 1, 2]. (References in this report to numbers in brackets are to articles cited in a numbered bibliography at the end of this report). No partisan information is used to construct this ensemble of maps; only the generally accepted districting criteria of approximately equal population per district, contiguous and relatively compact districts, reducing traversals, and keeping counties, precincts, and municipalities whole.

For both the NC House and NC Senate, we generate a *Primary Ensemble* whose non-partisan properties are close to those of the enacted plan. Because of this, we sometimes label this plan as the *Matched Ensemble*. For both the NC Senate and NC House, we produce a *Secondary Ensemble* which makes different policy choices concerning the preservation of municipalities. In a third ensemble built, we also consider the pairing of incumbents.

The ensembles are generated by using the Metropolis-Hasting Markov Chain Monte Carlo Algorithm in a parallel tempering framework which employs proposal from the Multiscale Forest RECOM algorithm [2, 1] and the single-node flip algorithm [7]. Using these proposals, the Metropolis-Hasting algorithm is then used to produce samples from the desired policy-informed, non-partisan distribution on redistrictings; such algorithms are widely accepted for sampling high-dimensional distributions. The Markov Chain Monte Carlo and Metropolis-Hasting algorithms are a cornerstone of modern computational statistics, protein folding and drug discovery, and weather prediction. They date back to at least the Manhattan Project in Los Alamos are used in a huge range of mathematical and statistical applications.

The distributions we use are defined to be concentrated on districting plans that contain districts near the ideal district population based on the one-person-one-vote principle (including the 5% population deviation acceptable for legislative districts). They are also designed to produce contiguous districts that are relatively compact and to reduce the number of counties and, in some cases, the number of people split out of a municipality. For the Primary Ensemble, the distribution on redistricting plans is tuned so that these non-partisan qualities, including the number of counties, municipalities, and precincts which are split, are similar to the enacted plan. We also respect the county-clustering requirement for State Legislative maps.

We will see that the enacted NC Senate preserves municipalities to a high degree; in a way consistent with the most municipality preserving distributions we could produce. Hence, we also provide a *Secondary Ensemble* for the NC Senate which does not explicitly preserve municipalities (thought compactness and the county preservation lead to a degree of municipality preservation.) It coincides with the primary ensemble properties in other resects.

For the NC house, we will see that the enacted plan is not as stringent in its municipality preservation, and that respecting the other criteria could naturally create many plans that better preserve municipalities than the enacted plan. Since we have tuned our primary ensemble to match the level of municipality preservation in the enacted plan, which include a Secondary Ensemble for the NC house we is better at preserving municipalities.

As the guidance from the legislature at the start of the redistricting process stated that one "may consider municipality preservation" (in contrast to other directives which were not optional), all four of these ensembles meet the guidance given by the legislature. As already mentioned, we also provide a third ensemble for both the NC house and NC Senate which is derived from the primary ensemble, but considers the double-bunking of incumbents.

In all cases using the Metropolis-Hasting Markov Chain Monte Carlo Algorithm, we can produce a mathematically representative sample of the redistricting plans that comply with the criteria described.

4.3 County Clusters for State Legislature

In Stephenson v. Bartlett, 562 S.E.2d 377 (N.C. 2002), the North Carolina Supreme Court ruled that North Carolina's state legislative districts should be clustered into groups of counties and that no district should cross between two of the "county clusters." As part of our non-partisan work concerning redistricting, we implemented the algorithmic part of the Stephenson Ruling in a publicly available open-source piece of software [4]. We used this computer software to produce the county clusterings used in this report. The resulting clusterings were described in our publicly released report which can be found here [5]. We understand that the NC Legislature also used this report to determine the possible clusterings. In any case, the clusterings we found coincide with those discussed by the legislature.

There is not a unique choice of statewide clustering. Rather there are parts of the state which can only be clustered in one way, while there are two ways to cluster the counties in other regions. In the state Senate, there are 17 clusters containing 36 of the 50 districts that are fixed based on determining optimal county clusters. These are represented by the color county groupings in Figure 4.3.1. The white numbers annotating each county clustering give the number of districts that the county cluster should contain. Ten of these clusters contain one district, meaning that ten of the 50 senate districts are fixed by the county clusters. The remaining county clusters must be further subdivided into legislative districts. The remaining 14 counties, shown in gray on the map in Figure 4.3.1 are distributed among four groups, each containing two clustering options. Following the nomenclature in [5], we will label the cluster groups by the letters A, B, C, and D . Each group consists of two different possible clusterings which we will label with the numbers 1 and 2. Thus, the first choice in cluster A is labeled A1, and the second choice A2. A complete choice of county clusters then consists of one choice from the A group, the B group, the C group, and the D group.

Similarly, in the NC State House, there are 33 clusters containing 107 of the 120 districts that are fixed based on determining optimal county clusters. These are represented by the color county groupings in Figure 4.3.2. Again, the white numbers annotating each county clustering give the number of districts that the county cluster should contain. Eleven of these clusters contain one district, meaning that eleven of the 120 house districts are fixed by the clustering process. The remaining clusters (shown in gray) are separated into three groups each containing two clustering options. As before, the groups will be demoted by the letters A, B, and C with each of the two options in each group labeled with the numbers 1 or 2.

More details can be found in [5] and [4]. It should be noted that the algorithm used to produce these clusterings only implements the algorithmic portion of the Stephenson v. Bartlett. In particular, it does not address any compliance with the Voting Rights Act.

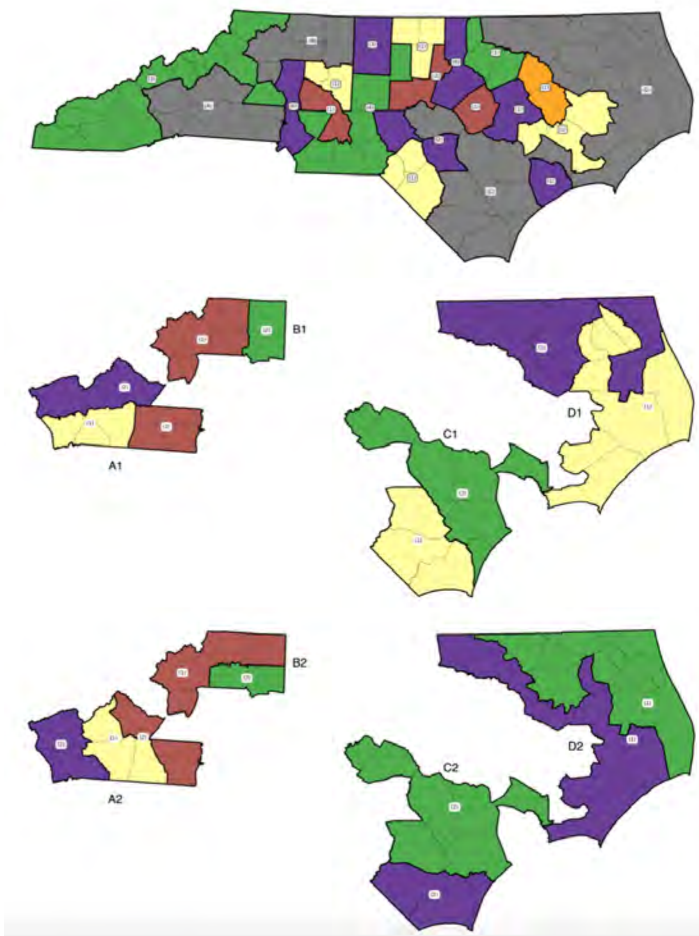


Figure 4.3.1: Senate

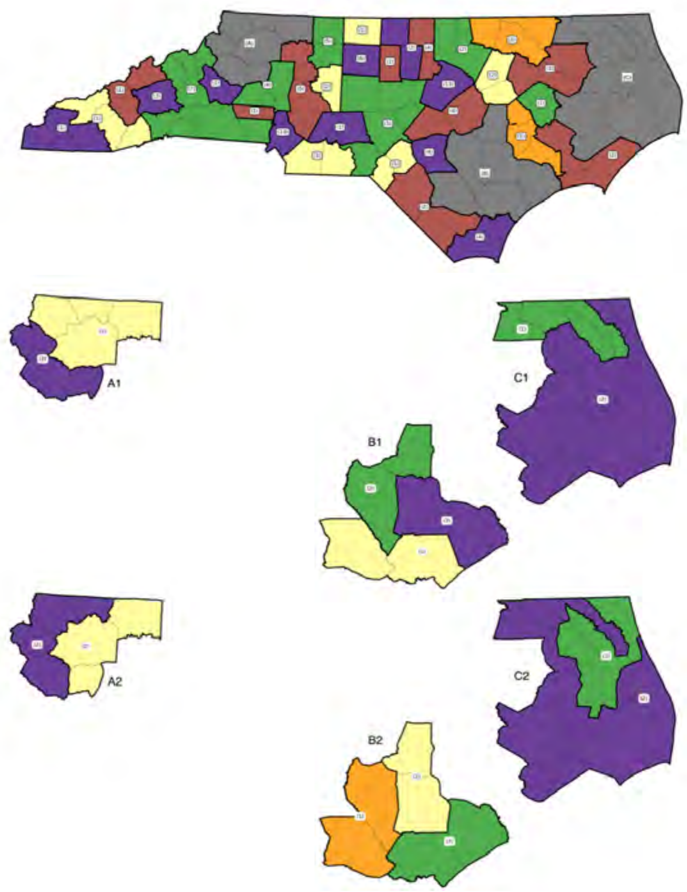


Figure 4.3.2: House

4.4 State Legislature: Ensemble Overview

We now give more details on the different distributions already sketched in Section 4.2. They represent different distributions that emphasize different policies consistent with the Legislature's guidance and historical presidents. All the distributions from which we build our ensembles respect the county clusters we derived in [6] by algorithmically implementing the ruling Stephenson v. Bartlett, 562 S.E.2d 377 (N.C. 2002). That is to say in both the State House and State Senate, the state is segmented into groups of counties referred to as *county clusters* so that the population of each county cluster can be divided into a number of districts each with a population within 5% of the ideal district population. The county clusters are different for the State House and State Senate as the number of districts, and hence the ideal district populations, are different. Each district is constrained to lay entirely within one county cluster.

Beyond the county cluster requirement all of our primary and secondary ensembles for both chambers also satisfy the following constraints:

- The maps minimize the number of split counties. The 2021 redistricting criteria state that "Within county groupings, county lines shall not be traversed except as authorized by Stephenson I, Stephenson II, Dickson I, and Dickson II."
- Districts traverse counties as few times as possible.
- All districts are required to consist of one contiguous region.
- Except for two exceptions, the deviation of the total population in any district is within 5% of the ideal district population. The two special cases are explained in Section 7.2.
- Voting tabulation districts (i.e. VTDs or precincts) are not split (see again the two exceptions with population deviation in Section 7.2)
- Compactness: The distributions on redistricting plans are constructed so that a plan with a larger total isoperimetric ratio is less likely than those with a lower total isoperimetric ratio. (See Section 7.2 and 8.1 for a definition of the isoperimetric ratio.) The total isoperimetric ratio of a redistricting plan is simply the sum of the isoperimetric ratios over each district. The isoperimetric ratio is the reciprocal of the Polsby-Poper score; hence, smaller isoperimetric ratio corresponds to larger Polsby-Poper scores. The General Assembly stated in its guidance that the plans should be compact according to the Polsby-Popper score or the Reock score [9]. We have found that while the Reock is useful when comparing two districts. However, the Polsby-Popper/isoperimetric score is a better measure when generating district computationally. In our previous work, we have seen that this choice did not qualitatively change our conclusions (see [7] and the expert report in Common Cause v. Rucho).

  We tuned our primary ensemble so that compactness scores of the ensemble were comparable to those of the enacted plan. See Section 7, for plots showing the compactness scores.

Municipality Preservation: We now come to the property which distinguishes the Primary and Secondary ensembles. In both chambers of the NC Legislature, we tune the primary ensemble to match the level of municipalities preservation to those seen in the enacted plan. Since municipality preservation is concerned with keeping the voters of a particular municipality together as a block, we concentrate on the number of *ousted voters*. Ousted voters are those who have been removed from the districts which primarily contain the other members of the municipalities. We construct the ensemble to control the total number of ousted voters across the entire state. More details are given in Section 7.2. As already mentioned, we tune the Secondary ensembles differently for the two chambers. Since the Enacted Senate plan was at the lowest end of municipality splitting we observed, we have included a secondary ensemble in the Senate which did not explicitly consider municipality reservation. In the NC House, since the enacted plan did not preserve municipalities to the level we found possible, we included a secondary ensemble which better preserved municipalities.

Incumbency: The effect of incumbency are addressed in a subsequent section of this report.

4.5 Construction of Statewide Ensembles for State Legislature

Statewide ensembles are created by drawing samples from a number of "sub-ensembles." Because of the county cluster structure, we can sample each county cluster independently of the other county clusters. In the house, we sample the Wake and Mecklenburg county cluster groups separately from the rest of the state as they have many more precincts and districts. In the Senate, we sample the Wake county cluster independently since it must split precincts to achieve the 5% population balance. There are several regions of the state that have multiple options for county clusters and we sample each of the county clustering options separately. We then sample the remainder of the state together.

We combine these sub-ensembles by first choosing which of the county clustering options will be used, treating all options equally. With these fixed, we then choose a map from each of the other sub-ensembles and combine them to produce a statewide map. We used this procedure to create an ensemble of 100,000 maps. These ensembles of statewide maps were used to generate the various figures. This number was chosen as it proved to be sufficient for the statistics of the quantities of interest to have converged. That is to say that adding additional maps to the ensemble did not change the results. See Section 7.1 for more details on the sampling method.

4.6 Election Data Used in Analysis

The historic elections we consider are from the year 2016 and 2020. We only consider statewide elections. We will use the following abbreviations: AG for Attorney General, USS for United States Senate, CI for Commissioner of Insurance, LG for Lieutenant Governor, GV for Governor, TR for State Treasure, SST for Secretary of State, AD for State Auditor, CA for Commissioner of Agriculture, and PR for United States President. We add to these abbreviations the last two digits of the year of the election. Hence CI16 is the vote data from the Commissioner of Insurance election in 2016.

5 State Legislature: Main Statewide Analysis

Our analysis shows that the enacted plan for the NC State House is an extreme gerrymander over a wide range of voter behavior seen historically in NC. The effect of this extreme gerrymander is to prevent the Democrats from winning as many seat as they would have had the maps been drawn in a neutral way without political considerations. This gerrymander is achieved by packing Democrats in a number of the most Democratic districts while depleting them from those districts which typically change hands when the public changes its expressed political opinon through the vote. The effect is particularly strong in situations where the Democrats would typically reduce a Republican supermajority to a a simple majority. The enacted map often denies this transition. Similarly the enacted map again behaves in an anomalous fashion by under electing democrats when the typical maps would almost always give the Democrats the majority in the House. This extreme outlier behavior is reflected in the behavior we see at the individual cluster level.

The effect in the Senate is less pronounced. At the cluster level there are a number of strong and extreme outliers signaling extreme partisan gerrymandering. At the statewide level, the structure of the map shows it to be an extreme outlier in the fashion in which Democrats are packed in certain districts and depleted from others. The effect at the statewide level is mostly seen when the Republicans are in danger of losing the supermajority in the Senate. Over this range the anomalous packing and cracking of Democrats leads to a number of extreme outlier behaviors which result in the Republicans maintaining the supermajority when they typically would have lost it under a non-partisan map from the ensemble.

Additionally we see that the reason that the Senate map is typical in many situations stems from the choice to highly conserve municipalities. The municipality preservation is at the extreme end of what we have observed. In contrast, the municipality preservation in the house is less extreme as we can easily create an ensemble which preserves municipalities to a higher degree. For the Senate plan, relaxing the requirement to preserve municipalities leads to an ensemble that is more favorable to the Democrats, meaning that the enacted plan would be an extreme outlier in more situations. Put differently, prioritizing municipality preservation in the Senate plan appears to enable more maps that favor Republicans. By contrast, for the House plan, where the enacted map does not prioritize preserving municipalities, my analysis finds that such a prioritization would not have favored the Republican party.

5.1 NC State House

Figure 5.1.1 shows the distribution of Democratic seats elected under a number of historical elections which capture plausible voting patterns in North Carolina elections. The elections are arranged vertically by the statewide Democratic vote share, from most Republican at the bottom to the most Democratic at the top. The Democratic seats elected under each election by the enacted plan is marked with a yellow dot.

It is important to remember that the single number of statewide vote fraction is not sufficient to categorize an election. Elections with similar statewide vote fractions can have dramatically different seat counts since the votes can be concentrated differently geographically. An example of this is shown in Figure 5.1.8 which shows the Collected Seat Histograms for an ensemble that places more weight on preserving municipalities that the enacted plan or the primary ensemble. Notice that the AG20 votes produce more democratic seats typically than either AG16 or GV16 even though the statewide vote fraction of AG20 is sandwiched between AG16 and GV16. (Recall the definitions of these abbreviations given in Section 4.6.)

Returning to Figure 5.1.1, we see that the enacted map is atypical in its favoring of the Republican party in every one of the elections considered and an outlier or extreme outlier in the vast majority of the elections. Additionally, the enacted plan is an extreme outlier when the Republicans are likely to lose either the Super-majority or control of the chamber. Observe that in the vast majority of plans in the primary ensemble (Figure 5.1.1) the votes in PR16, LG20 and CL20 produce a simple majority for the Republican party in the NC State House (and not a supermajority). Yet under the enacted plan, the Republican Party maintains the supermajority in all three cases.

Similarly, in a large number of the ensemble plans the Democrats hold the majority in the chamber under the voting patterns given by AD20, SST20, and GV20. (Under GV20 the Democrats have the majority most of the time, under AD20 roughly half the time and under SST roughly 75% of the time.) Yet, under the enacted plan the results are extreme outliers, giving the Republicans the majority with a safety margin of a few seats in all cases.

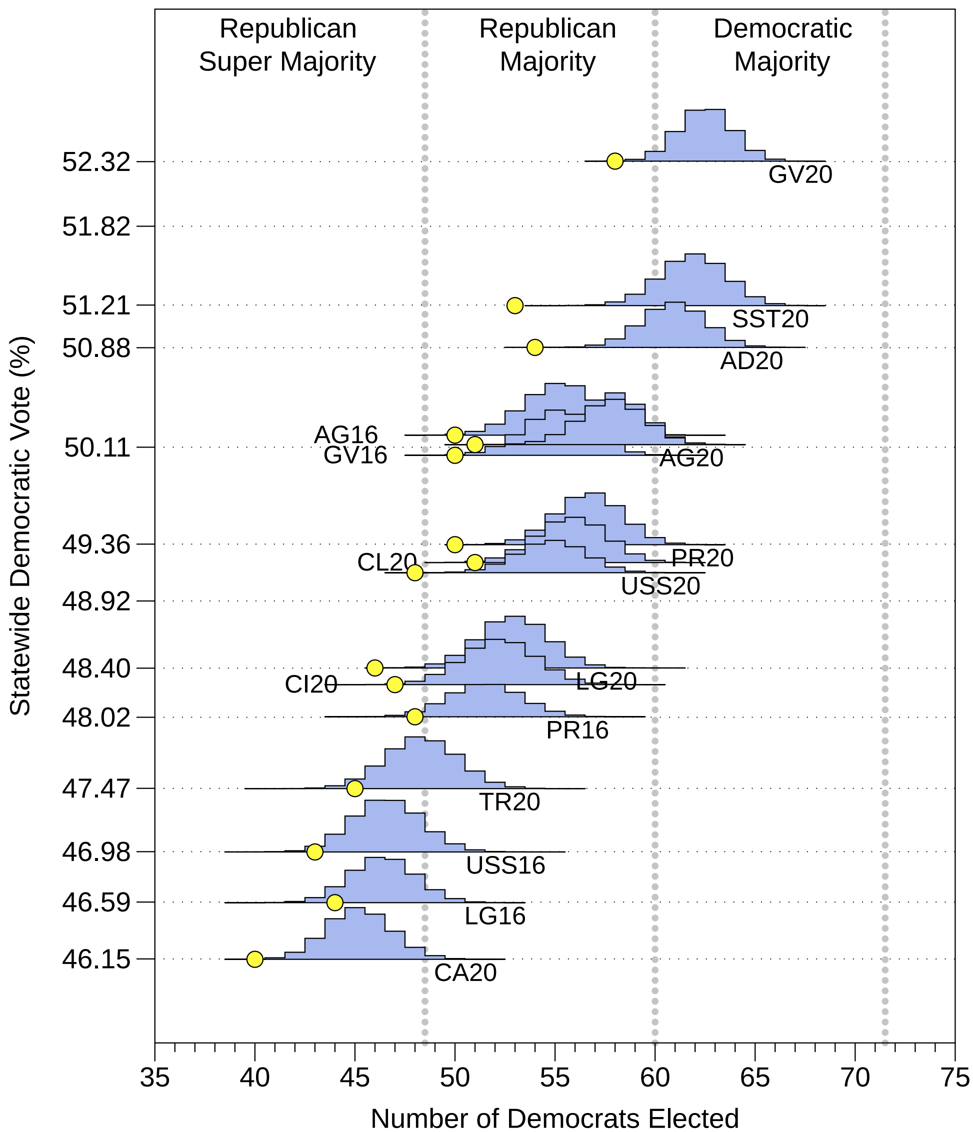


Figure 5.1.1: The Collected Seat Histogram for the Primary Ensemble on the NC House. The individual histograms give the frequency of the Democratic seat count for each of the statewide elections considered from the years 2016 and 2020. The histograms are organized vertically based on the statewide partisan vote fraction for each election. The more Republican elections are placed lower on the plot while more Democratic elections are placed higher. Three dotted lines denote the boundary between where the supermajorities and simple majorities are in force. The yellow dot represents the enacted plan.

As already observed, Figure 5.1.1 helps to identify the properties of the Enacted Map under different electoral environments. There is a clear trend as one moves to more Democratic elections, the atypical results (already tilted to toward

% Dem	Election	% Outlier	# Outlier	# Samples
52.32%	GV20	0.118%	118	100000
51.21%	SST20	0.000%	0	100000
50.88%	AD20	0.007%	7	100000
50.20%	AG16	0.451%	451	100000
50.13%	AG20	0.005%	5	100000
50.05%	GV16	0.399%	399	100000
49.36%	PR20	0.007%	7	100000
49.22%	CL20	0.759%	759	100000
49.14%	USS20	0.012%	12	100000
48.40%	LG20	0.009%	9	100000
48.27%	CI20	0.461%	461	100000
47.47%	TR20	5.569%	5569	100000
46.98%	USS16	3.066%	3066	100000
46.59%	LG16	11.778%	11778	100000
46.15%	CA20	0.094%	94	100000

Table 1: NC House Collected Seat Histogram Outlier Data. Starting from the left, the first column gives the statewide partisan makeup of the of the election under consideration whose abbreviation is given in the second column from the left. The right most column gives the total number of plans in the ensemble considered which is 100,000. The second column from the right gives the number of those 100,000 plans which elect the same or less Democrats under the given election. These are the plans which are as much or more of an outlier than the enacted map. The middle column is the percentage of plans which are more or equal of an outlier. (It is calculated by dividing the 2nd column from the right by 100,000 and multiplying by 100 to make a percentage.) The extremely low percentages in the middle column shows that the enacted plan is an extreme outlier across many different electoral settings.

the Republican party) in the more Republican elections in Figure 5.1.1 trend into extreme outliers as we shift to the more Democratic leaning elections.

To make the above table more quantitative, in Table 1 we tabulated the number of maps which produced the same or fewer seats for the Democrats in each of the elections we consider. We see that the enacted map is an extreme outlier. Across the vast majority of elections, the house map behaves as an extreme outlier in favor of the Republican party.

In the three elections where the results are not an extreme outlier (TR20, USS16, and LG16), the enacted plan is still atypically tilted to favor the Republican party. These three elections have a strong statewide Republican vote fraction. Hence, there is no need for a gerrymander as the Republicans have the needed votes to often keep a supermajority under even a typical map.

We will see in Figure 5.1.2 and 5.1.3 below that when these three elections are shifted (using the uniform swing hypothesis) to produce plausible voting fractions at a larger statewide Democratic vote fraction, then the results are also extreme outliers.

It is also worth noting that the bias in the enacted plan from what non-partisan map would produce systematically is the favor of the Republican party. Not once is the tilt even mildly in the favor of the Democrats.

To better control for other variation, we now include a number of Collected Seat Histograms built from a single election which has been shifted to create a sequence of elections with different statewide partisan vote fractions but the same spatial voting patern.

In Figures 5.1.2 and 5.1.3, we see that the same phenomena from Figure 5.1.1 is repeated again and again. As the vote share increases to the point where the primary ensemble for the NC House would typically break the Republicans supermajority, the enacted plan under elects Democrats to an extent which makes it an extreme outlier. This exceptional under-electing of Democrats persists past the point where almost all of the ensemble maps would have given the majority to the Democrats. In many cases the enacted map fails to respond to the shifting will of the electorate, leaving the control in the Republican hands. In addition to presenting these figures, we have also animated this affect with movies that have been submitted.

To better understand the structures responsible at the district level for the extreme outlier behavior seen in Table 2 and Figures5.2.1 to 5.2.2, we now turn to the rank-order-boxplots as described in Section 3.4. It is easy to see the abnormal structures of the enacted plan which are responsible for its extreme outlier behavior. The pattern revealed is one often seen in gerrymandered maps; namely *packing and cracking*. This refers to the depleting of one party from districts which typically would be competitive but often elect a representative from their party and instead place them in districts which were already overwhelmingly safe for either party. In Figures 5.1.4, 5.1.5, and 5.1.6, a version of this pattern is repeated. The number of Democrats seen in the districts which usually would be moderate in their partisan makeup has been decreased with a corresponding increase in the number of Democrats in the more Democratic districts where their presence has little effect on the election outcome. We give the specifics in the captions of each figure. We will see that this type of structure will be repeated in many of the individual clusters which are analyzed in Section 6.1. In addition to presenting these figures, we have also animated this affect with movies that have been submitted.

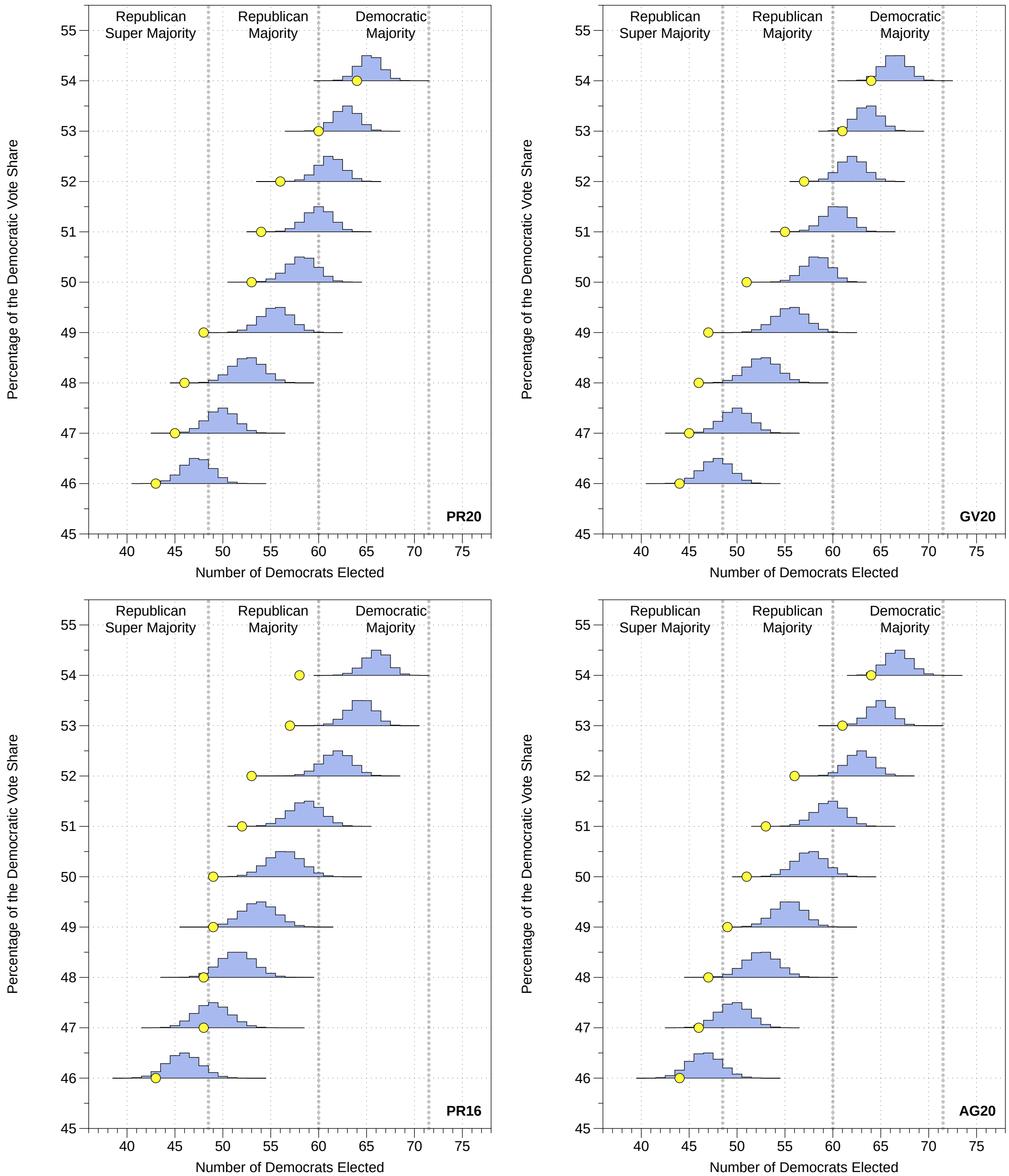


Figure 5.1.2: The individual histograms give the frequency of the Democratic seat count in the ensemble for each of the shown statewide elections, with a uniform swing. The histograms are organized vertically based on the statewide partisan vote fraction. The more Republican swings are placed lower on the plot while more Democratic swings are placed higher. Three dotted lines denote the boundary between where the supermajorities and simple majorities are in force. The yellow dot is the enacted plan.

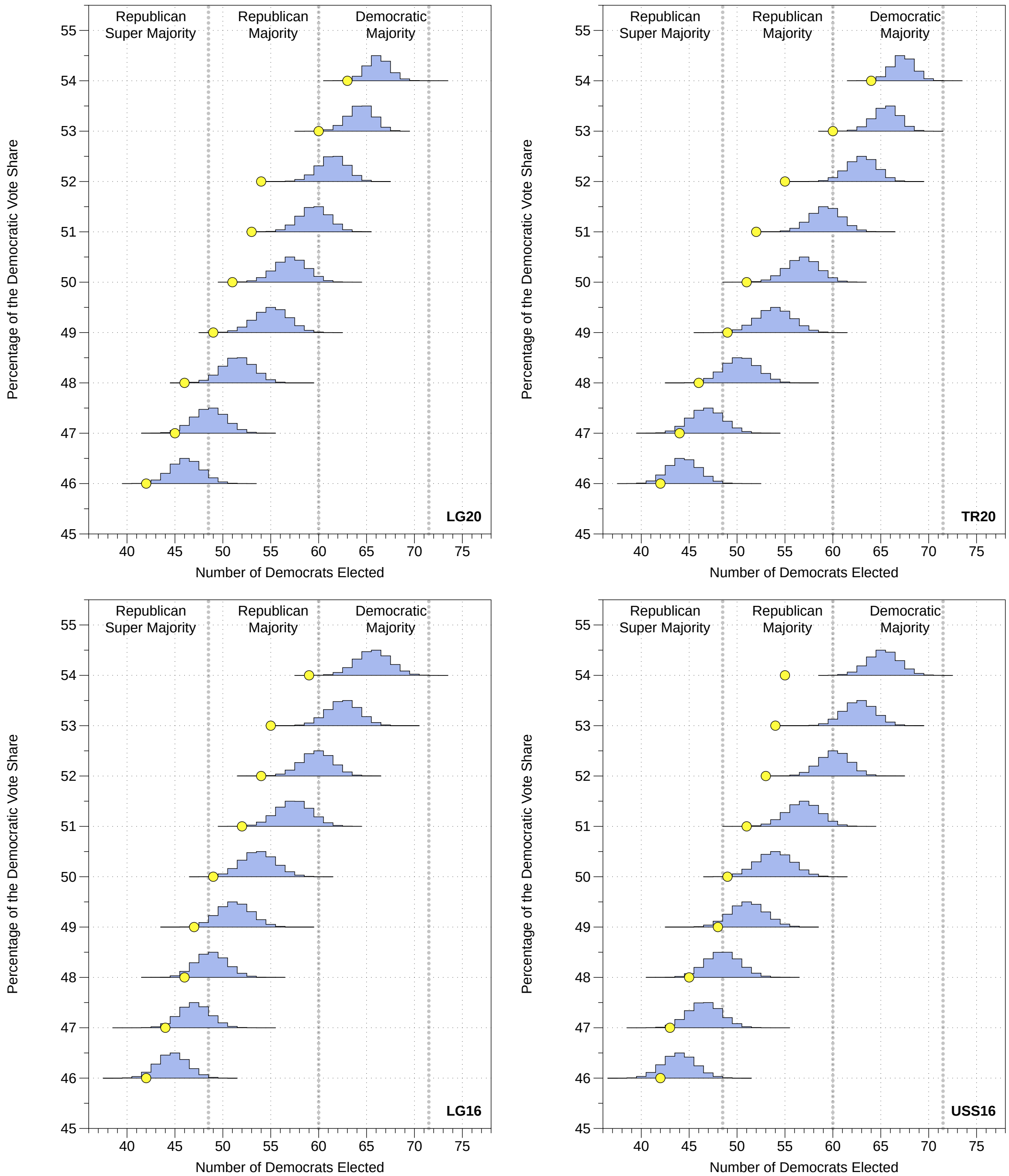


Figure 5.1.3: The individual histograms give the frequency of the Democratic seat count in the ensemble for each of the shown statewide elections, with a uniform swing. The histograms are organized vertically based on the statewide partisan vote fraction. The more Republican swings are placed lower on the plot while more Democratic swings are placed higher. Three dotted lines denote the boundary between where the supermajorities and simple majorities are in force. The yellow dot is the enacted plan.

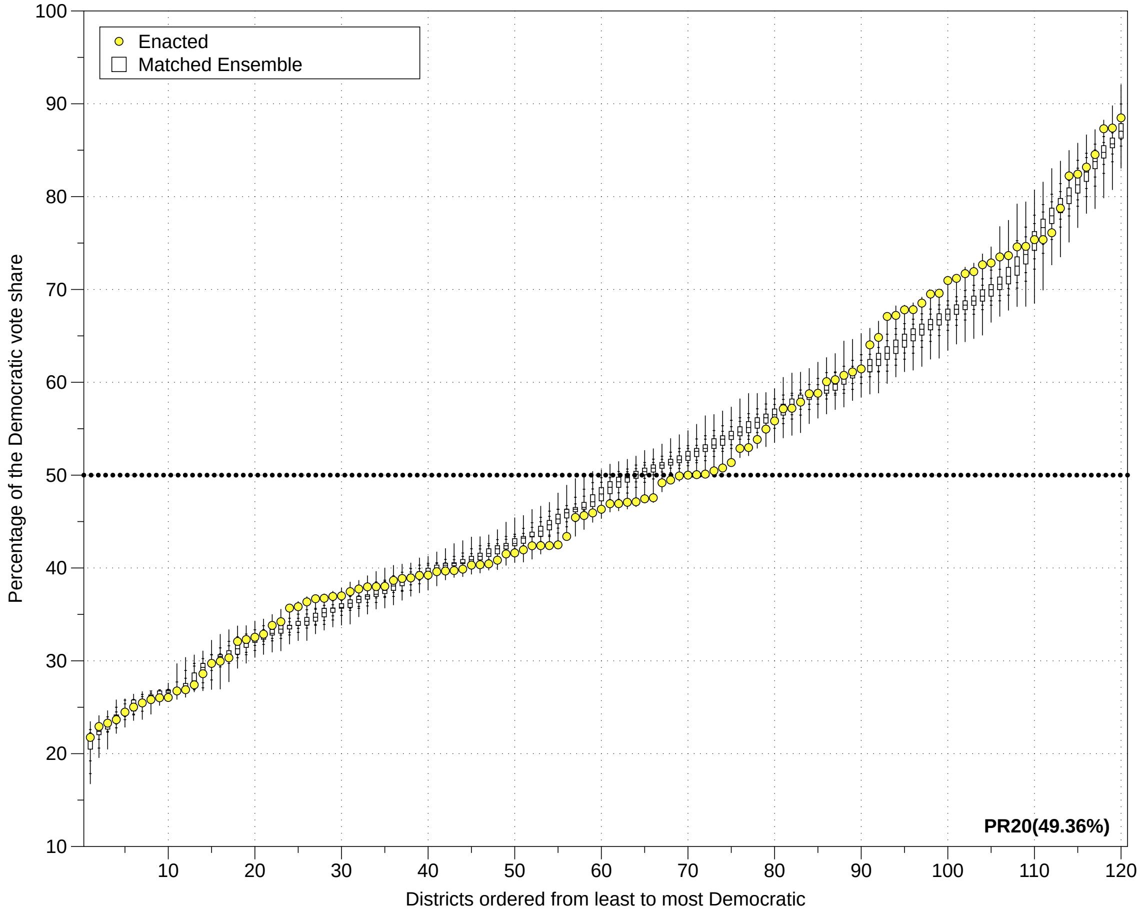


Figure 5.1.4: The yellow dots represent the democratic vote fraction of the enacted map under the PR20 vote count when the district are ordered from most Republican on the left to most Democratic in vote share on the right. The box-plots show the range of the same statistic plotted over the primary ensemble. From around the 60th to 80th district the yellow dots all well below the boxplots of the ensemble. This result is that many dots fall well below the dotted 50% line than usually would; and hence more Republicans are elected than typical. To achieve this effect, the fraction of Democrats is increased in the already strongly democratic districts ranging from the 90th to 105th most Democratic districts. This structure does not exist in the non-partisan ensemble and is responsible for the map's extreme outlier behavior.

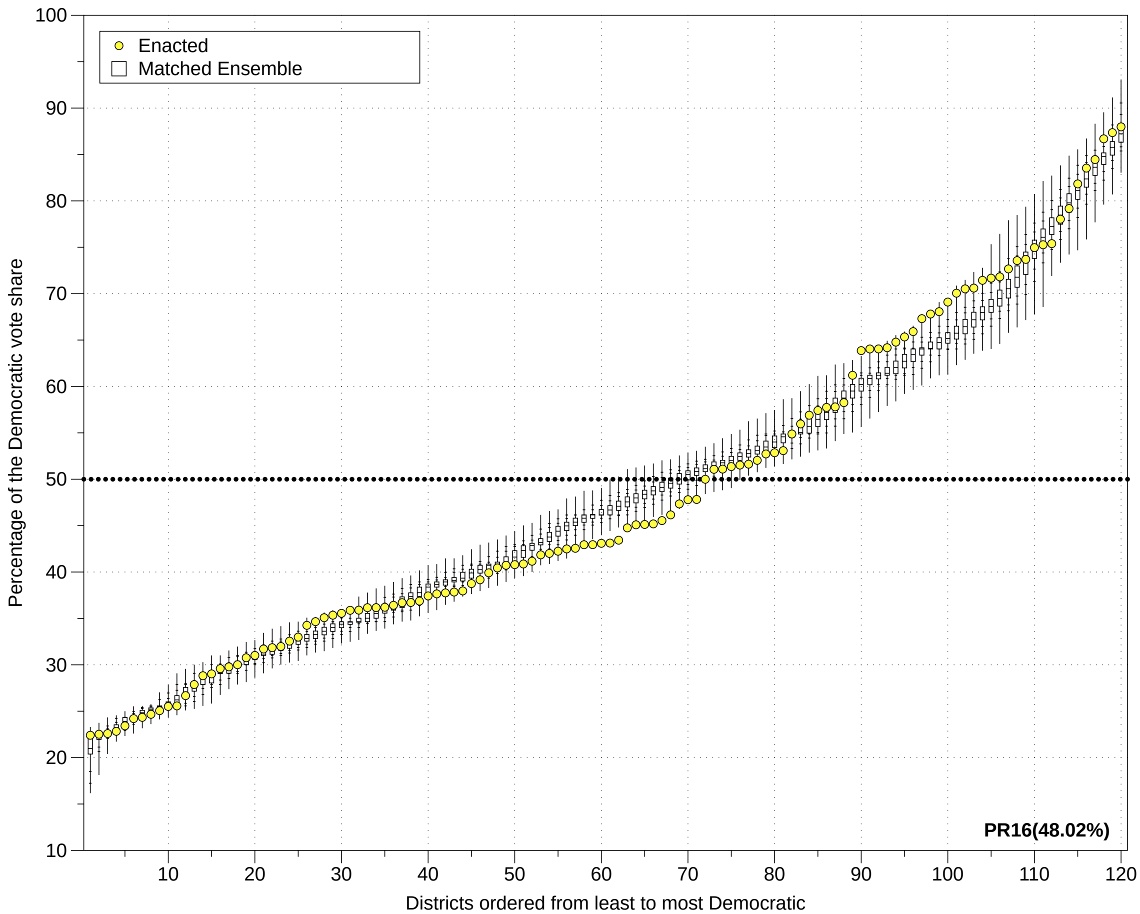


Figure 5.1.5: A similar structure to that seen in Figure 5.1.4 is repeated here. The low 50s to the high 70s have had the number of democrats depleted while the districts from the high80s to around 105 have an excess of Democrats.

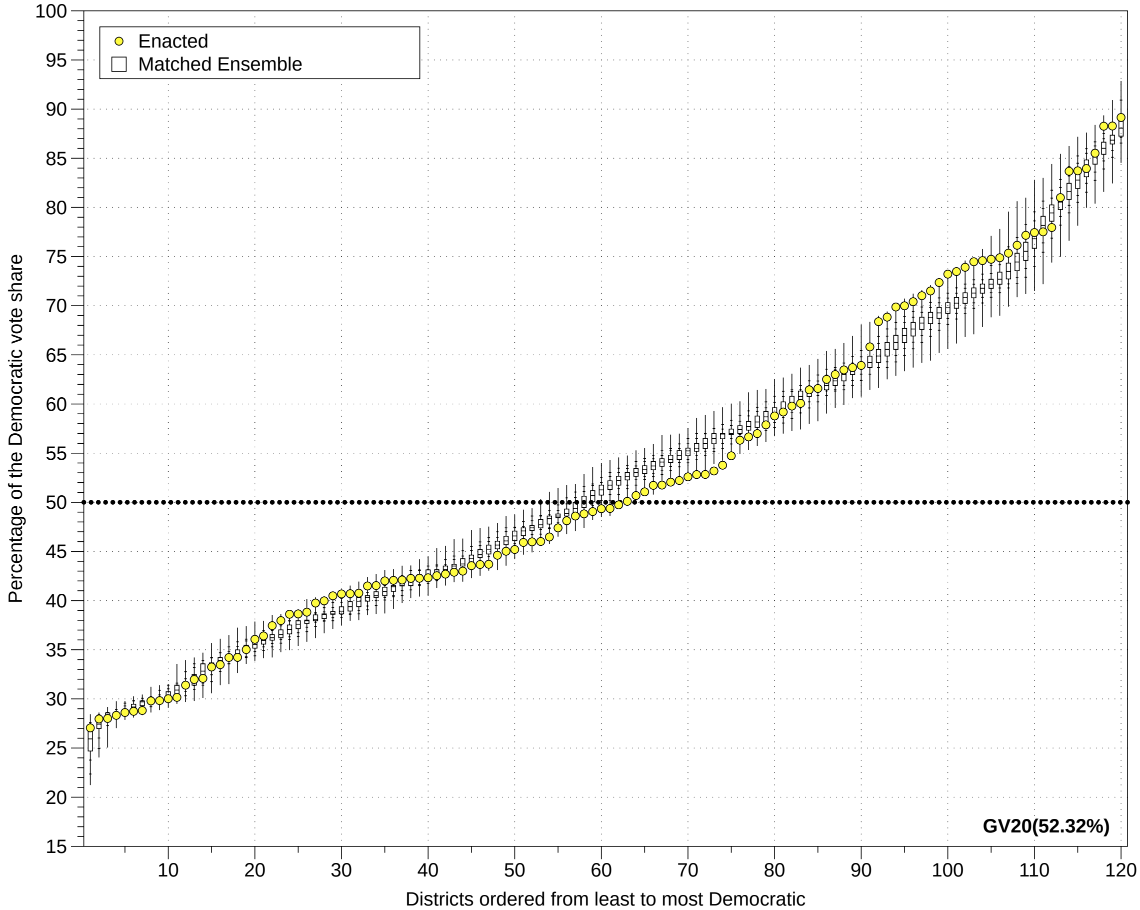


Figure 5.1.6: Mirroring what was seen in Figure 5.1.4 and Figure 5.1.5, we have abnormally few Democrats from around the 60th to the 80th most Republican and abnormally many Democrats packed in the districts in the low 90s to the just below 110.

NC House: Primary Ensemble considering Incumbency.

Figure 5.1.7 shows the Collected Seat Histogram analogous to Figure 5.1.1, but for an ensemble which pairs the same or fewer incumbents than the enacted plan. The other considerations are left unchanged from the Primary ensemble. Comparing the two figures, we see no qualitative change in the behavior of the ensemble. Hence the previous conclusions continue to hold. In particular, a desire to prevent the pairing of incumbents cannot explain the extreme outlier behavior of the enacted plan.

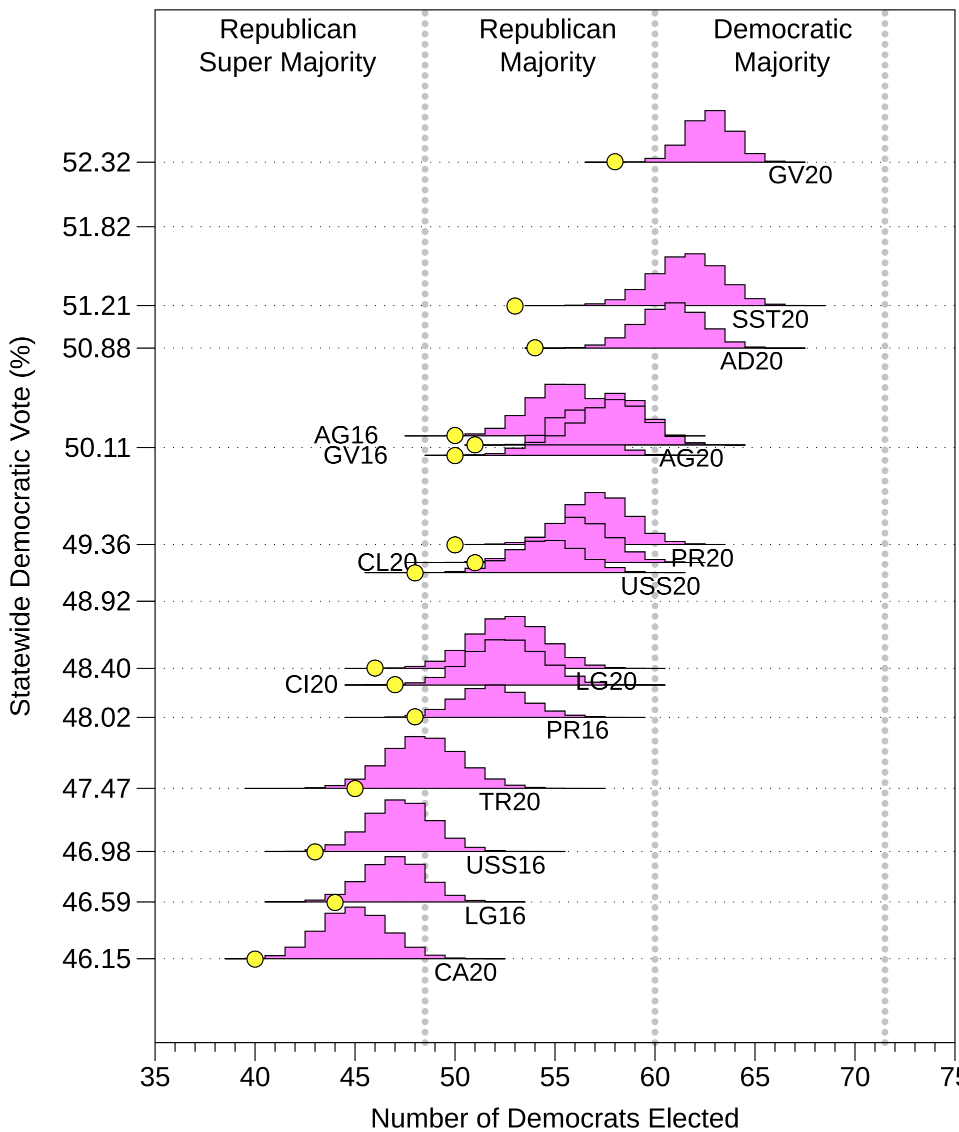


Figure 5.1.7: The Collected Seat Histogram for the Primary Ensemble on the NC House with incumbency considerations added. See Figure 5.1.1 for full description.

NC House: Secondary Distribution

The ensemble used to produce Figure 5.1.8, put more weight on preserving municipalities than either the enacted plan or the Primary Ensemble, which is tuned to match the enacted plan. This enacted plan is still an extreme outlier with respect to this secondary ensemble. We still see that the enacted map resists relinquishing the supermajority under PR16, CI20 and LG20 when this secondary ensemble almost always does. Similarly as the elections become more Democratic in AD20, SST20 and GV20 and the ensemble regularly would give the majority to the Democrats the enacted map dramatically under elects Democrats. In other words, we find that if the mapmakers had made an effort to prioritize preservation of municipalities in the House, that effort would not have led to a map that was more likely to favor Republicans.

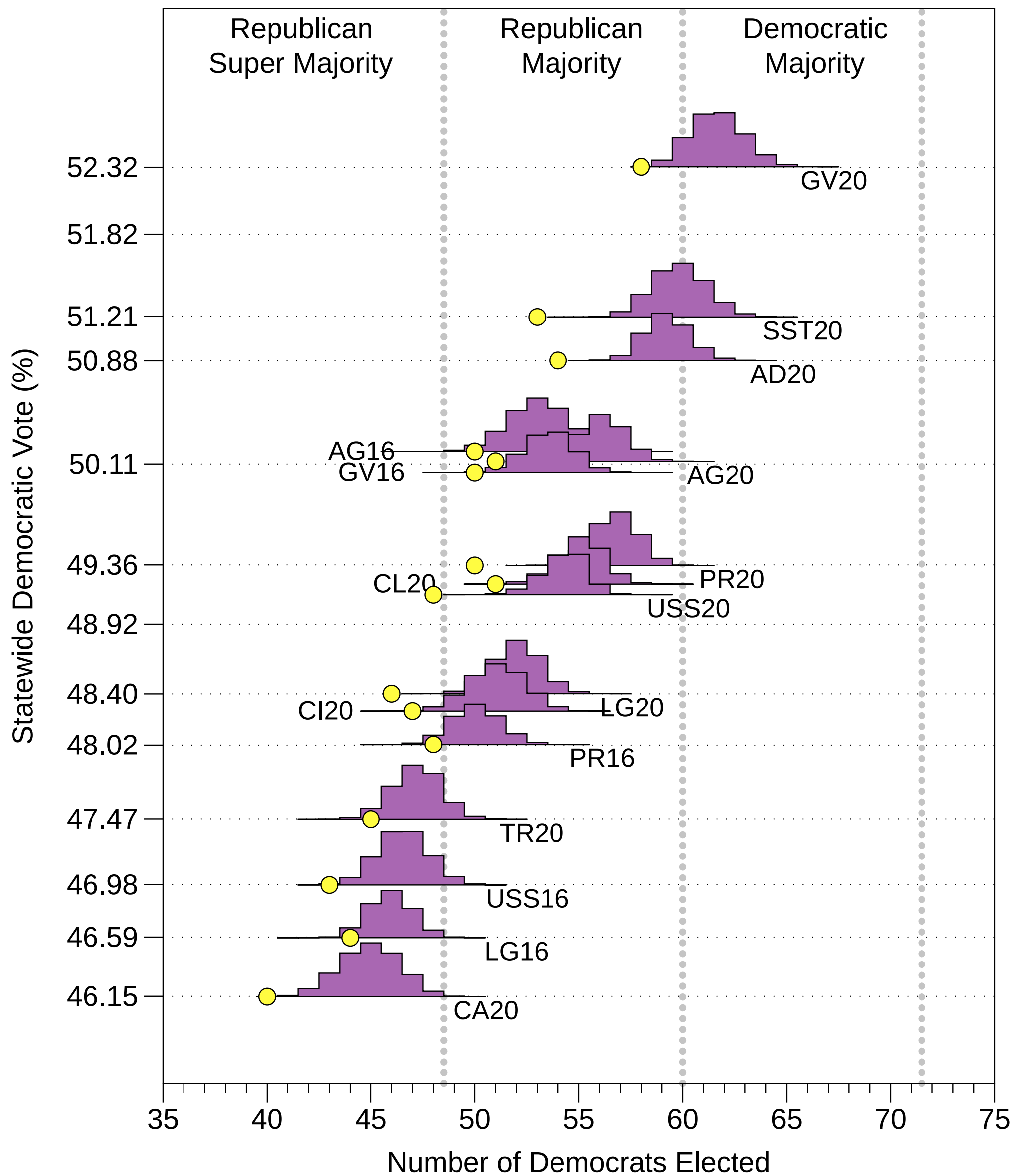


Figure 5.1.8: The Collected Seat Histogram for the Secondary Ensemble on the NC House. The Secondary Ensemble for the NC House is centered on distributions which better preserve municipalities than the enacted plan. See Figure 5.1.1 for full description.

% Dem	Election	% Outlier	# Outlier	# Samples
52.32%	GV20	16.343%	16343	100000
51.21%	SST20	35.184%	35184	100000
50.88%	AD20	42.880%	42880	100000
50.20%	AG16	12.129%	12129	100000
50.13%	AG20	4.332%	4332	100000
50.05%	GV16	0.075%	75	100000
49.36%	PR20	6.220%	6220	100000
49.22%	CL20	5.365%	5365	100000
49.14%	USS20	14.052%	14052	100000
48.40%	LG20	0.000%	0	100000
48.27%	CI20	0.322%	322	100000
47.47%	TR20	5.726%	5726	100000
46.98%	USS16	43.176%	43176	100000
46.59%	LG16	44.943%	44943	100000
46.15%	CA20	1.123%	1123	100000

Table 2: NC Senate Collected Seat Histogram Outlier Data. Starting from the left, the first column gives the statewide partisan makeup of the election under consideration whose abbreviation is given in the second column from the left. The right most column gives the total number of plans in the ensemble considered which is 100,000. The second column from the right gives the number of those 100,000 plans which elect the same or less Democrats under the given election. These are the plans which are as much or more of an outlier than the enacted map. The middle column is the percentage of plans which are more or equal of an outlier. (It is calculated by dividing the 2nd column from the right by 100,000 and multiplying by 100 to make a percentage.) The number of fairly small to extremely small percentage in the middle column between 50.13% (AG20) and 47.47% (TR20) are another signature of the anomalous behavior seen visually in Figure 5.2.1 over the same range of vote percentages.

5.2 NC State Senate

We will see in our cluster-by-cluster analysis that the NC Senate map has a number of clusters that are outliers. Their structures are systematically in favor of the Republican party. As discussed in Section 3.2, we often see maps that express their outlier status under a specific voting climate; often when one party is in danger of losing the majority or super-majority. The enacted map for the NC Senate shows this behavior.

Figure 5.2.1 is the plot for the NC Senate analogous to Figure 5.1.1, which was for the NC House. Most of the outlier behavior at the state level for the enacted NC Senate map is concentrated in the interval between 47.5% statewide Democratic vote share and around 50.5% statewide Democratic vote share. In this range, the enacted map is always an outlier and often an extreme outlier under the votes considered. This range is significant for a number of reasons. First, this is a range of statewide vote fraction where many North Carolina elections occur. Secondly, looking at Figure 5.2.1 we see that over this range the ensemble shows that one should expect the Republican super-majority (less than 21 Democratic Seats) to switch to a simple Republican majority (between 21 and 24 Democratic Seats). Yet the enacted map often resists this switch, breaking the supermajority only when the PR20 and CL20 votes are considered. In both of these elections, the ensemble places the typical number of Democratic seats well away from the supermajority line and centered between it and the simple majority line.

To make Figure 5.2.1 more quantitative, we have included Table 2 which shows the number of maps where the primary ensemble elects less democrates in that election than the enacted map.

Looking at Table 2 we see that a number of the elections in the critical partisan range of around 47.5% to 50% are extreme outliers (GV16, LG20, and CI20) while other (AG20, PR20, and TR20) show atypical behavior all favoring the Republican candidates. It is again important to notice that the enacted plan is never seen to favor the Democratic party relative to what is expected from the Primary non-partisan ensemble. The enacted map ranges between tilted to the Republican party to being an extreme partisan outlier. The importance of the range of statewide Democratic between 47.5% to 50% by looking at Figure 5.2.1. The primary ensemble shows that is within this range that one expects a Republican supermajority to become a simple majority. The effect of the enacted plan is to suppress this by under electing Democrats.

We will in the cluster-by-cluster analysis in Section 6.2 that a number of individual clusters are extreme outliers in their partisan structure.

To better control for other variation we now include a number of Collected Seat Histograms built from a single election which has been shifted to create a sequence of elections with different statewide partisan vote fractions but the same spatial voting pattern.

The large jump that we see in Figures 5.2.3 to 5.2.5 between the 33nd most Republican district and the 35th most

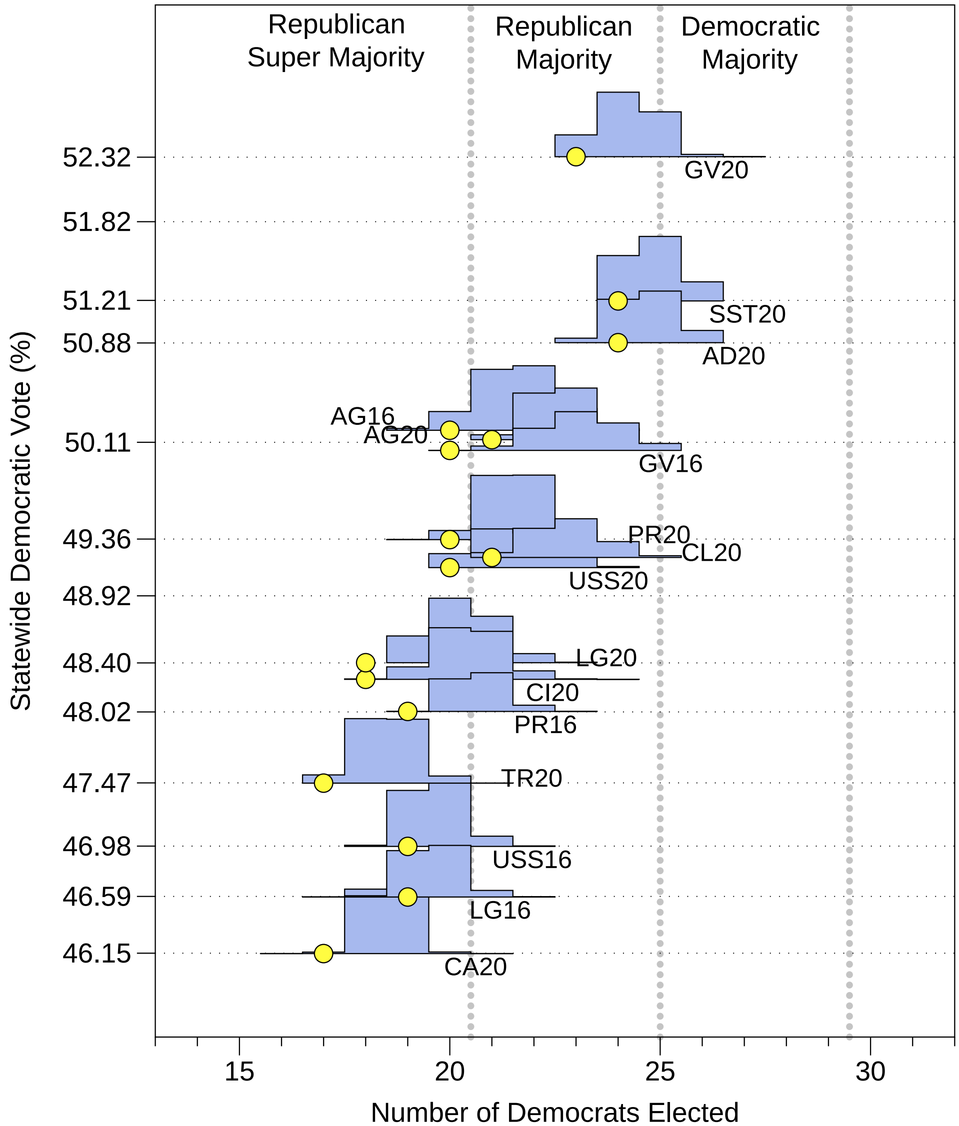


Figure 5.2.1: The Collected Seat Histogram for the Primary Ensemble on the NC Senate. The individual histograms give the frequency of the Democratic seat count for each of the statewide elections considered from the years 2016 and 2020. The histograms are organized vertically based on the statewide partisan vote fraction for each election. The more Republican elections are placed lower on the plot while more Democratic elections are placed higher. Three dotted lines denote the boundary between where the supermajorities and simple majorities are in force.

Republican district means that over a large range of swings in the partisan character of the election the outcome will change at most by one seat.

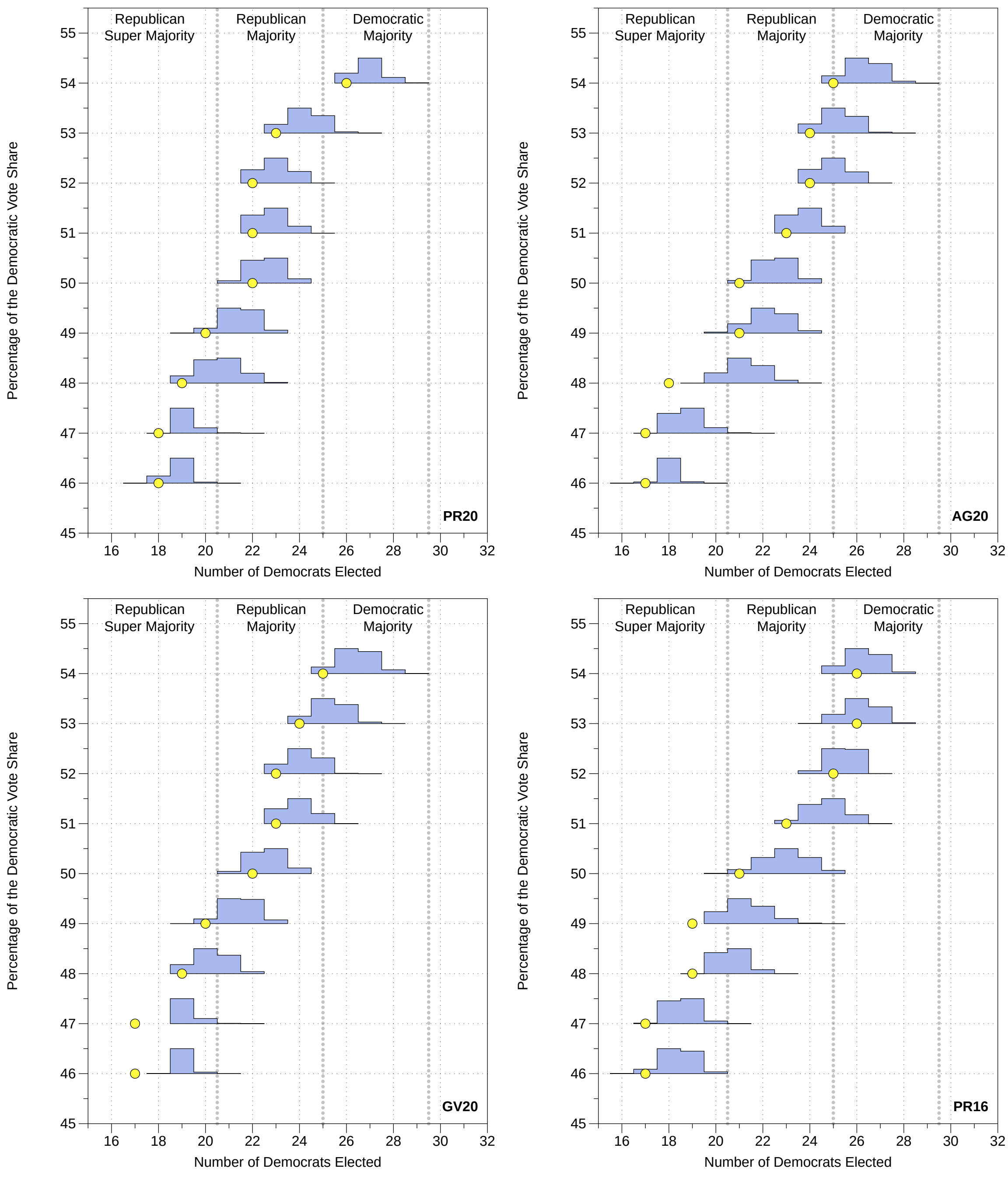


Figure 5.2.2: The Collected Seat Histograms for the Primary Ensemble on the NC House built from a collection of voting data generated via uniform swing.

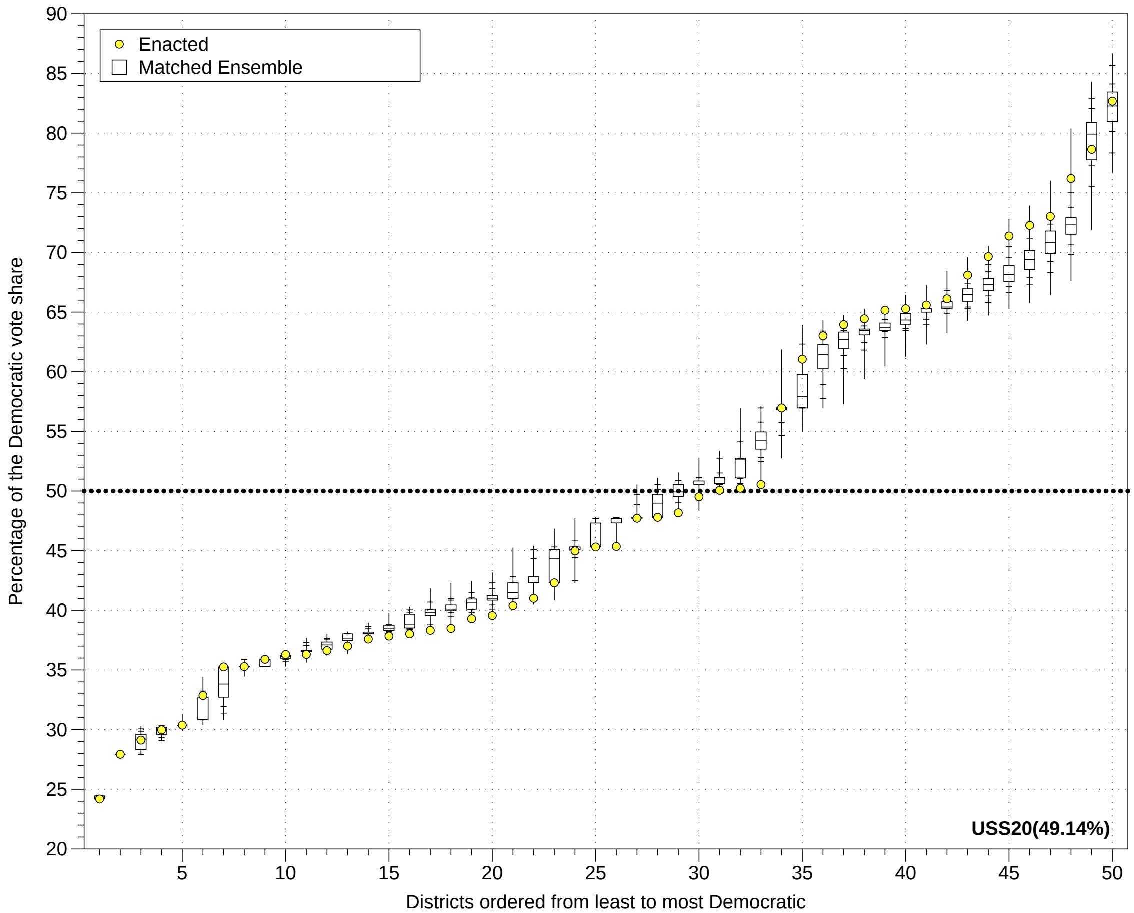


Figure 5.2.3: The yellow dots represent the democratic vote fraction of the enacted map under the USS20 vote count when the district are ordered from most Republican on the left to most Democratic in vote share on the right. The box-plots show the range of the same statistic plotted over the primary ensemble. Essentially all of the districts between the 15th most Republican and the 33rd most Republican have abnormally few Democrats. This is compensated by packing abnormally many Democrats the 35th to the 47th most Republican districts. This structure is an extreme outlier and does not occur in the ensemble.

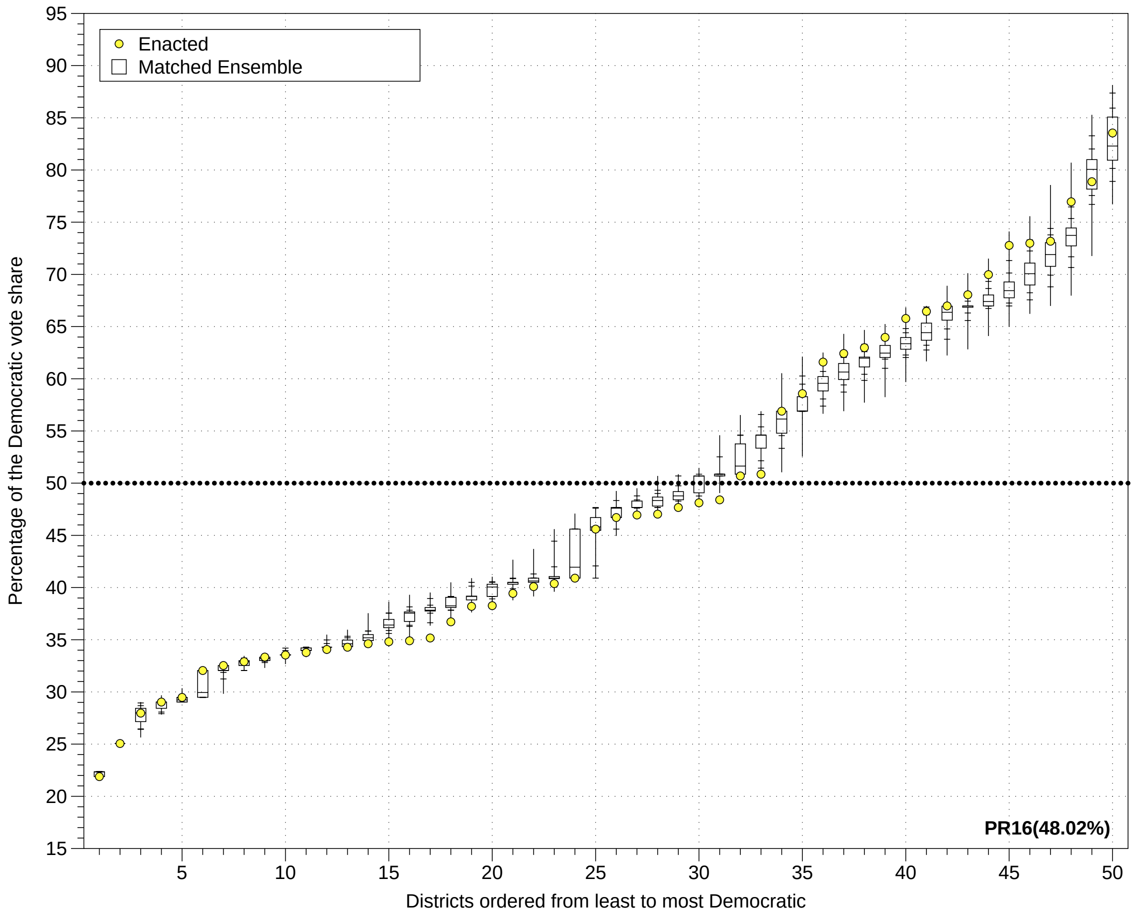


Figure 5.2.4: A similar structure to that seen in Figure 5.2.3 is repeated here over a nearly identical range of districts.

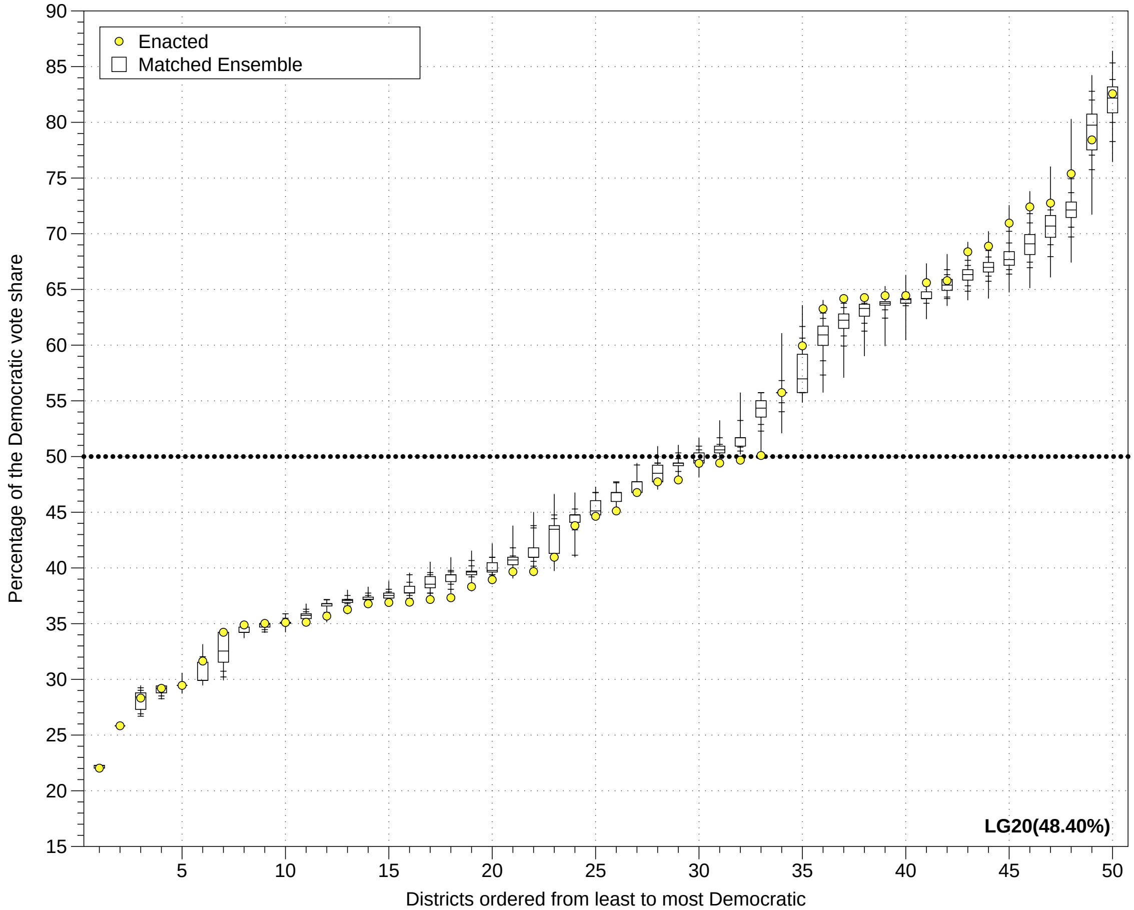


Figure 5.2.5: A similar structure to that seen in Figure 5.2.3 is repeated here.

NC Senate: Primary Ensemble considering Incumbency.

Preserving incumbency has little qualitative effect on the observations we have made. Looking at 5.2.6, we see that the election between and including GV16 and TR20 in the Figure 5.2.6 are all extreme outliers. This is in fact more extreme that the enacted map was under the Primary ensemble. It reinforces that this gerrymander seems to be most efective at the statewide level when the Republican supermajority is possible but in question.

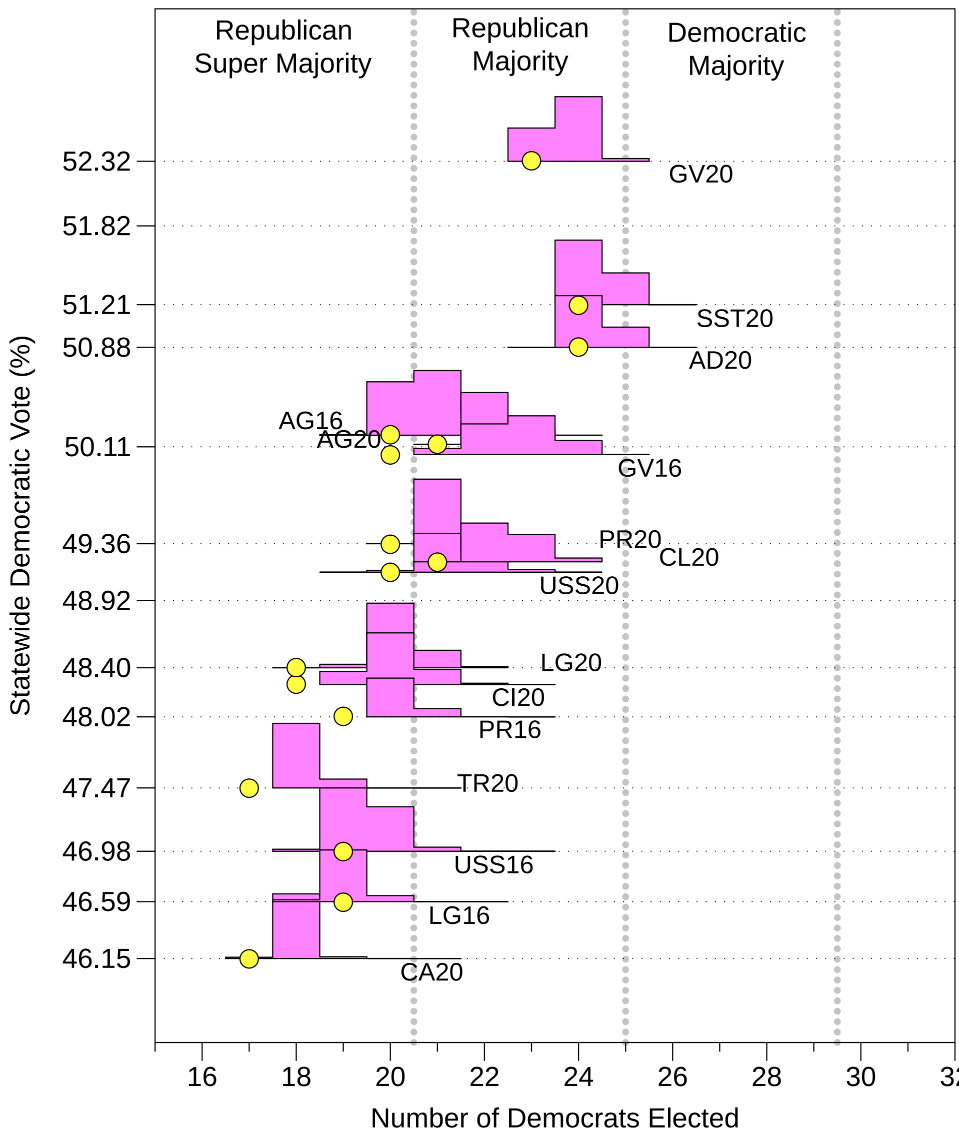


Figure 5.2.6: The Collected Seat Histogram for the Primary Ensemble on the NC Senate with incumbency considerations added. See Figure 5.1.1 for full description.

NC Senate: Secondary Distribution

When municipal preservation is not prioritized, the enacted plan becomes an outlier in all but the two most Republican elections as shown in Figure 5.2.7. Additionally, in most cases it was an extreme outlier when municipal preservation is not considered.

In other words, when municipal preservation is not prioritized, the ensemble produced is more favorable to the Democrats, meaning that the enacted plan appears as an extreme outlier in more situations than in the ensemble that matched the enacted map in prioritizing municipality. Put differently, the decision to prioritize municipality preservation in the Senate plan appears to have enabled more maps that favor Republicans.

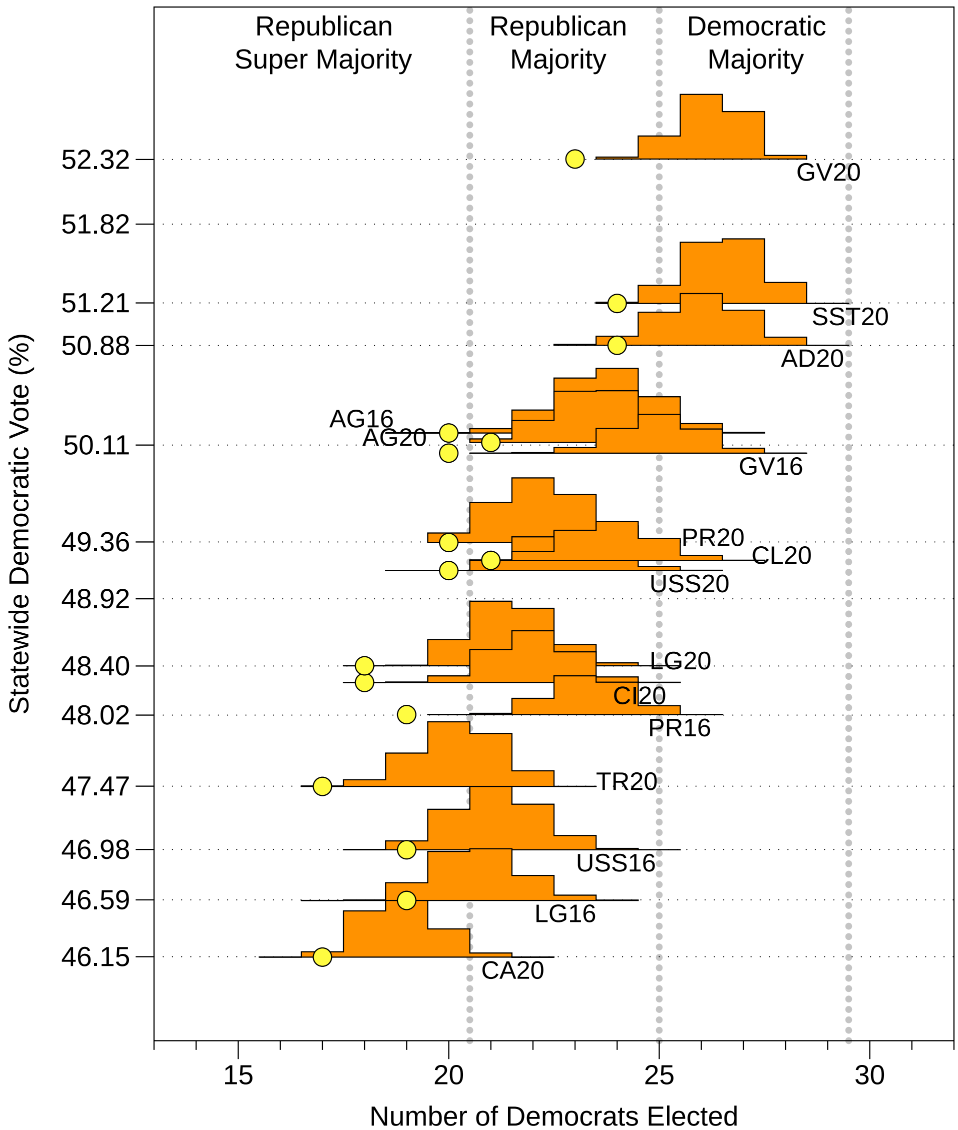


Figure 5.2.7: The Collected Seat Histogram for the Secondary Ensemble on the NC Senate. The Secondary Ensemble for the NC Senate is centered on distributions which do not explicitly consider municipality preservation. See Figure 5.1.1 for full description.

6 State Legislature: Selected Cluster by Cluster Analysis

Using the same tools, we now turn our analysis to the individual cluster. We find that a number of cluster demonstrate significate cracking and packing. In some cases this leads to changes in the partisan make of the representative typically elected from the region. In other cases, it makes the districts insensitive to changes in the voters political outlook as expressed in their votes.

6.1 NC State House

6.1.1 Mecklenburg

The ranked ordered histogram for the Mecklenburg cluster using the primary ensemble (which matches the number of people displaced from municipalities) is given in Figure 6.1.1. Across all of the voting patterns considered, we see that the two most Republican Districts (districts 98 and 103) have exceptionally few Democrats. This has the effect of making them more likely to elect a Republican when many (and often almost all) ensemble plans elect a Democrat in those districts. Specifically, that is the case under LG20, AG20, USS20, CL20, AD20 and SST20. Under GV20 and PR20, the two most Republican districts barely elect Democrats even though the majority of the ensemble plans safely elect Democrats. Under CA20 and TR20, the enacted plan safely elects two Republicans while under the ensemble the races are much closer, swinging in both directions under different plans. In these two elections, the enacted map elects a third Republican (in District 104) when the ensemble of maps typically would not. All of this is achieved by packing exceptionally many Democrats into the 6th through 9th most Democrat district, as shown in Figure 6.1.1 where the enacted plan is consistently at the extreme top of the range seen in the ensemble. All of these facts make the plan an extreme outlier in this cluster.

In fact, ranging over all of the elections considered, the Democratic fraction in the four most republican districts in the ensemble is greater than that in the enacted plan in less than 1.7% of the plans with it dipping as low as around 0.5% in a few cases. More dramatically, the percentage of plans in the ensemble where the fraction of Democrats, in the four most Democratic districts, is always less than 0.11% with it often dipping as low as 0.02% or lower.

As already discussed, it was possible to oust many less people from municipalities than the enacted plan does. Figure 6.1.2 shows the secondary ensemble which constrains municipalities much more strongly. We seen that structures highlighted above persist in this ensemble; again making the enacted map an extreme outlier.

Municipal Splits and Ousted Population: In Figure 6.1.3, we see that the enacted plan ousts people from municipalities at a number that is comparable to the primary ensemble but typically more than the Secondary House ensemble.

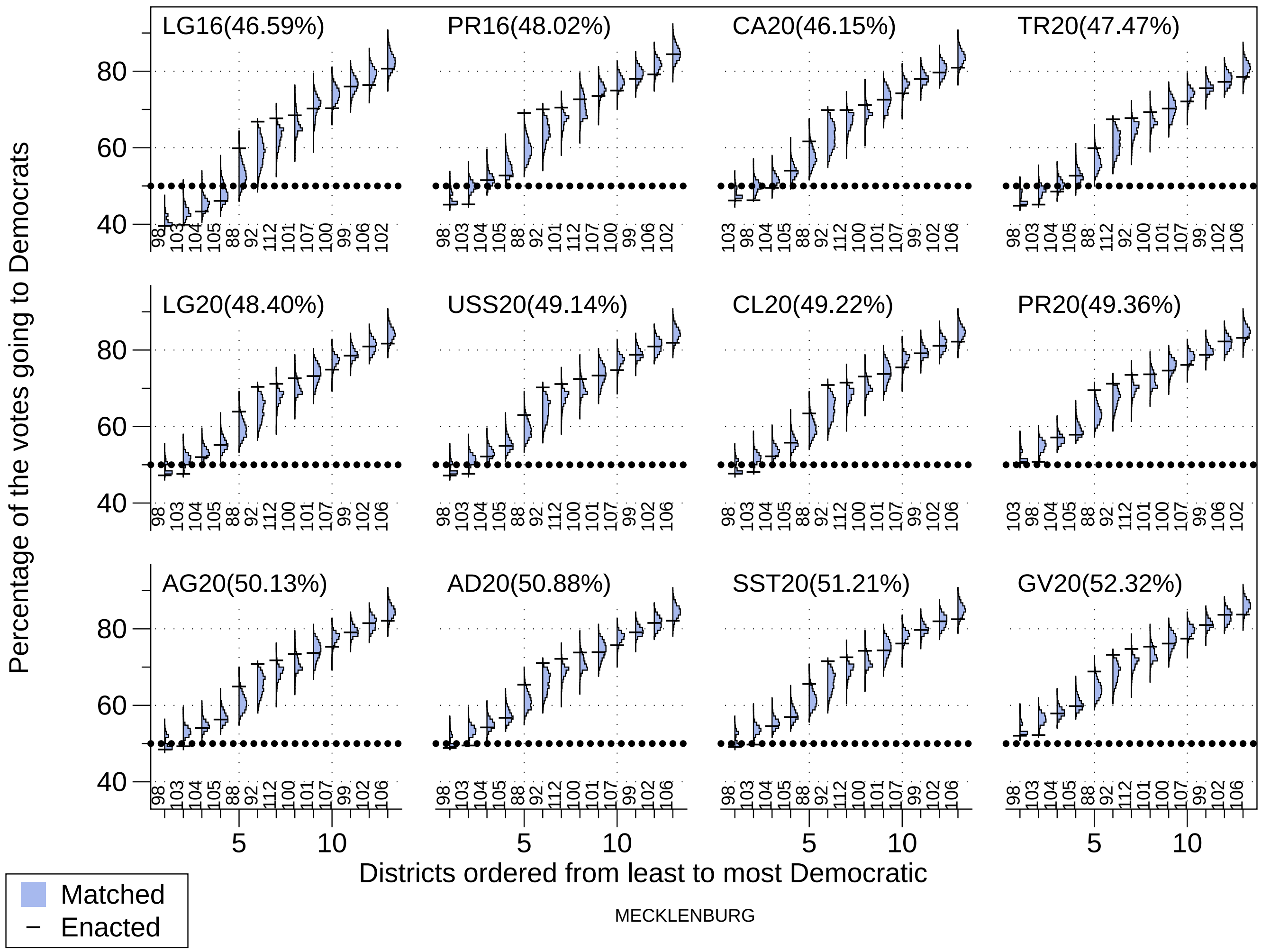


Figure 6.1.1: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

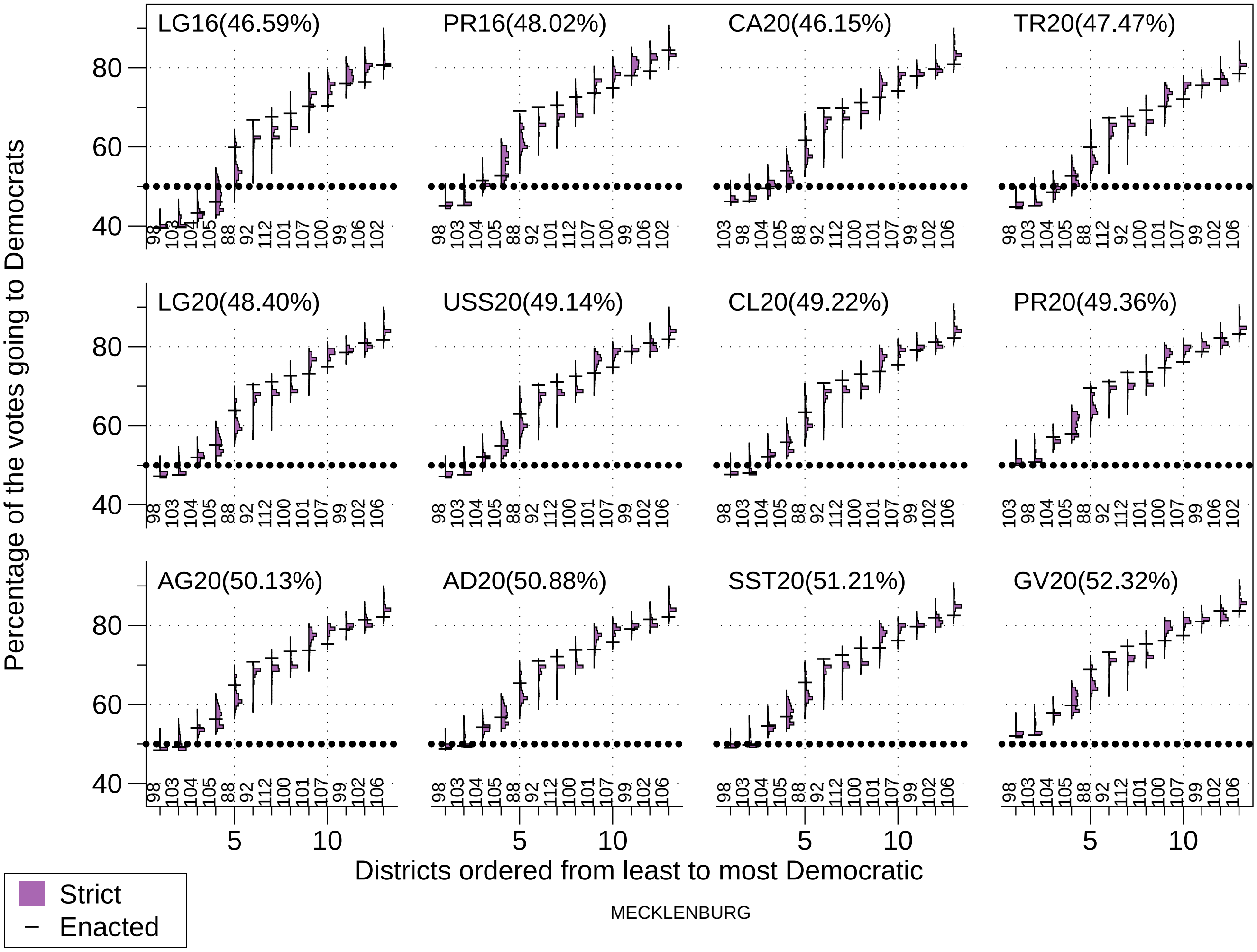


Figure 6.1.2: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

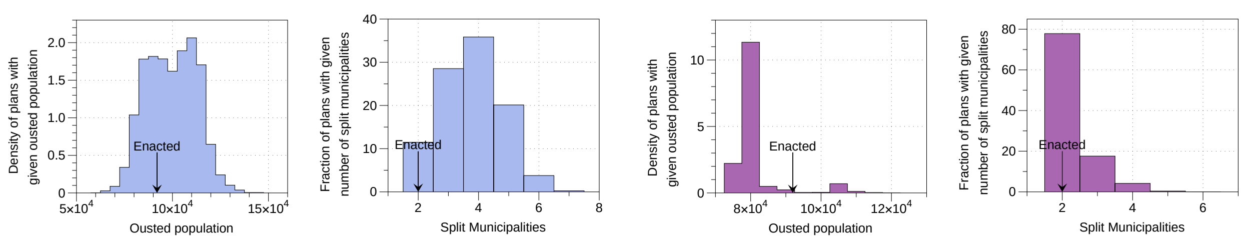


Figure 6.1.3: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.2 Wake

In the Wake cluster, we again see the depleting of Democrats from the two most Republican districts (Districts 37 and 35) while packing Democrats into the next several districts, as in the Mecklenburg cluster. The effect is to swing the two most Republican districts into play in elections where they would not be under the ensemble. Furthermore, the enacted plan makes them safer for Republicans in situations when the ensemble maps would typically have it as a toss-up.

Across all of the elections considered, the number of maps in the ensemble which have a lower Democratic vote fraction in the two most Republican districts than in the enacted plan is less than 0.42% except for the CA20 election where it is 1.2%.

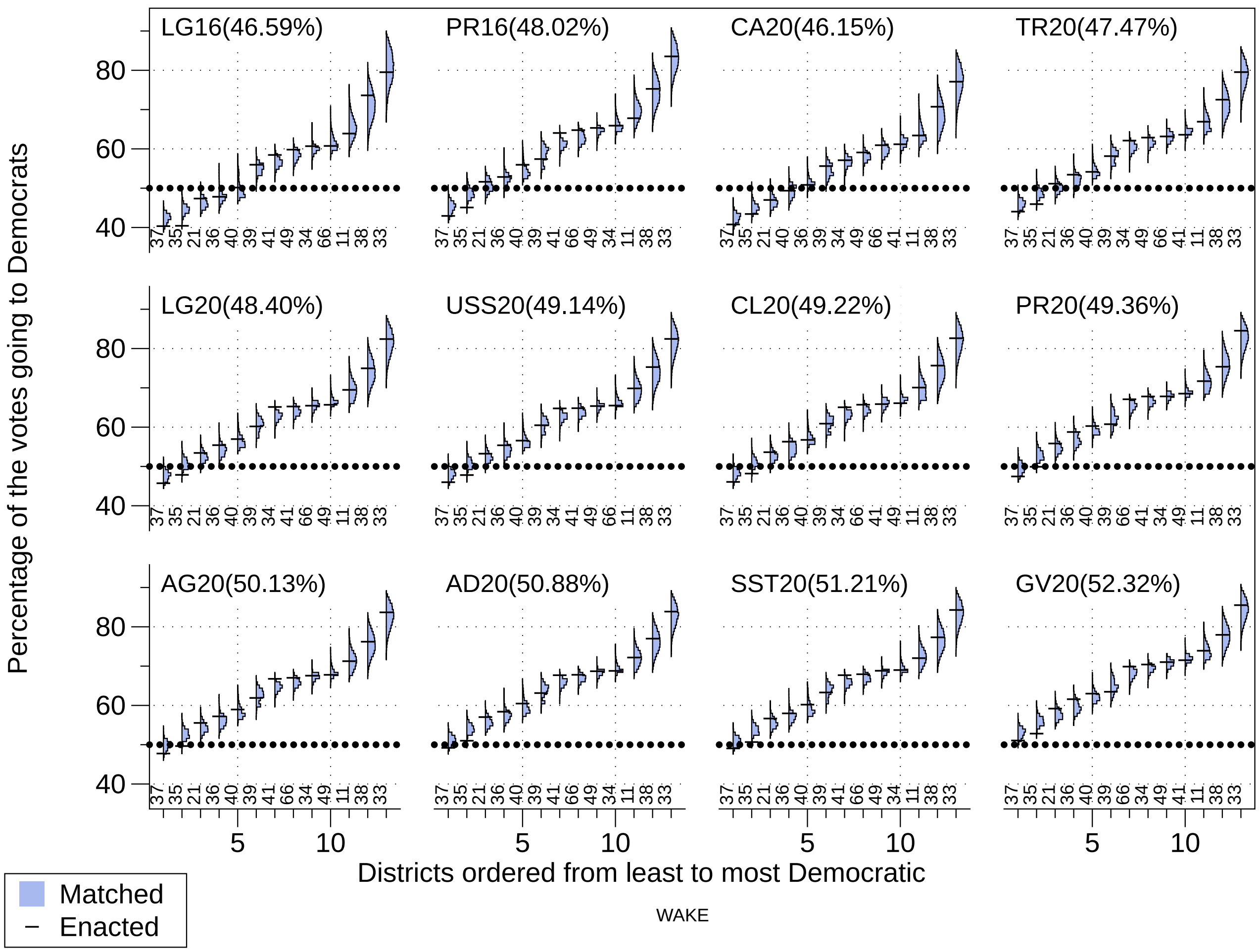


Figure 6.1.4: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

As shown in Figure 6.1.5, the trend continues under the secondary ensemble which better preserves municipalities.

Municipal Splits and Ousted Population:

In Wake we see from Figure 6.1.6 that the enacted plan consistently ousts more people than the primary ensemble and significantly more than the secondary ensemble.

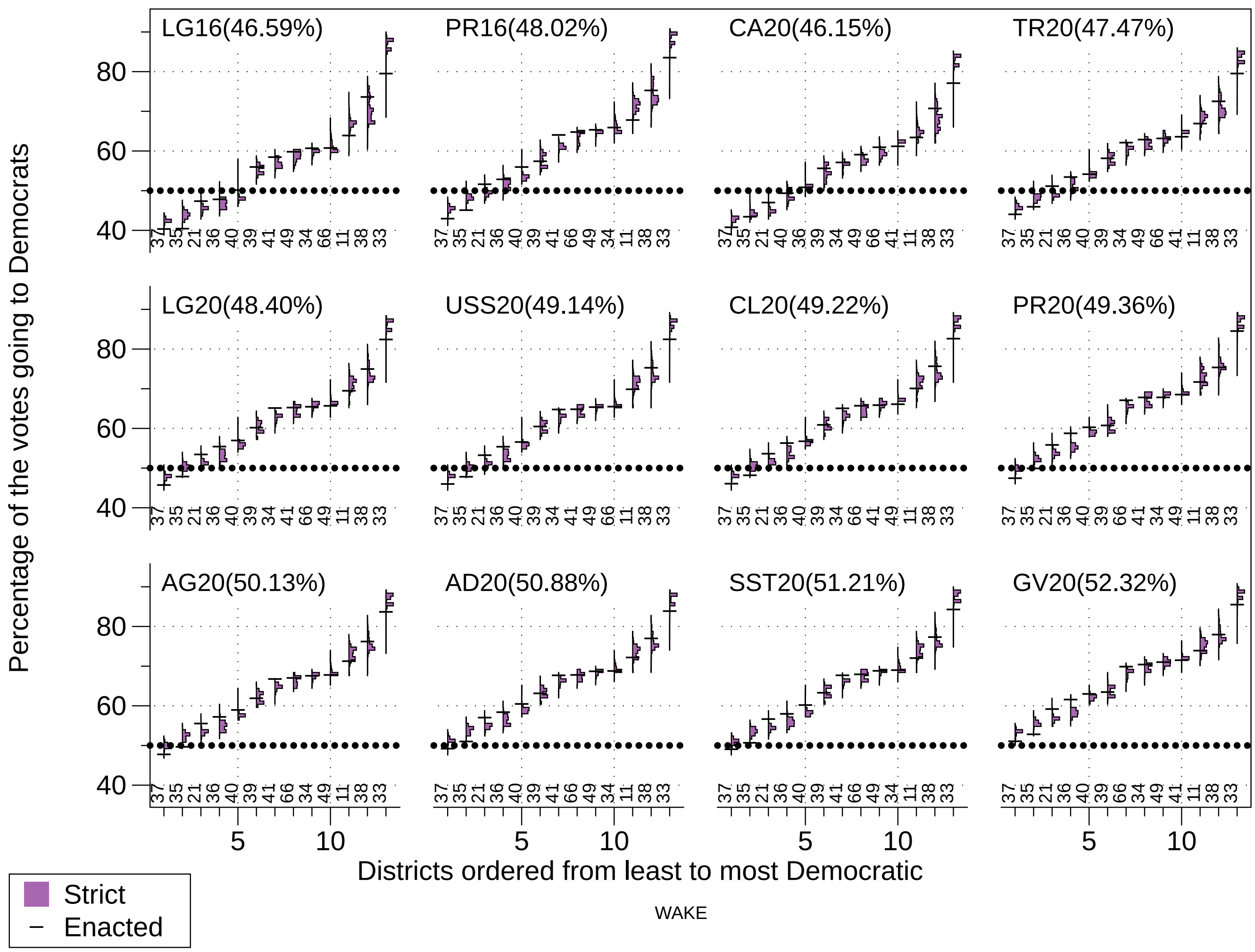


Figure 6.1.5: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

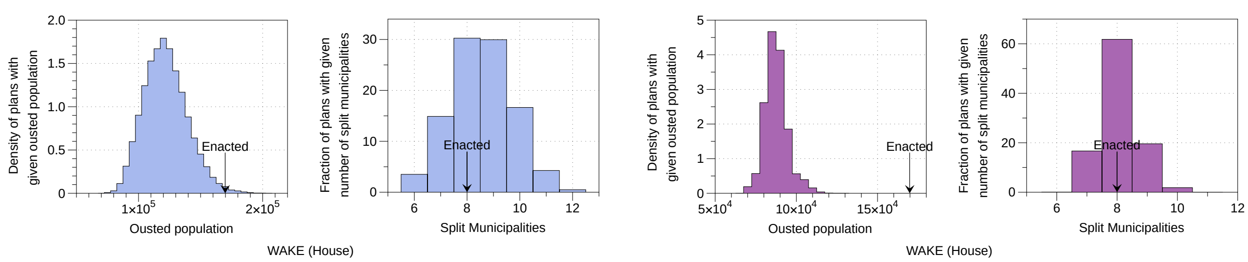


Figure 6.1.6: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.3 Forsyth-Stokes

Again in Figure 6.1.7, showing the primary ensemble in the Forsyth-Stokes cluster, we see the most Republican districts depleted of Democrats while excess Democrats are packed in safe democratic districts and in the safest Republican district are moved to competitive districts. The effect is apparent in all of the elections, but varies slightly across different voting patterns. In all cases, we see the Democratic makeup of the 3rd most Republican district pulled below the range typically seen in the ensemble often resulting in this district electing a Republican when it would not typically. In the three elections where the 3rd-most Republican district still elects a Democrat (GV20), the map's depletion of Democrats from the second most Republican district is enough to reliably elect a Republican in that district when typically the election would vary between being close and strongly favoring the Democrats.

Ranging over all of the elections considered, less than 0.02% of the plans in the ensemble have a lower Democratic fraction in the three most Republican districts than the enacted plan signaling extreme cracking. Additionally, less than 1.3% of the plans in the ensemble have a larger Democratic in the two most Democratic districts than the enacted plan.

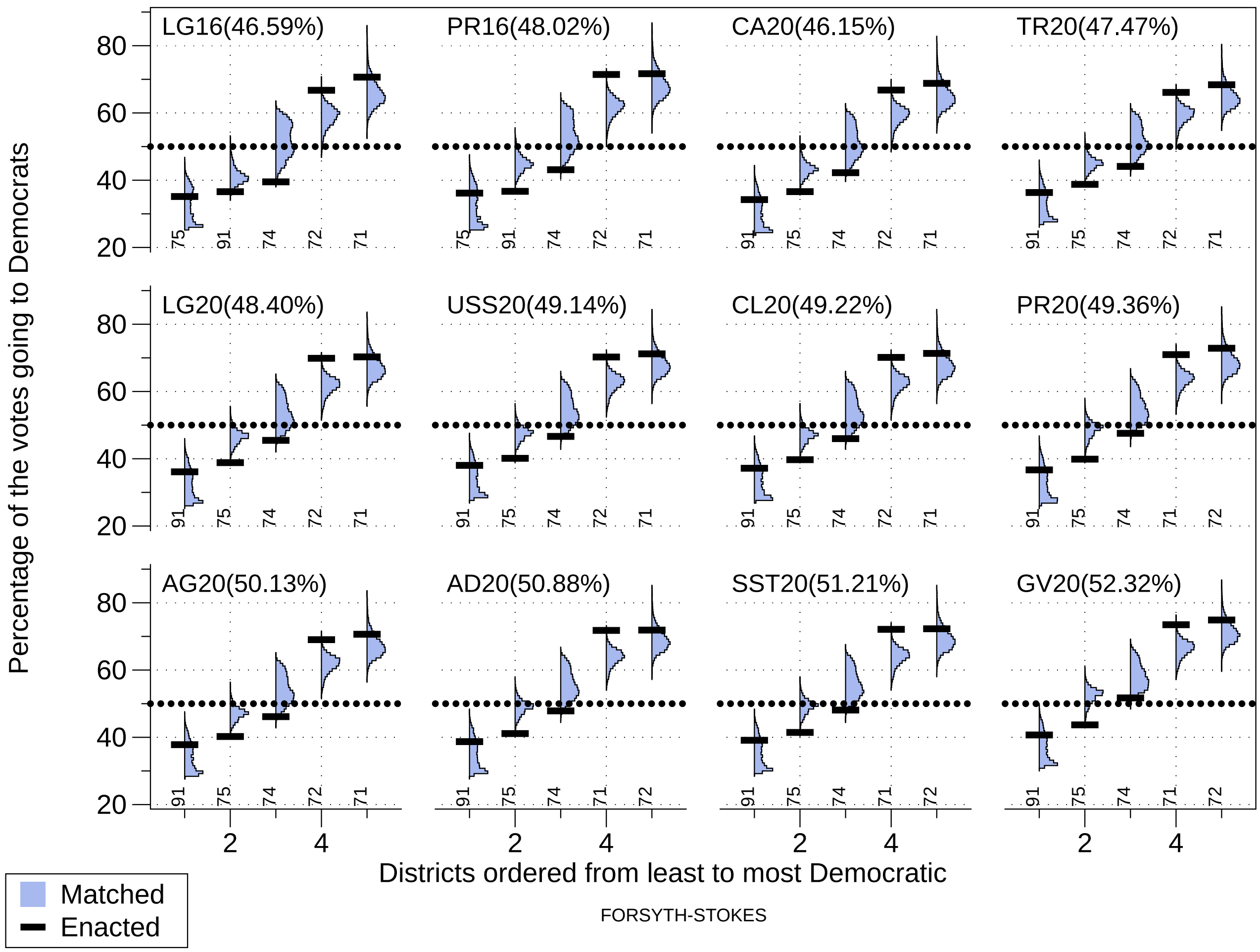


Figure 6.1.7: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

As shown in Figure 6.1.8, the trend continues under the secondary ensemble which better preserves municipalities. Some of the effects are more extreme and in this cluster, this ensemble leads to more partisan districts. Nonetheless, the enacted map still regularly elects a Republican in the third most Republican district even thought it is typically more firmly Democratic under this ensemble.

Municipal Splits and Ousted Population:

From Figure 6.1.9, we see that in Forsyth-Stokes the enacted plan ousts a number of people comparable to the primary ensemble but consistently more than the secondary ensemble.

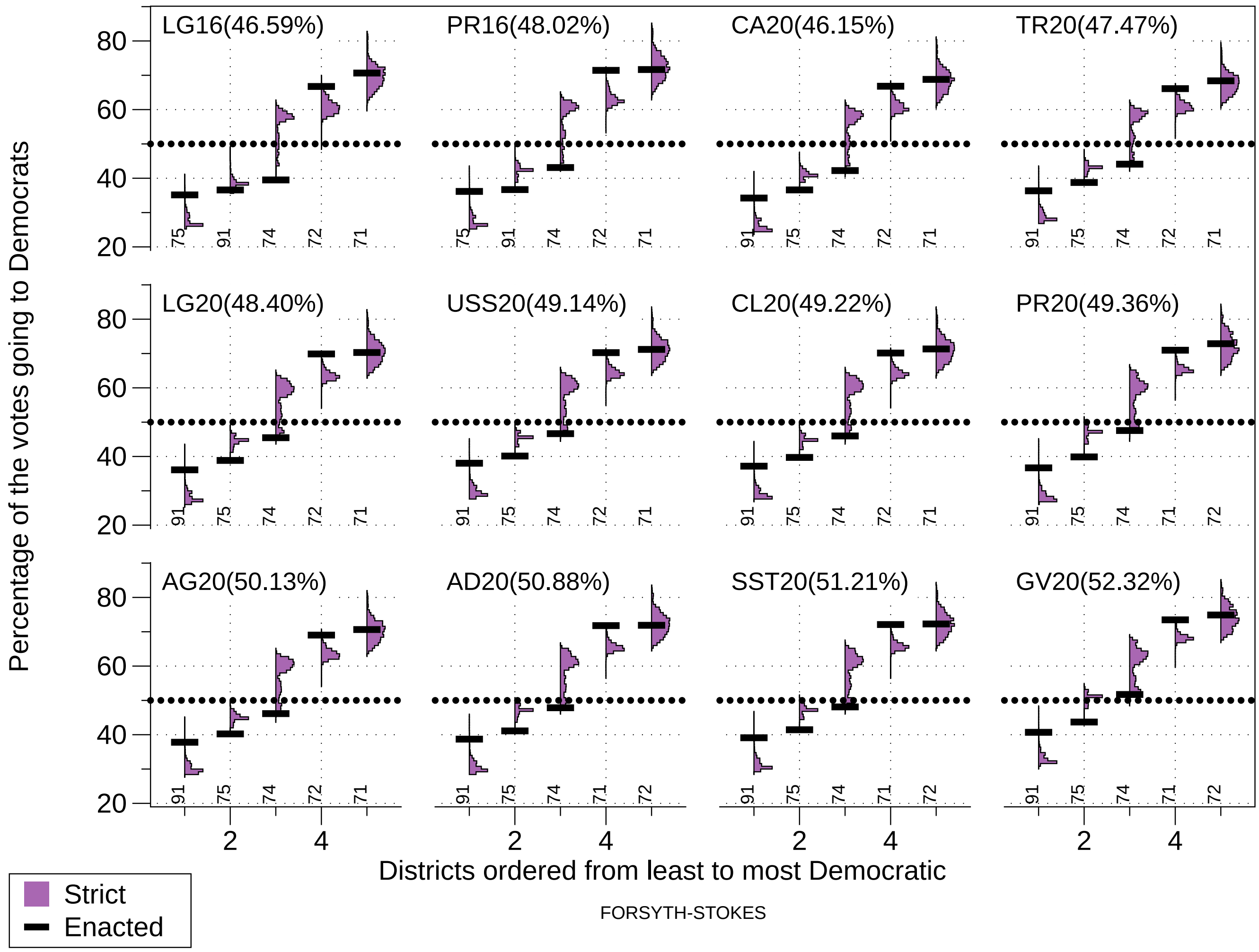


Figure 6.1.8: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

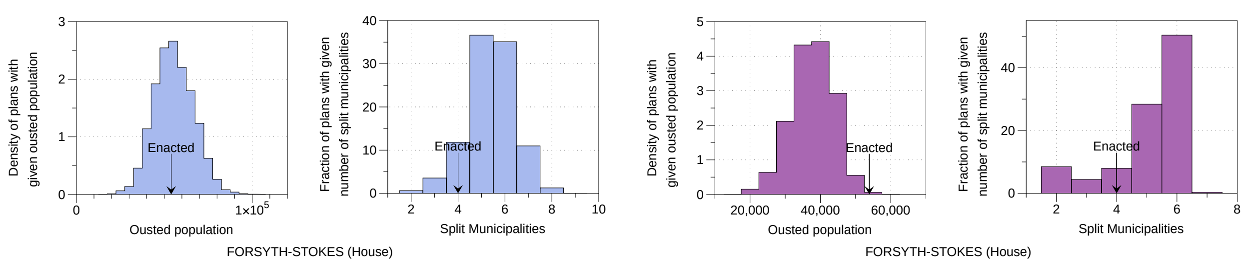


Figure 6.1.9: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.4 Guilford

The pattern seen previously is again repeated in an extreme fashion in the Guilford County. The two most Republican Districts (districts 59 and 62) have abnormally few Democrats when compared to what is seen in the primary ensemble and the more Democratic districts (numbered 57, 58, 60, and 61) have exceptionally many Democrats packed into them. The effect is that the enacted plan regularly (and often safely) elects two Republicans under election climates which would rarely or never do so.

Over all of the elections considered and all of the around 80,000 plans in the ensemble, none of the plans have a higher Democratic fraction in the four most Democratic districts or a lower Democratic fraction in the two most Republican districts, in comparison to the enacted plan. . In other words, this cluster shows more cracking and packing of Democrats than every single plan in the nonpartisan ensemble.

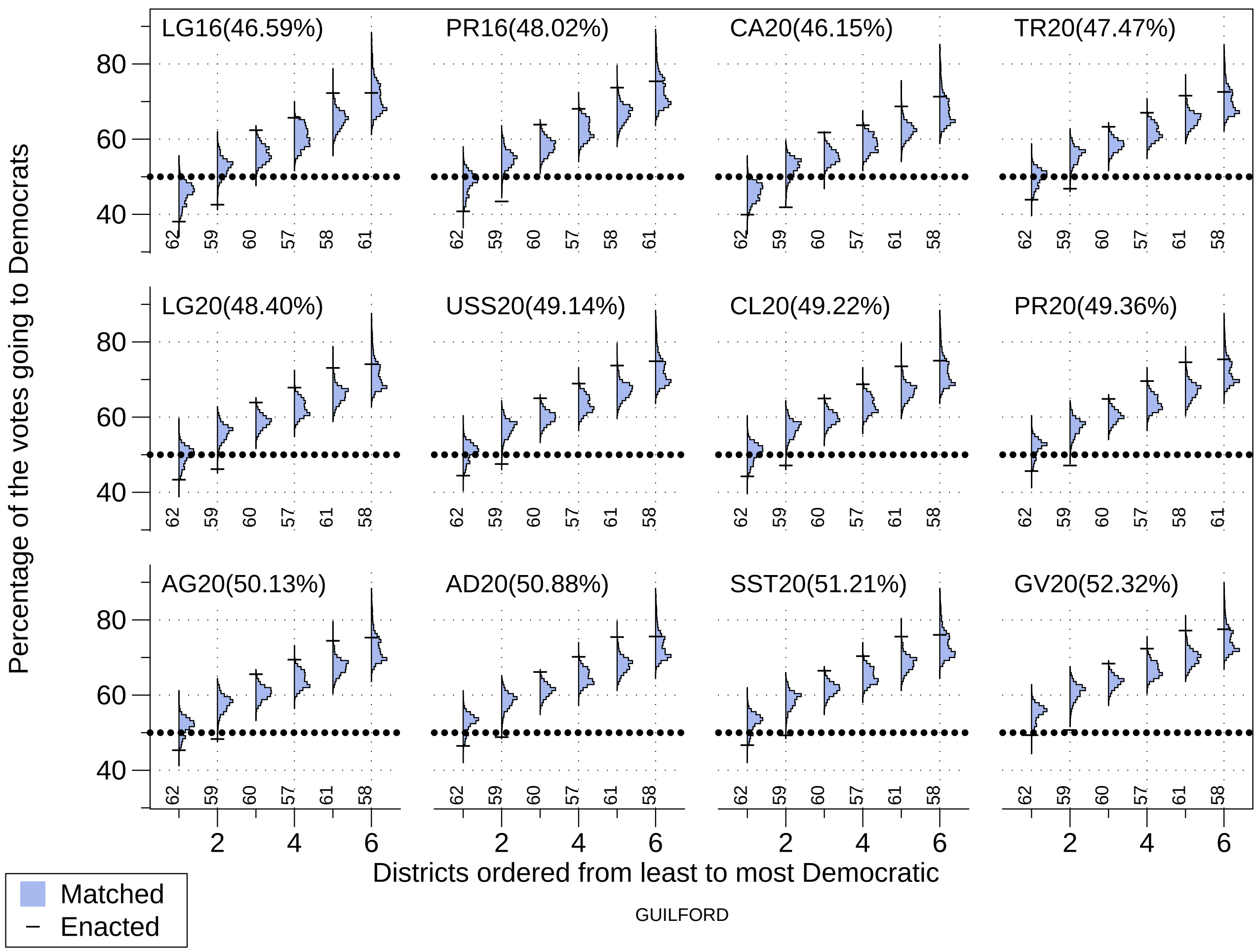


Figure 6.1.10: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

In Figure 6.1.11, we see the effect of considering the the ensemble that more strongly preserves municipalities than the enacted plan. The ensemble reliably has four democratic districts and a 5th which typically leans Republican but sometimes is competitive. Yet, the enacted plan gives one clearly Republican district and one which is often safely Republican and at times competitive.

Municipal Splits and Ousted Population: From Figure 6.1.12, we see that in Guilford the enacted plan ousts a number of people comparable to the primary ensemble but constantly more than the secondary ensemble.

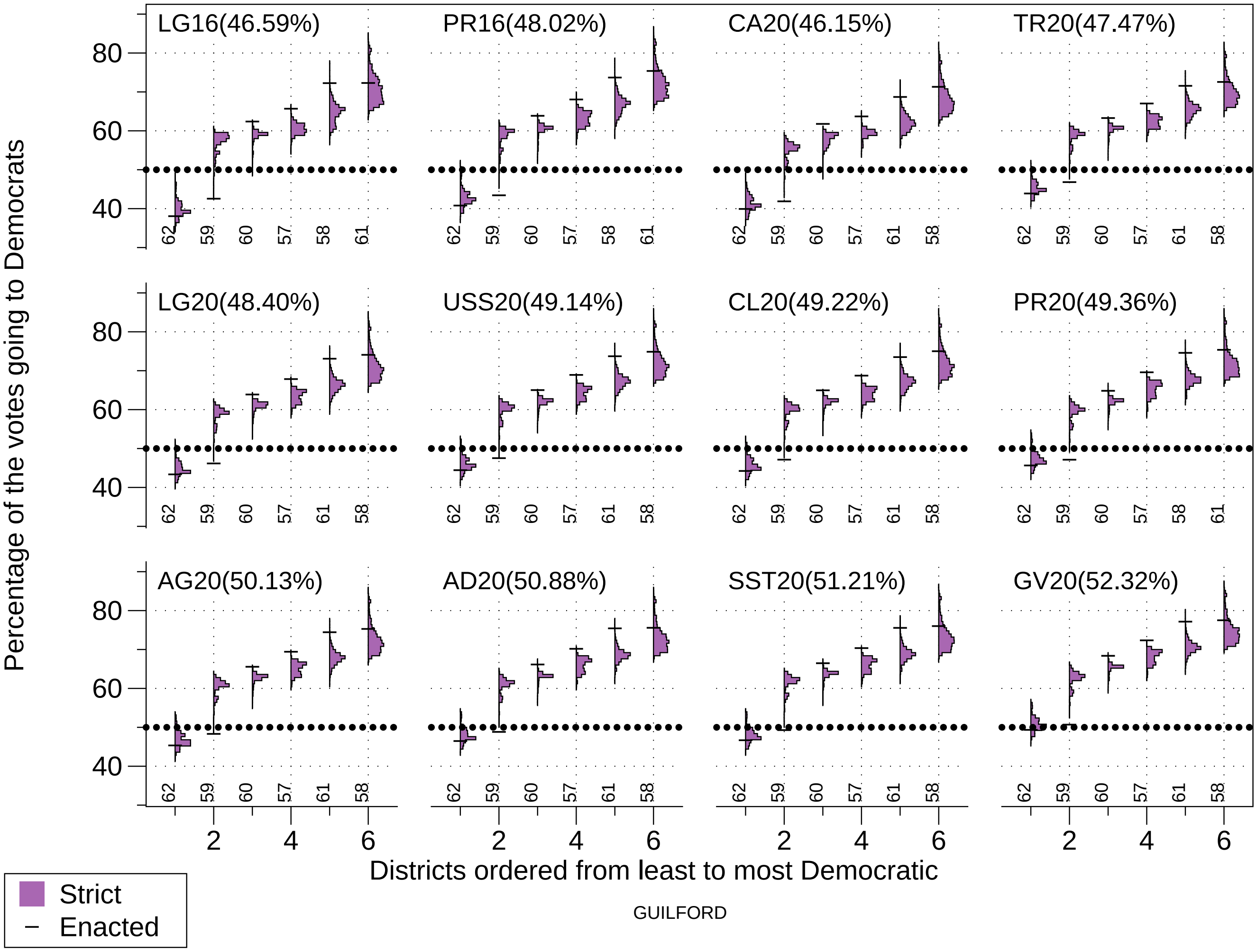


Figure 6.1.11: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

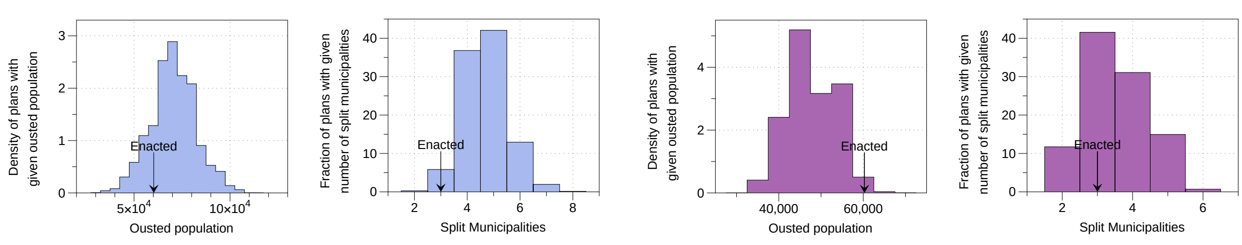


Figure 6.1.12: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.5 Buncombe

As seen in Figure 6.1.13, the primary ensemble shows two Democratic districts with a third typically leaning Democratic but sometimes in play. However, the enacted map produces one district which is typically Republican. This is achieved by packing unusually many Democratic in the most Democratic district (district 114) leaving abnormally few Democrats for the most Republican district (district 116).

Ranging over the elections considered, at most 1.2% of the plans in the ensemble have a lower democratic fraction in the most Republican district in the ensemble than the enacted plan does. The percentage of plans with a larger Democratic fraction in the most Democratic district in the ensemble fluctuates around 5%.

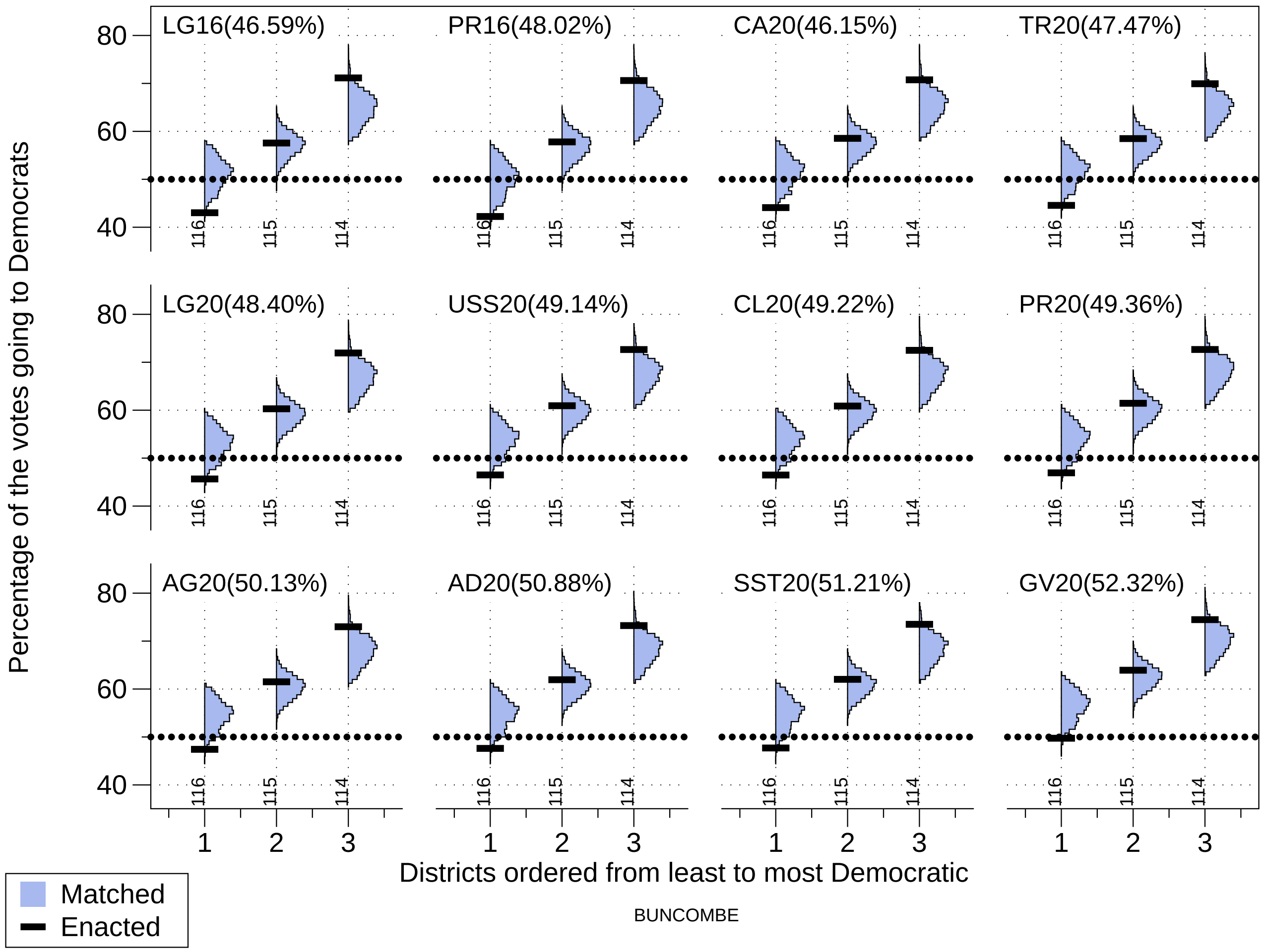


Figure 6.1.13: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

The same pattern of depleting Democrats from the most republican district so that it often elects a Republican when it typically would not under the ensemble is again seen in Figure 6.1.14 which shows the results under the secondary ensemble.

Municipal Splits and Ousted Population: From Figure 6.1.15, we see that there is not a lot of difference between the two ensembles in the number of ousted people. Both are comparable to the enacted map.

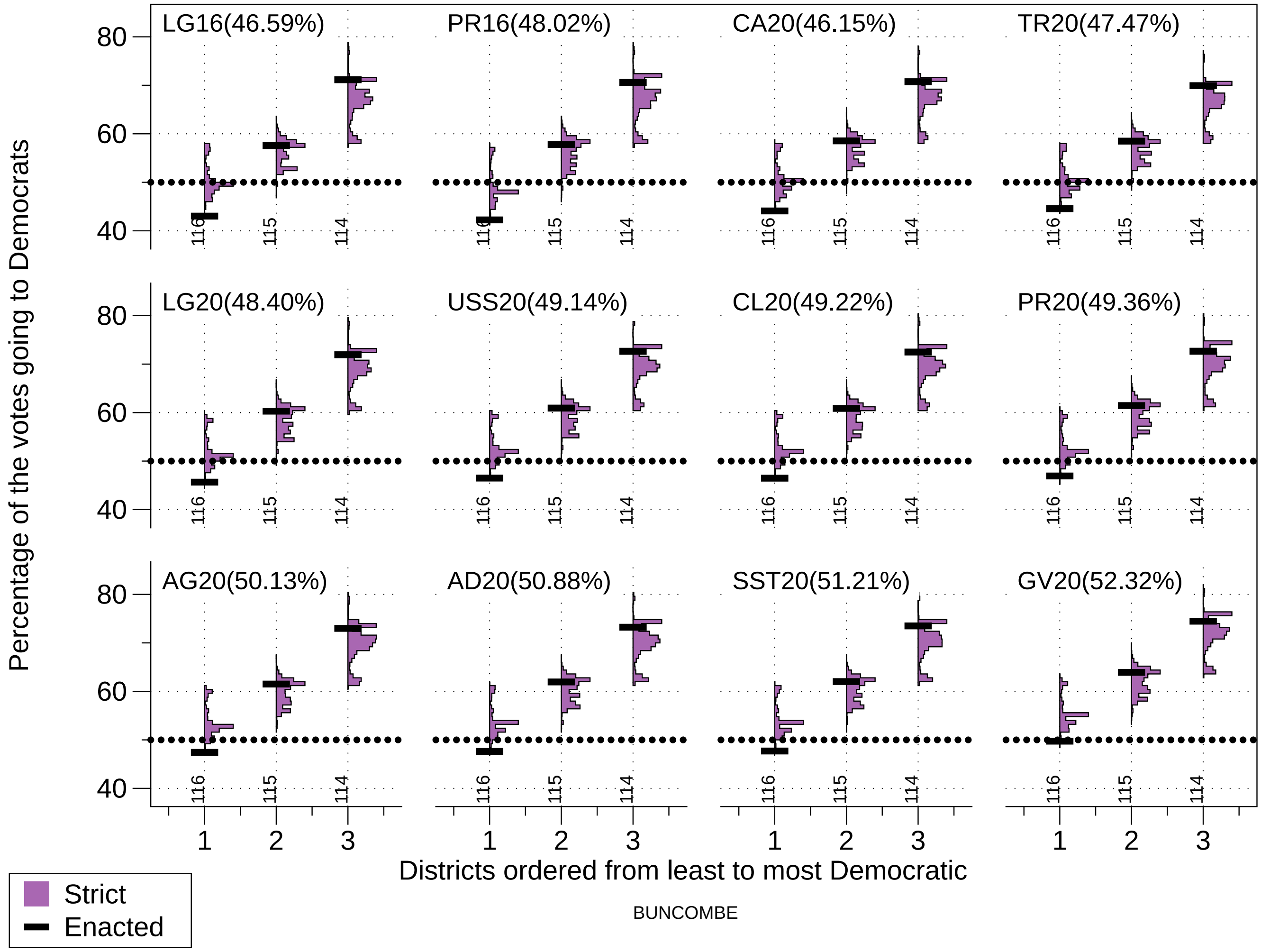


Figure 6.1.14: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The “ – ” on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the “ – ”. This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

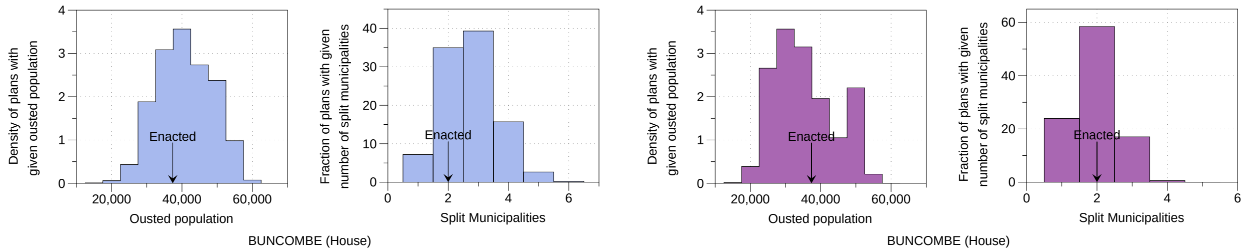


Figure 6.1.15: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.6 Pitt

Pitt County only has two districts. The enacted places atypically many Democrats in the most Democratic district (district 8) while placing atypically few in the most Republican district (district 9). This maximizes the chance that the second district will elect a republican. In many cases, it does when many of the ensemble maps would not. By maximizing the difference in the partisan makeup of the two districts, the enacted map minimized the degree to which the enacted map responds to the shifting opinions of the electorate.

Across the elections considered, the percentage of plans in the ensemble which have a higher fraction of Democrats in the most Democratic district than the enacted plan fluctuates between 1.1% and 5.3%.

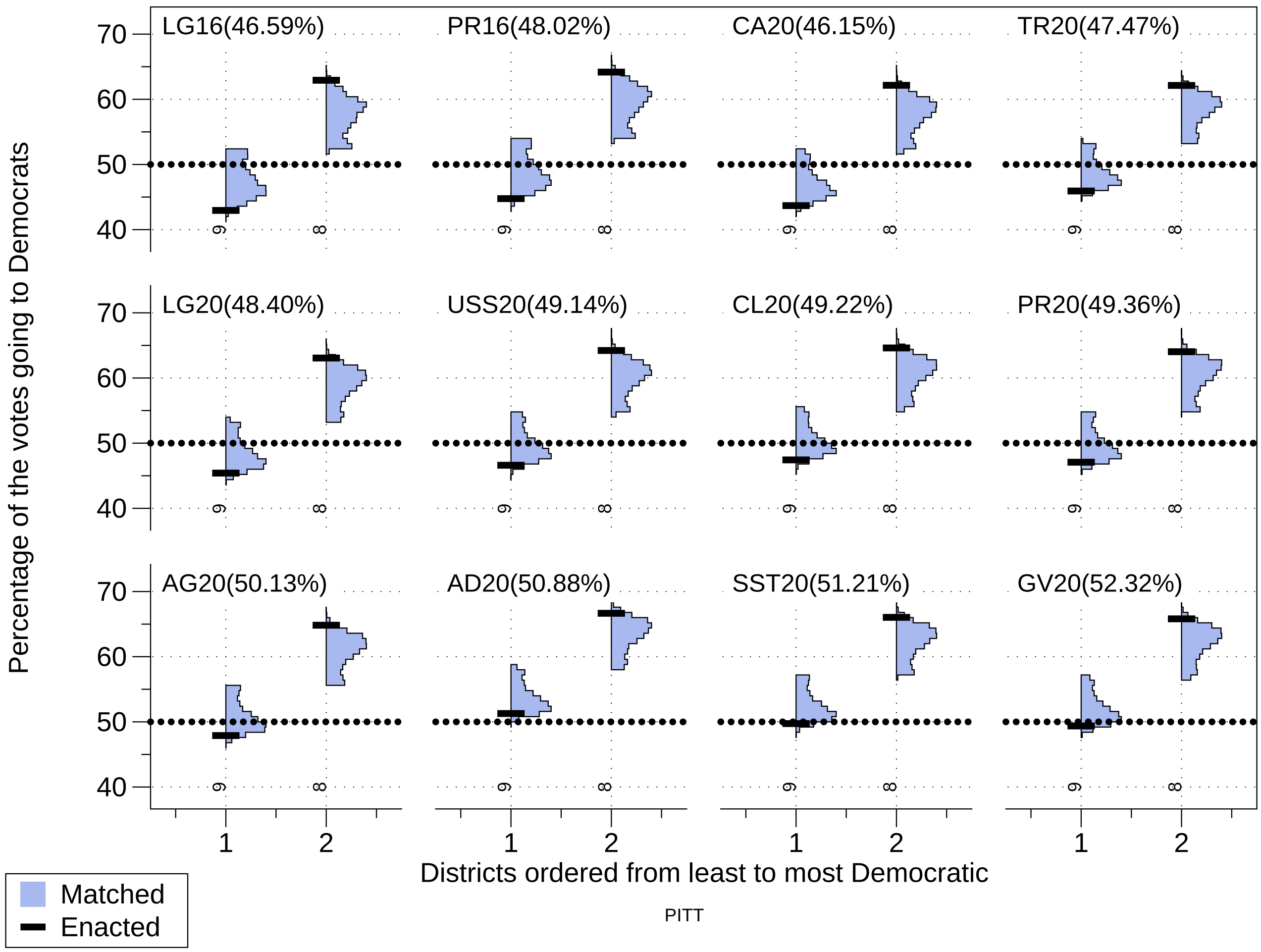


Figure 6.1.16: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

The same pattern is repeated in Figure 6.1.17 which uses the secondary ensemble which better preserves municipalities than the enacted map.

Municipal Splits and Ousted Population: From Figure 6.1.18, we the number of ousted people in the primary ensemble is comparable to the enacted plan but more than the secondary ensemble.

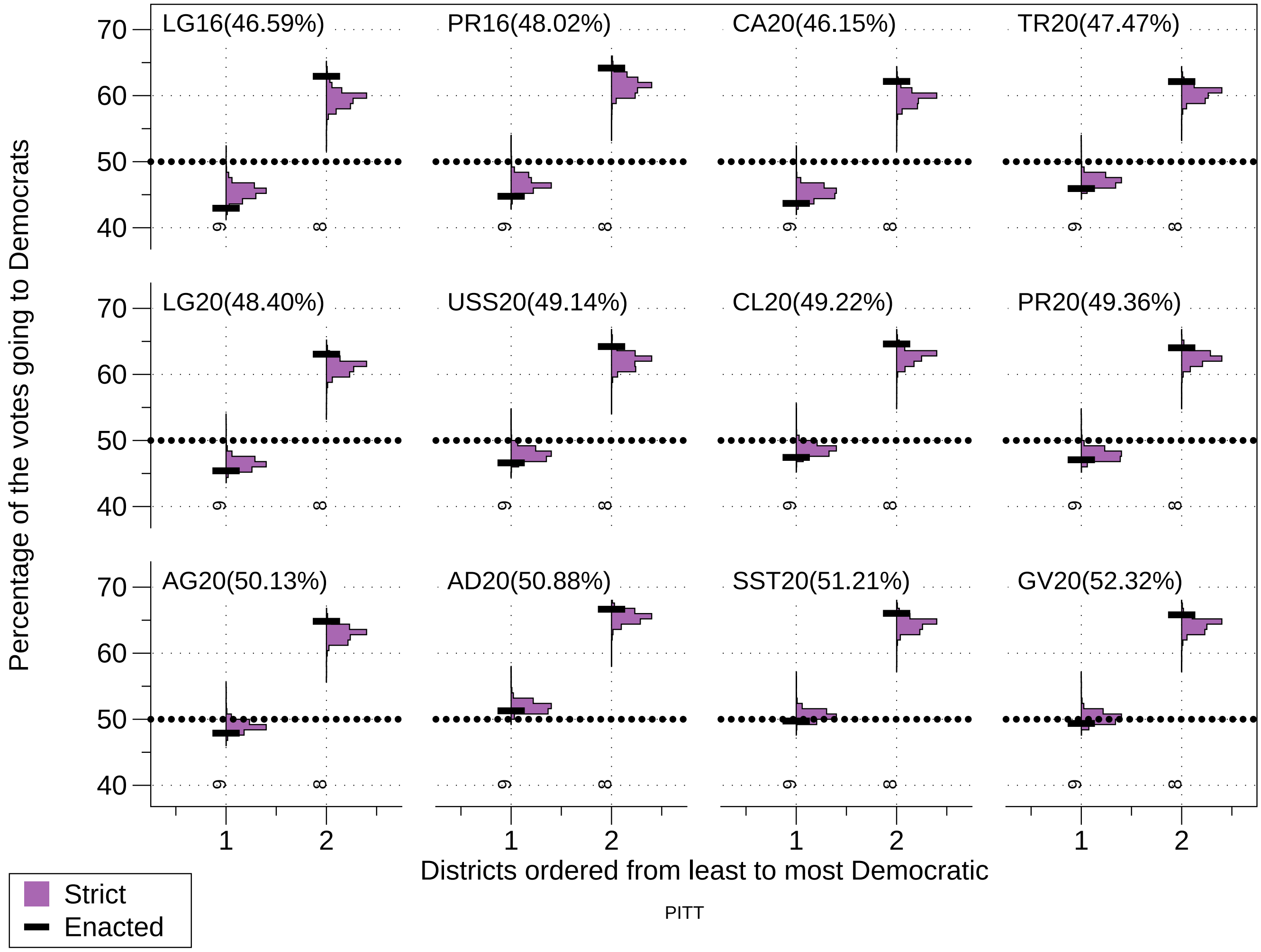


Figure 6.1.17: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The “ – ” on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the “ – ”. This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

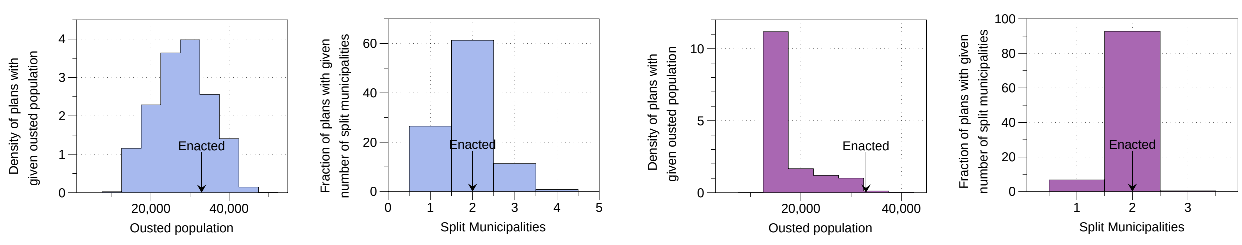


Figure 6.1.18: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.7 Duplin-Wayne

In the Duplin-Wayne county cluster the two districts are safely Republican under the elections considered. The enacted map is typical, falling in the middle of the observed democratic fraction on the Histograms.

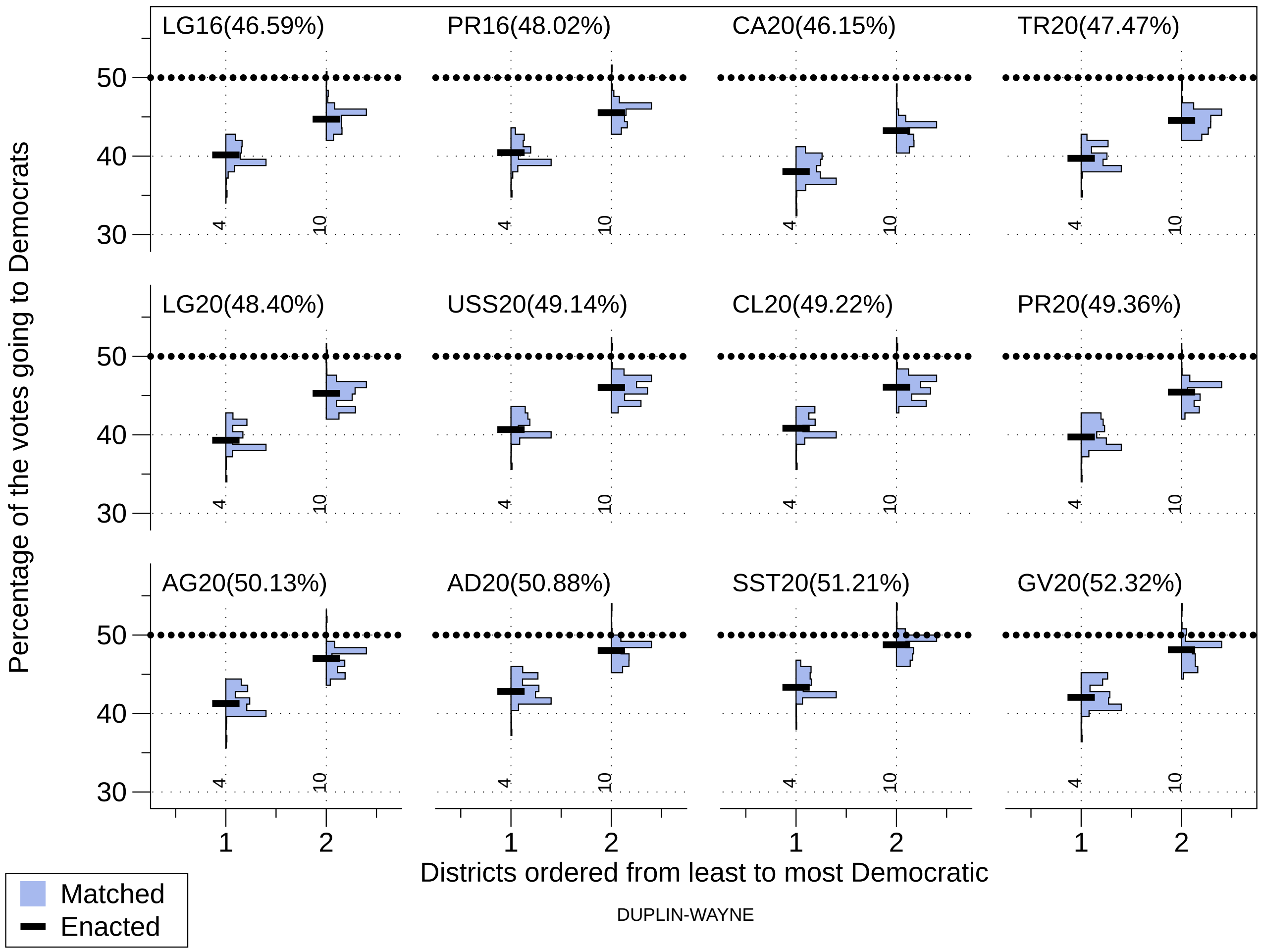


Figure 6.1.19: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

As seen in Figure 6.1.20, the distribution has extremely small variance when municipalities are better preserved. Here there seem to be a little less Democrats in the most Democratic district than typical, but this has little effect as the two districts are firmly Republican and the distribution is highly concentrated.

Municipal Splits and Ousted Population: From Figure 6.1.21, we seen that the number of people ousted by the enacted plan is at the lower end of the typical amounts seen in the Primary ensemble or the secondary ensemble.

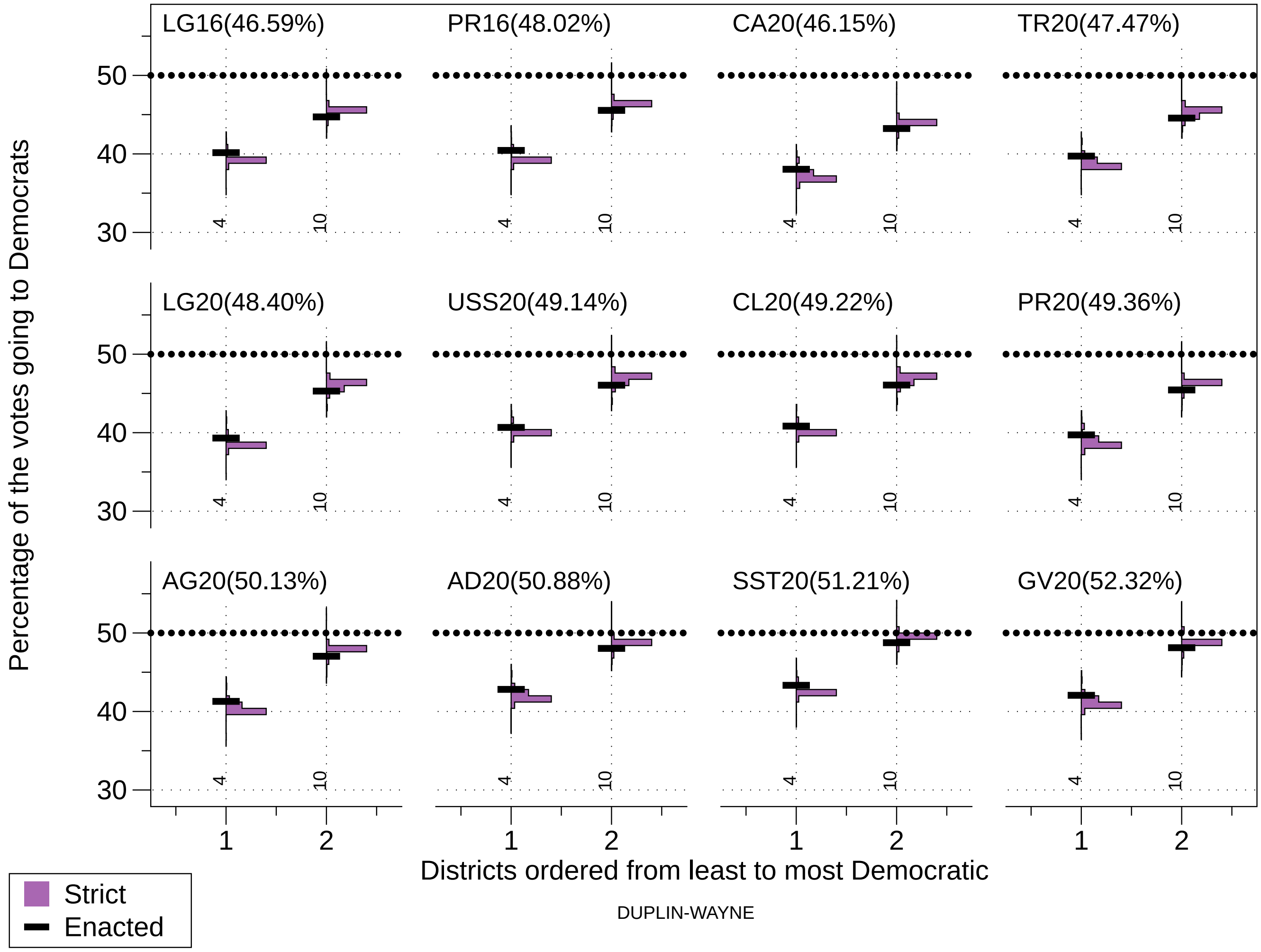


Figure 6.1.20: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

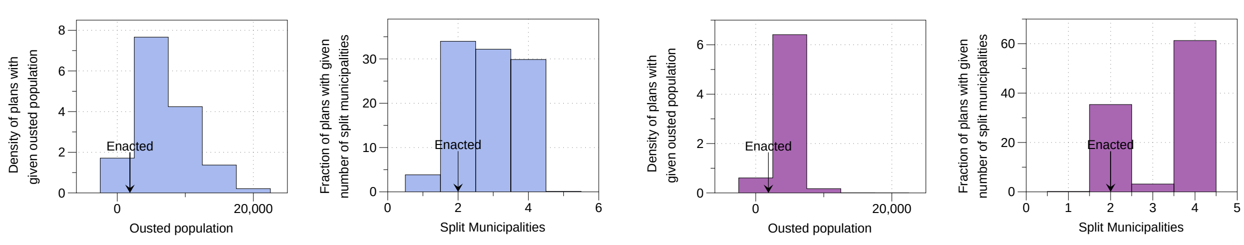


Figure 6.1.21: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.8 Durham-Person

As seen in Figure 6.1.22, under the primary ensemble Durham-Person cluster typically has three exceedingly Democratic districts and one more moderately Democratic district. The enacted plan places abnormally few Democrats in the most Republican district (district 2). This is accomplished by packing more Democrats in the most Democratic districts (districts 29 and 30). The effect is sufficient to pick up a Republican seat in a few elections where the seat typically would have remained democratic according to the non-partisan primary ensemble.

Not a single map in the non-partisan ensemble across any of the elections considered has a smaller fraction of Democrats in the most Republican district than the enacted plan does. This signals extreme cracking. In all but two elections the fraction of plans which have a higher Democratic vote fraction than the enacted plan is less than 0.62%. The two exceptions are LG16 (3.5%) and CA20 (1.2%).

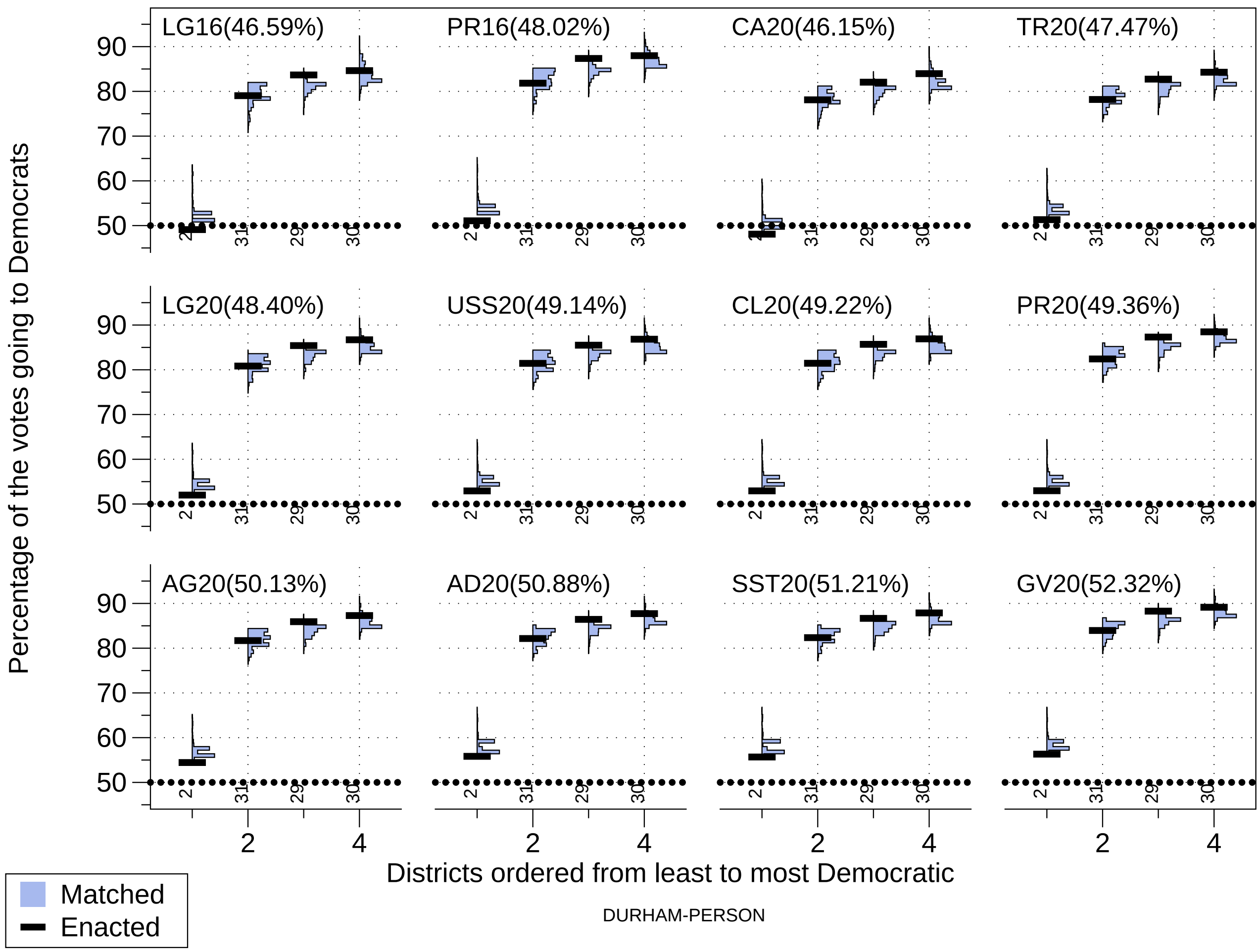


Figure 6.1.22: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

A similar effect is seen in 6.1.23, for the ensemble which better preserves municipalities.

Municipal Splits and Ousted Population:

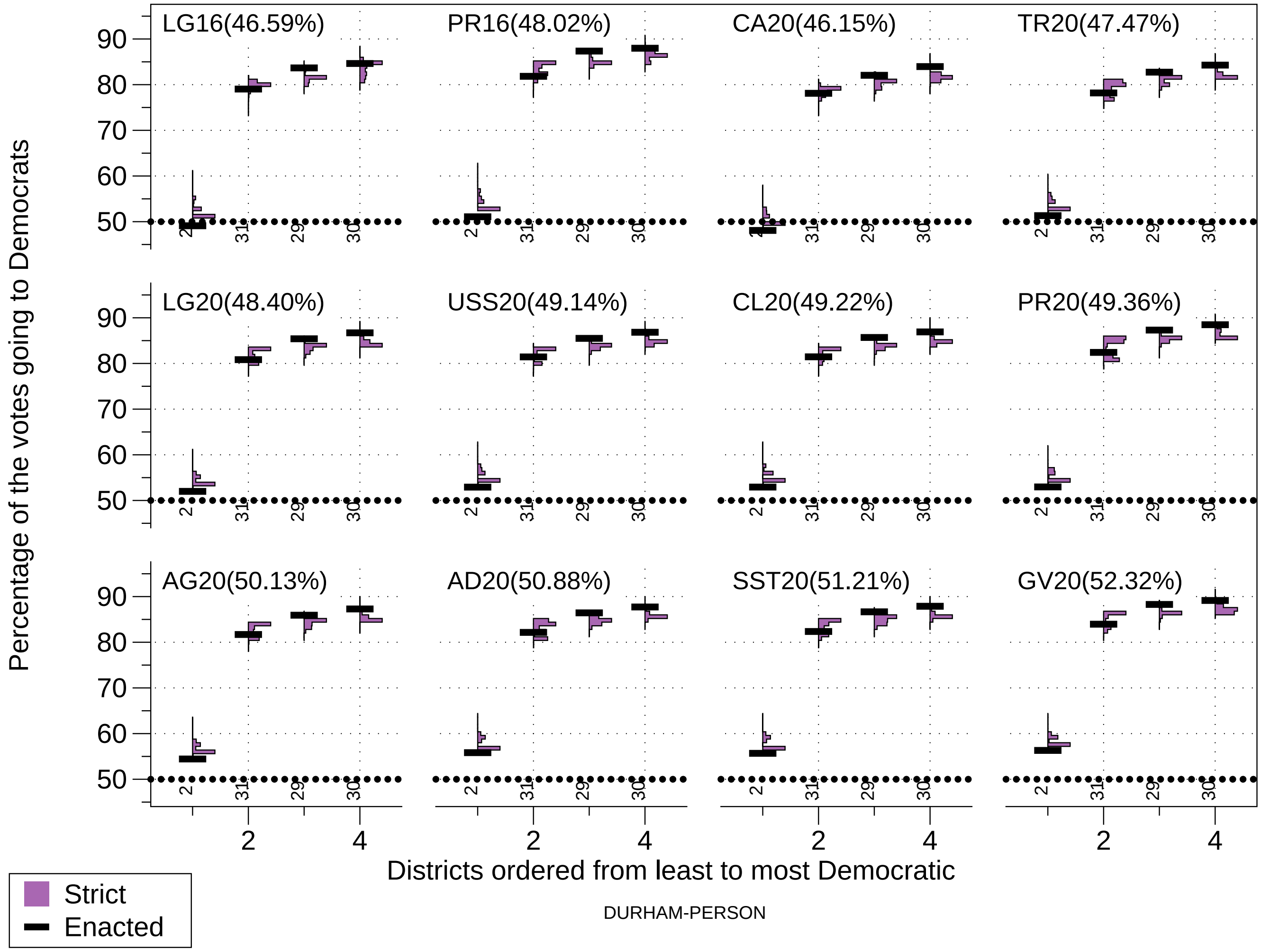


Figure 6.1.23: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

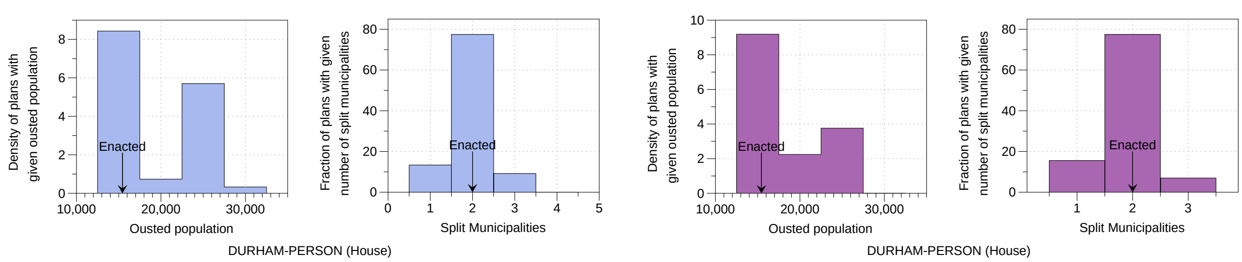


Figure 6.1.24: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.9 Alamance

From Figure 6.1.25, we see that though the enacted map tends have more Democrats in the more Democratic district and less in the less democratic district it not an outlier on its own.

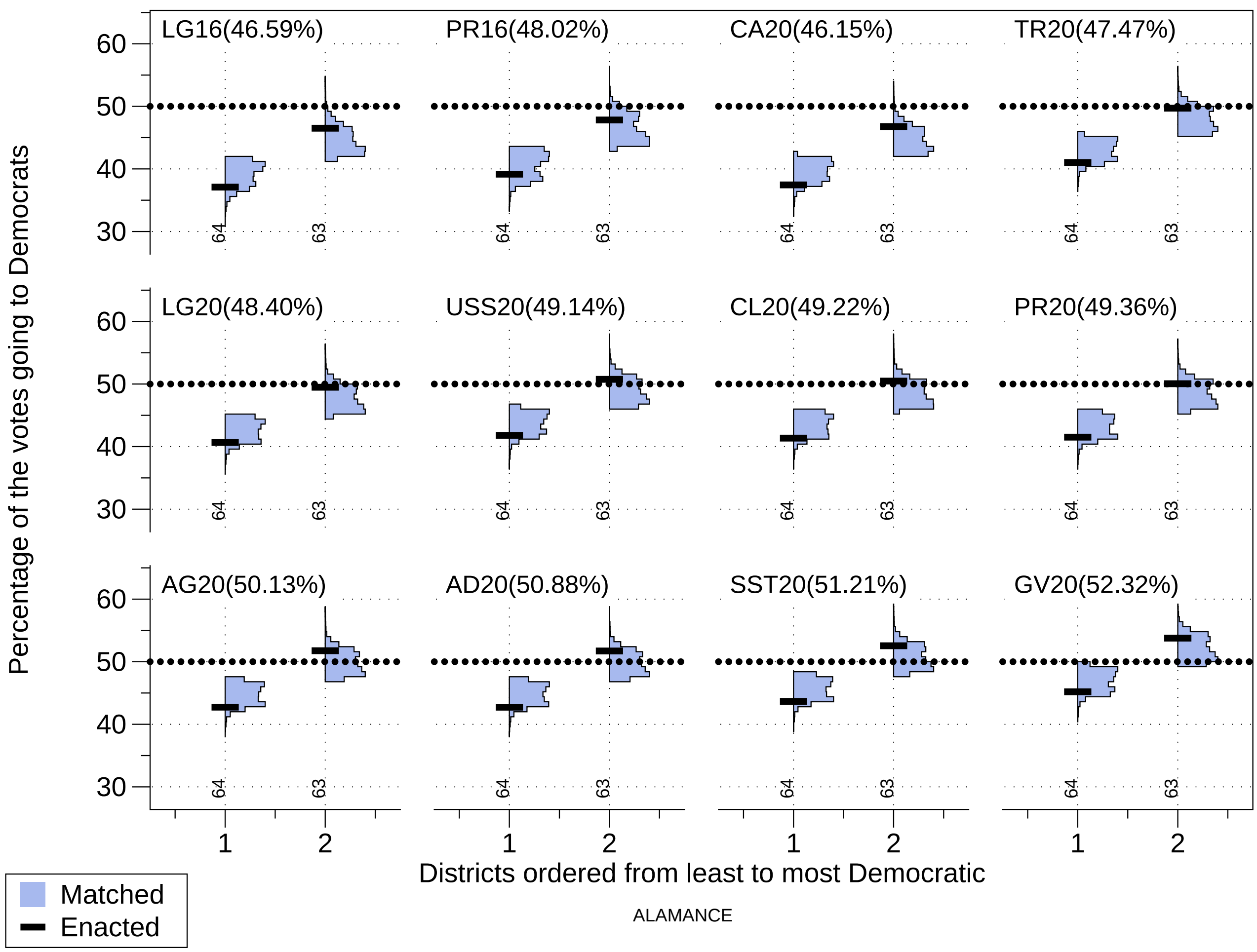


Figure 6.1.25: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

Figure 6.1.26 tells a similar story to Figure 6.1.25,

Municipal Splits and Ousted Population:

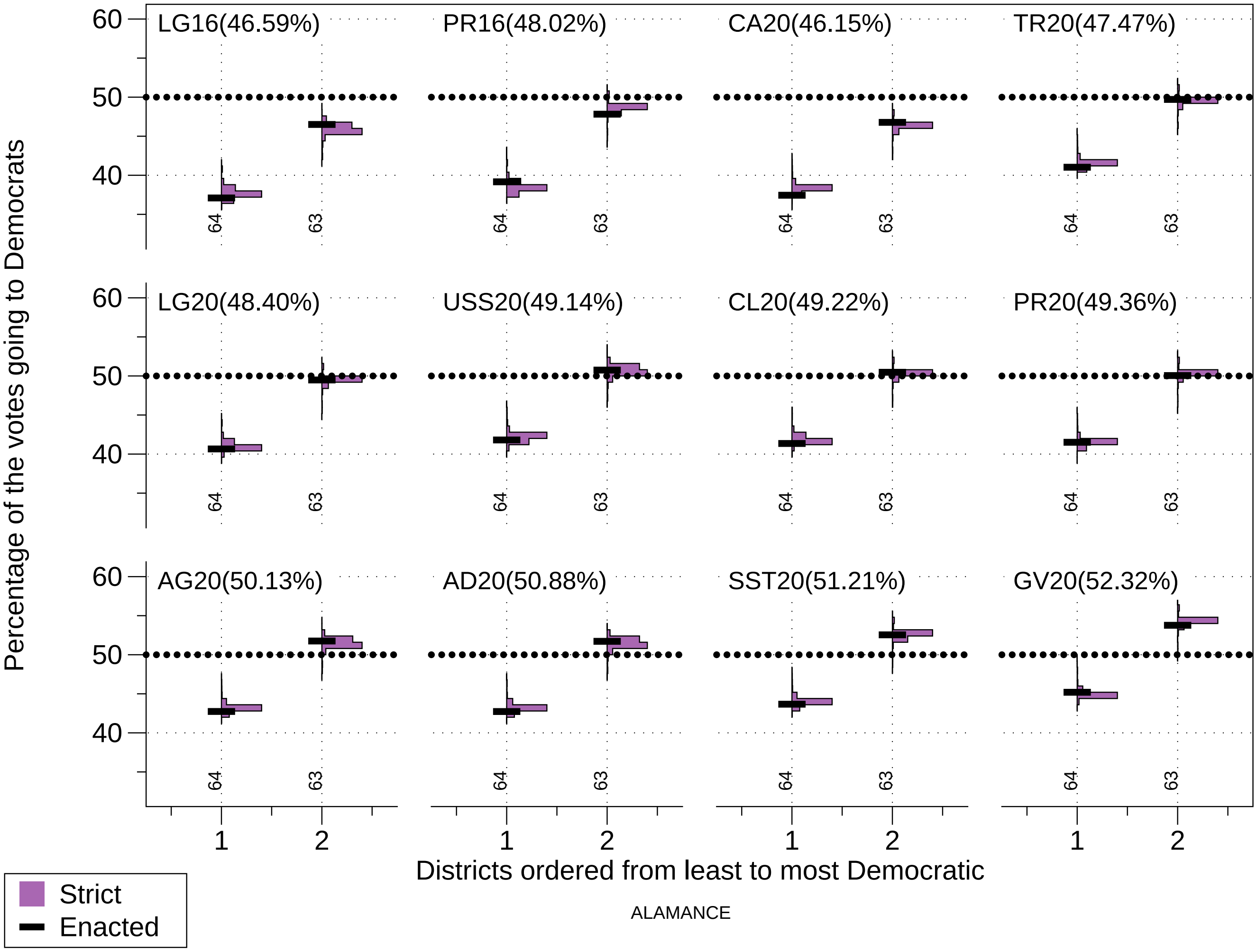


Figure 6.1.26: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

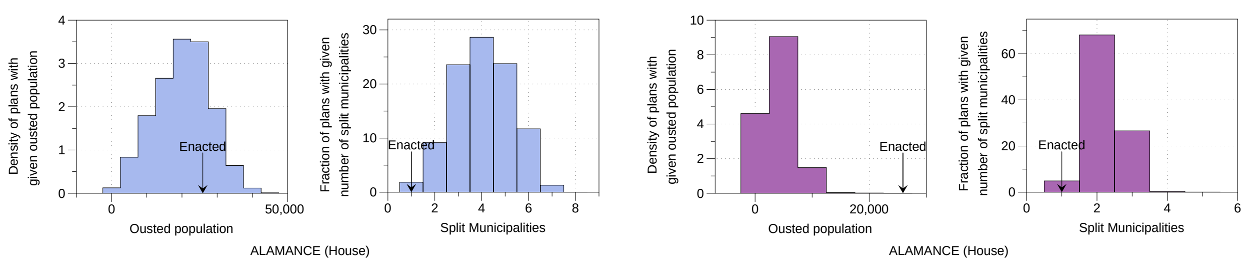


Figure 6.1.27: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.10 Cumberland

Looking at Figure 6.1.28, we again see outlier behavior in Cumberland County. We see that the districts in the enacted plan have been constructed so that the two most Republican districts (district 43 and 45) have a similar partisan makeup. Typically, one is more Democratic and one is more Republican. This is achieved by removing republicans from the most republican district and Democrats from the most democratic two districts. While the effect on the most Republican district individually is within the typical range, the combined effect creates an enacted cluster which is an strong outlier.

For each of the elections considered, the number of plans in the ensemble with smaller fraction of democrats in the second most republican district is typically around 1% with, for a few elections, the percentage reaching as high as 7% or as low as 0.4%.

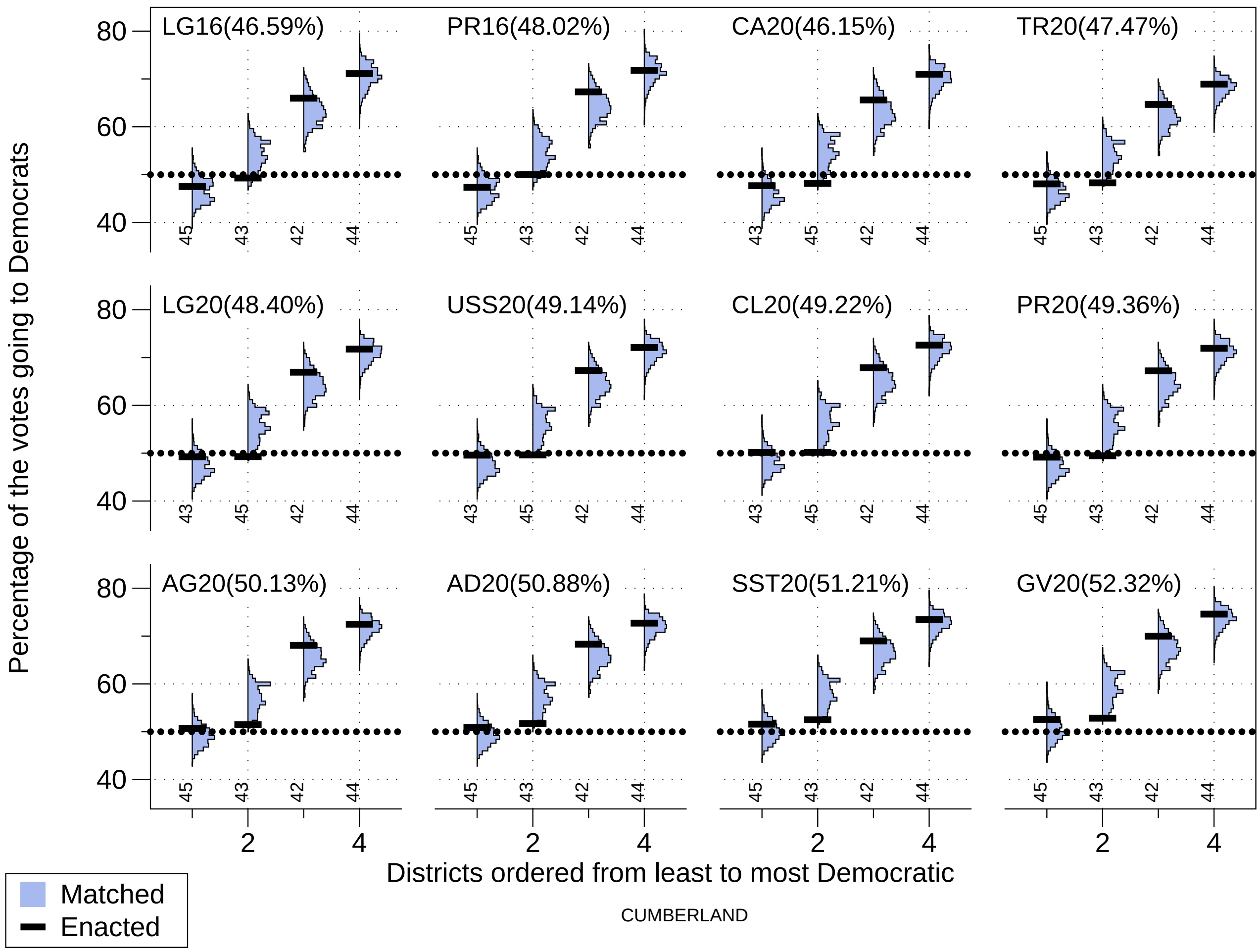


Figure 6.1.28: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

Looking at Figure 6.1.29, we see that the structure of the enacted map is a more extreme outlier for the secondary ensemble which better preserves municipalities. In an ensemble that better preserves municipalities, the most Republican district is typically more republican and the second most Republican district more Democratic. This makes the enacted plan which squeezes the two together with an large outlier.

Municipal Splits and Ousted Population:

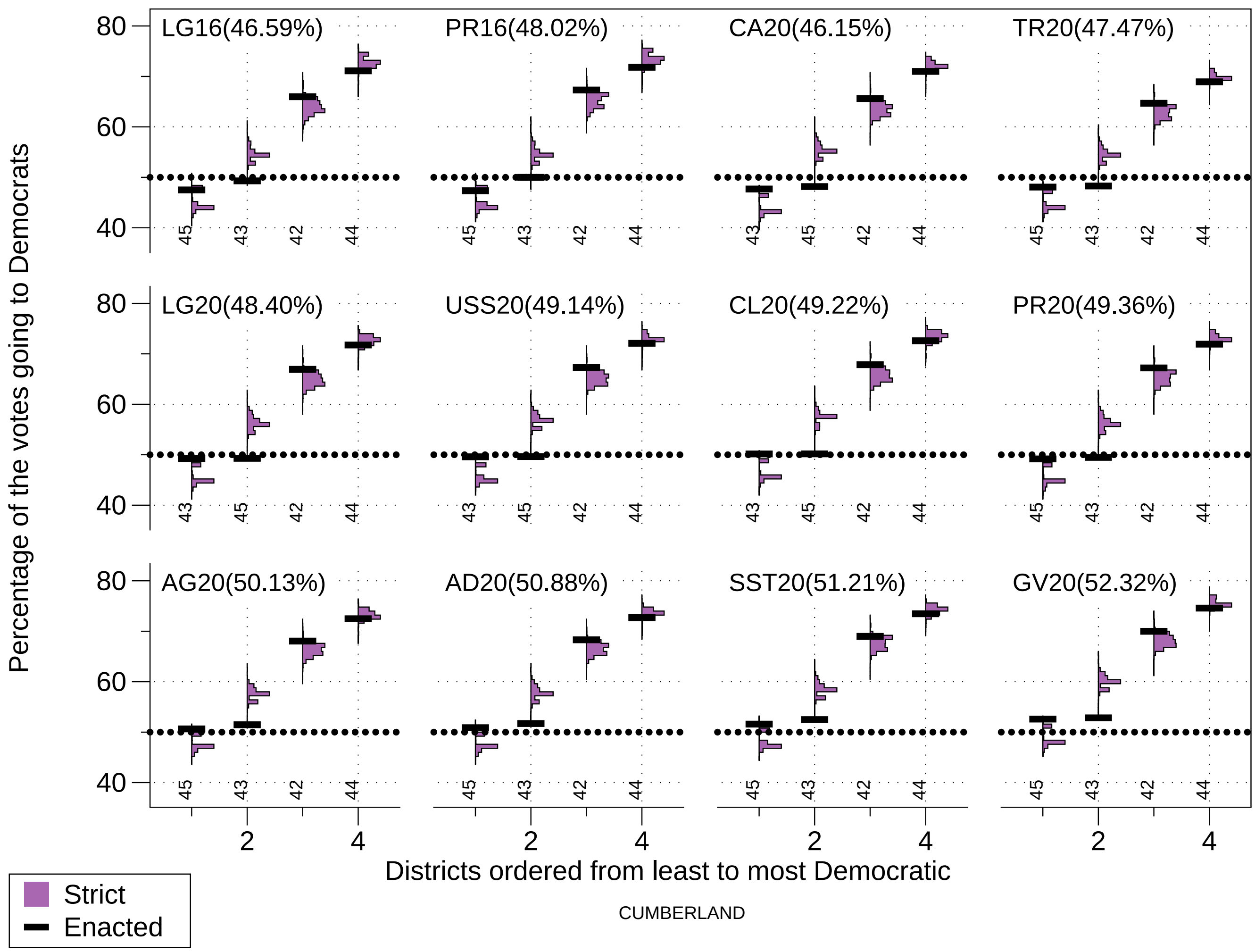


Figure 6.1.29: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

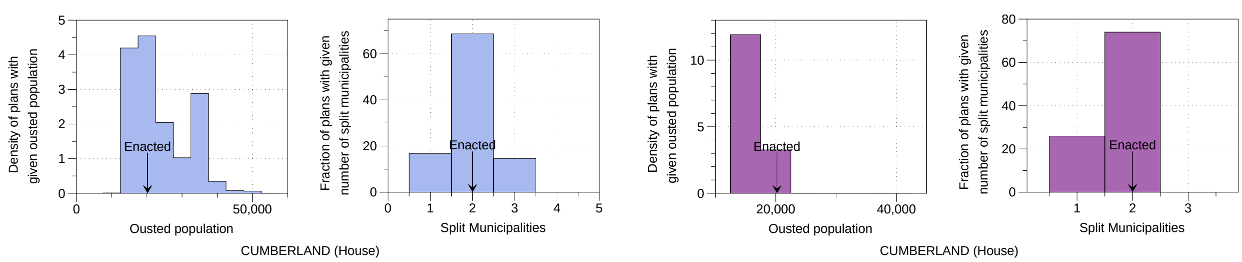


Figure 6.1.30: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.11 Cabarrus-Davie-Rowan-Yadkin

In the Cabarrus-Davie-Rowan-Yadkin county cluster, there are abnormally few Democrats in the most Democratic district (district 82). This is accomplished by placing abnormally many Democrats in the next three most democratic districts (districts 73, 76, and 83 – all of which are safe Republican districts). The effect is to make the most Democratic district a relatively reliable Republican seat (being won by the Republicans in all of the elections considered). Under the ensemble, it would switch parties in a number of the elections and regularly be a close contest.

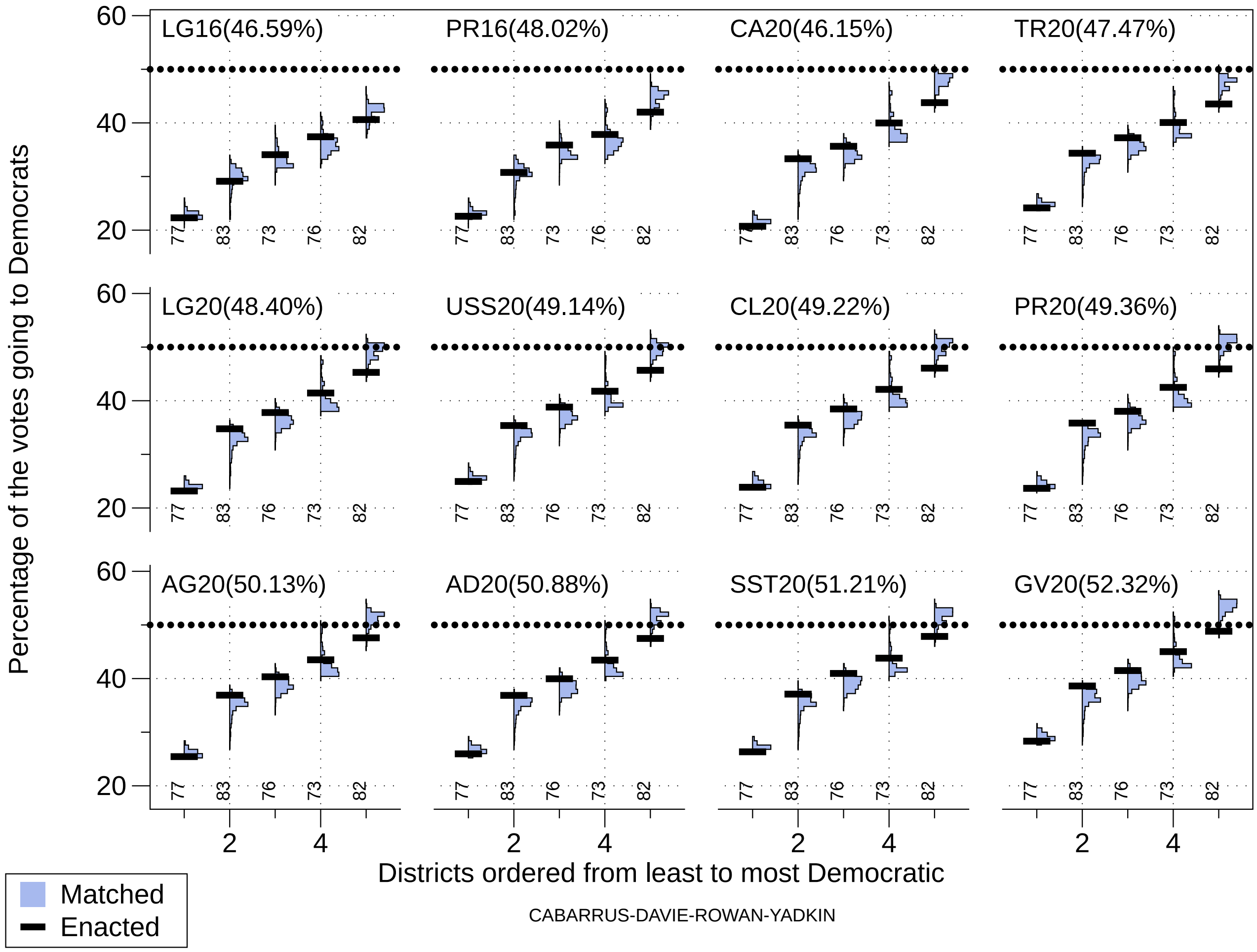


Figure 6.1.31: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

Looking at Figure 6.1.32, we see that the same pattern persists under the secondary ensemble which better preserves municipalities.

Municipal Splits and Ousted Population:

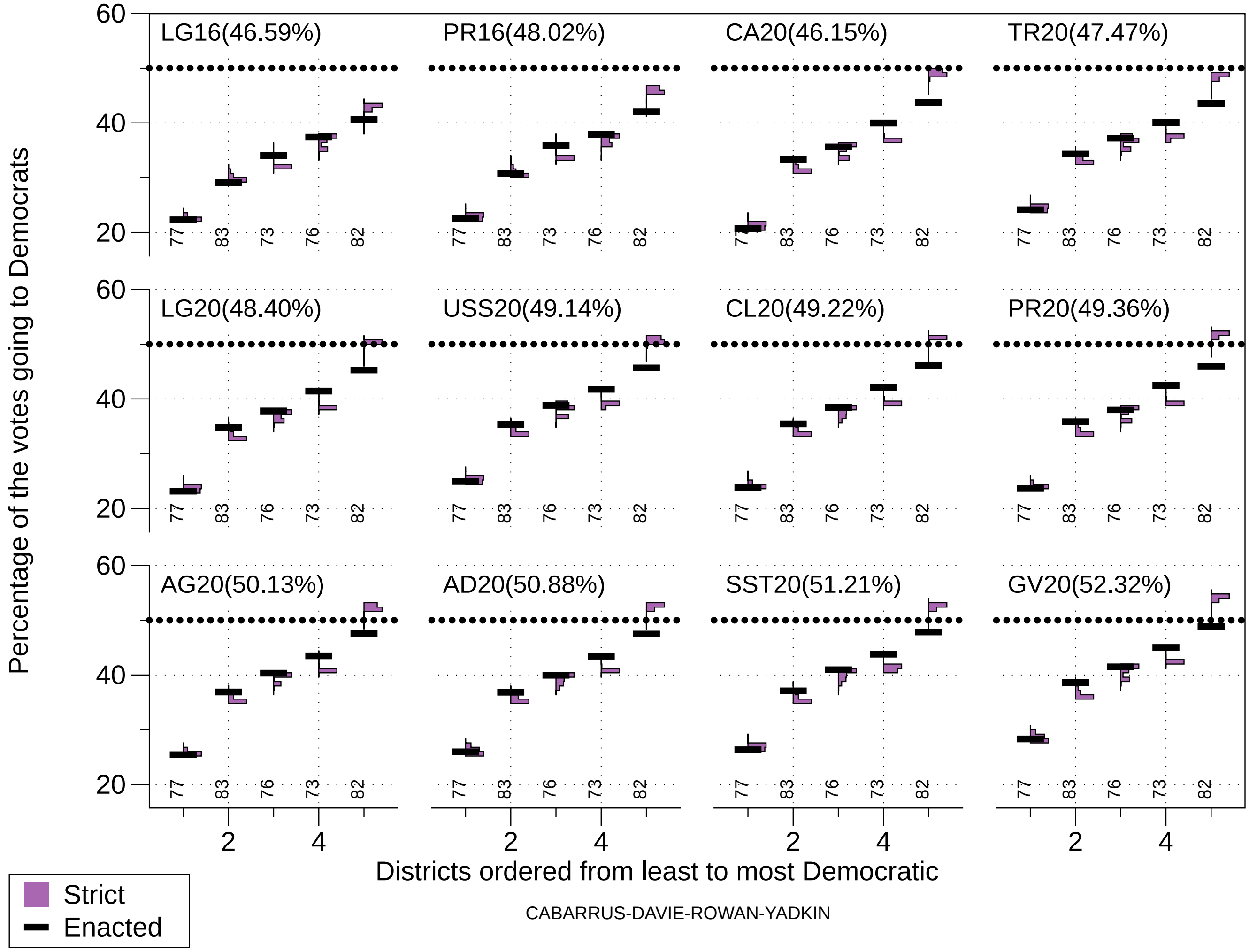


Figure 6.1.32: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The “ – ” on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the “ – ”. This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

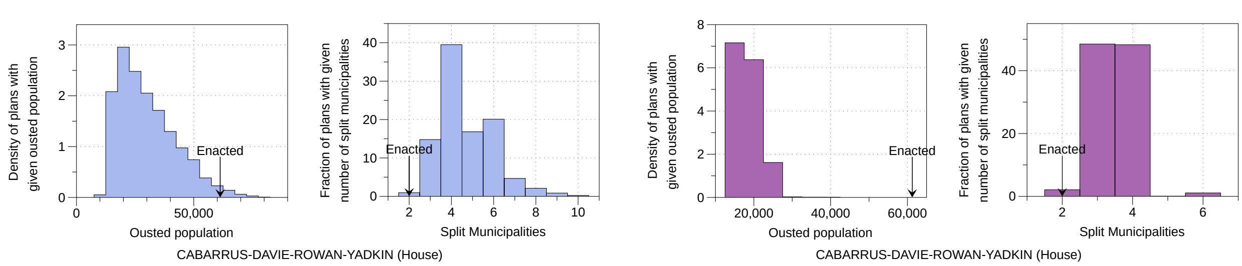


Figure 6.1.33: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.1.12 Brunswick-New Hanover

In the Brunswick-New Hanover county cluster, Figure 6.1.34 shows that the most Democratic district (district 18) has had abnormally many Democrats packed into it and the most Republican has had abnormally few Republicans placed in it, while the second-most Democratic district (district 20) has been depleted of Democrats. This makes the enacted plan much less responsive to changes in the the enacted plan preferences of the voters. The Republican party typically wins the second most democratic district in the enacted plan even though it would go to the Democrats under a number of elections when the neutral maps in the primary ensemble are used. Over each of the elections considered, the fraction of plans in the ensemble

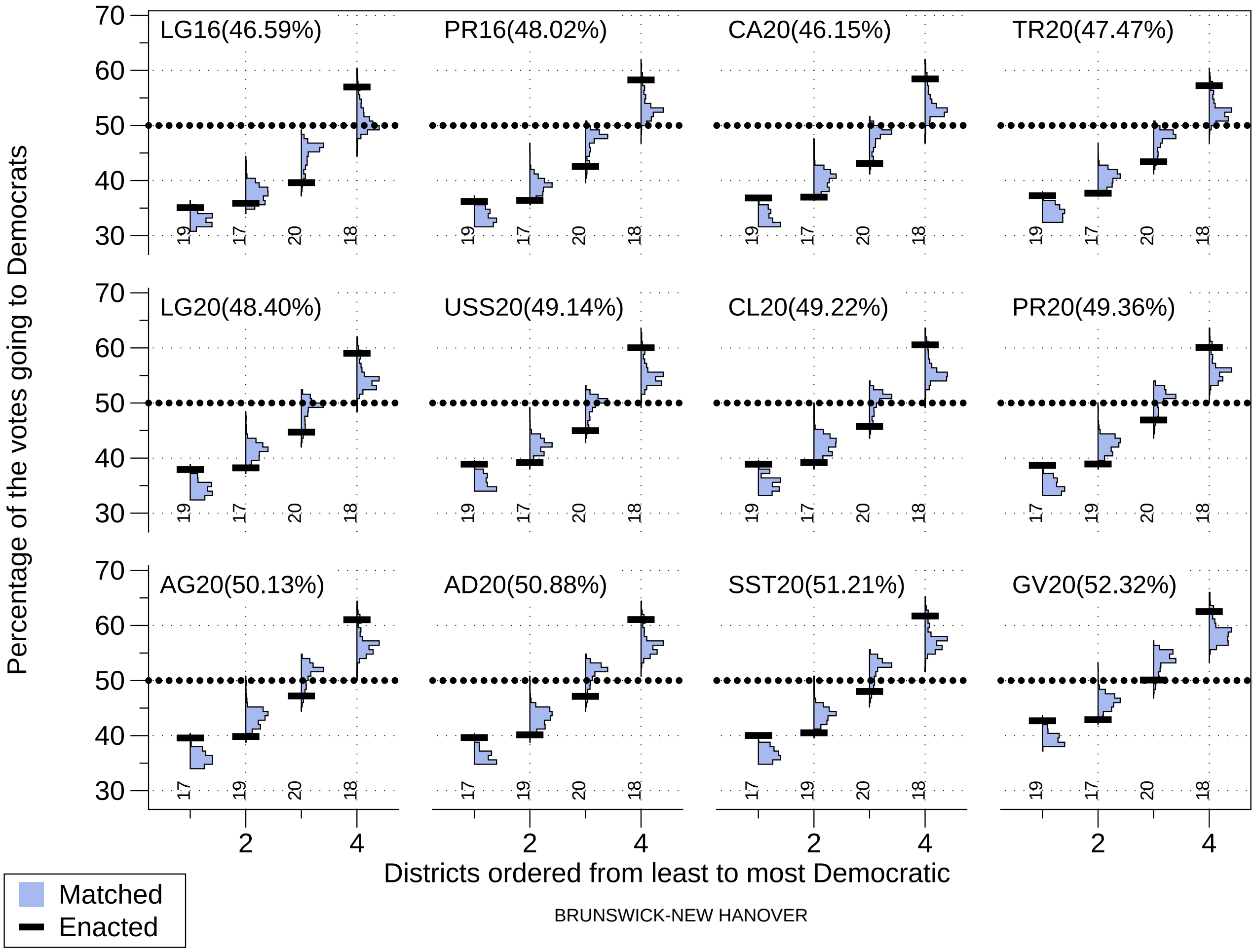


Figure 6.1.34: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

when a lower Democratic vote fraction in the second and third most Republican districts in the ensemble compared to the enacted plan map is always less than 0.5% and often much smaller.

Under the secondary ensemble which better preserves municipalities shown in Figure 6.1.35, we see that the same structure persists. The enacted map becomes a more extreme outlier since this ensemble reduced the variance of the marginals and aligns the outcome gradual progression which ensures the map is fairly responsive to changes in the voter's preference, a property not shared by the enacted map.

Municipal Splits and Ousted Population:

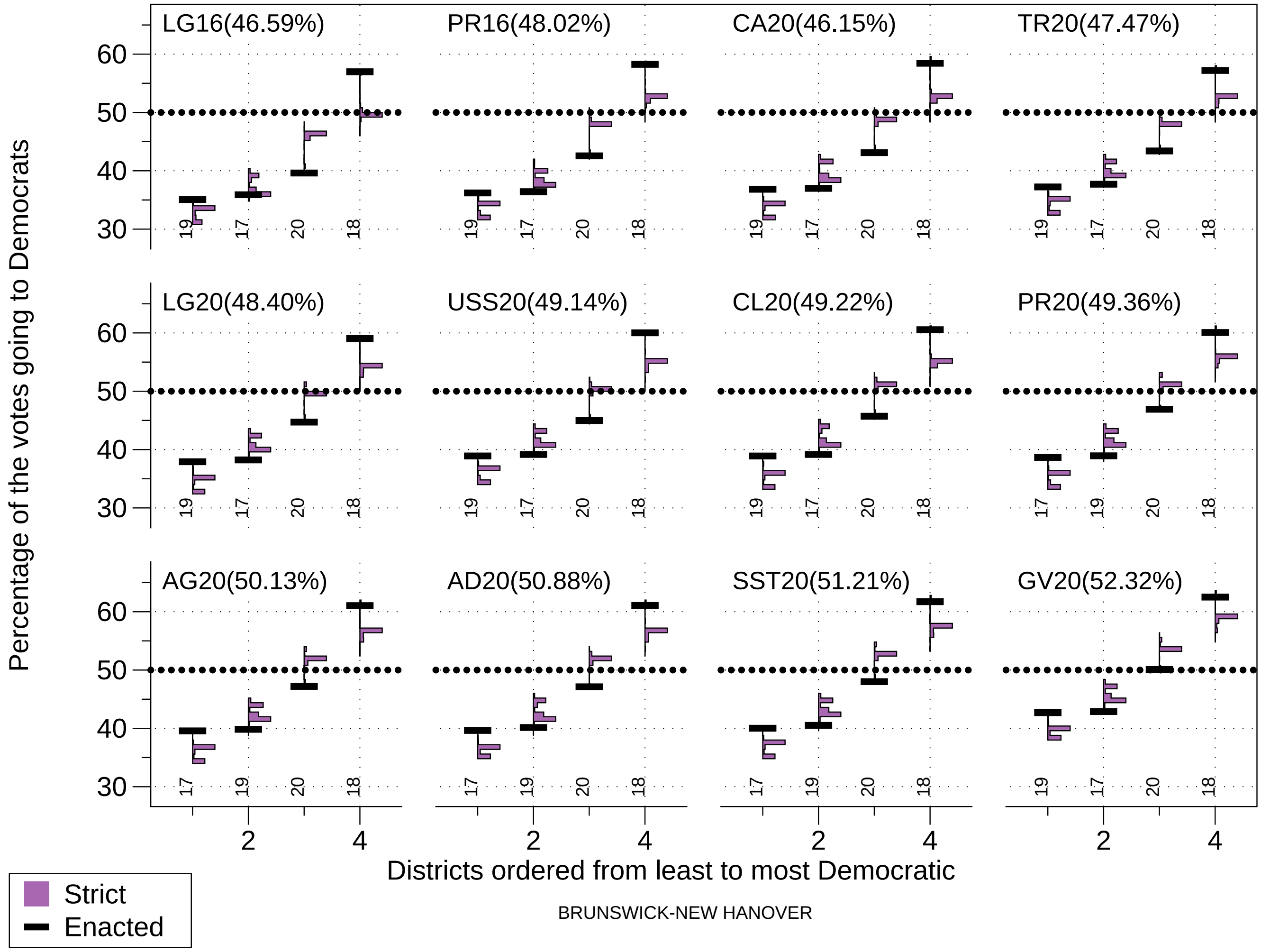


Figure 6.1.35: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Secondary ensemble which better preserves municipalities than the enacted plan.

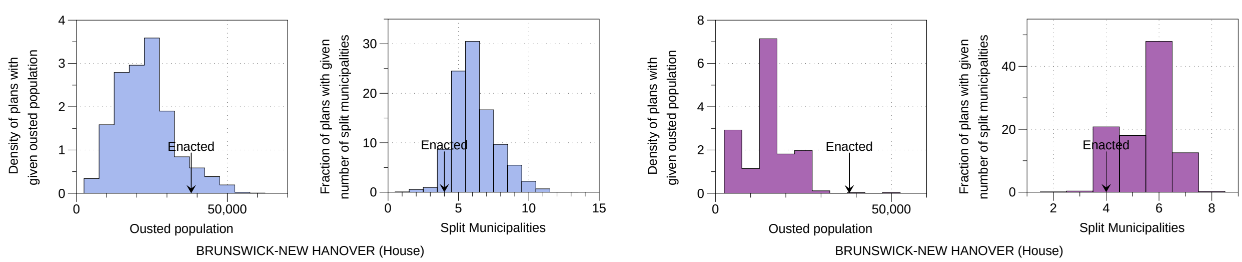


Figure 6.1.36: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.2 NC State Senate

Though the principal Senate ensemble, which prioritizes municipality preservation in line with the enacted plan, does not have as dramatic a shift towards the Republicans at the statewide level in comparison to the House, we still see a number of cases of extreme packing and cracking at the individual cluster level. Without exceptions, the effect is to minimize the effect of the Democratic votes and make the outcome of the election insensitive to a wide range of swings in the partisan vote fraction.

In the NC Senate, we again see the effect of prioritizing municipal preservation in our ensemble. When municipal preservation was not prioritized, there are two major effects. First, the enacted maps become extreme outliers, as the typical results swings are much less tilted to the Republican Party. Second, the two parties are much less separated. Requiring a high level of municipal preservation often leads the separation of the two political parties between disjoint districts. This in turn produces maps that are much less responsive to swinging public opinion. In other words, the results of the elections do not change over a wider range of statewide vote ranges.

6.2.1 Iredell-Mecklenburg

In this cluster, the second most Republican district (District 41 in the enacted plan) is the principal district whose outcome varies from election to election. In the enacted plan, unusually few democrats have been placed in this district to maximize the chance that the district elects a Republican. See Figure 6.2.1. In many elections, this means that the Republican wins this district under the enacted plan, whereas a Democrat would win the district under the a majority of ensemble plans.

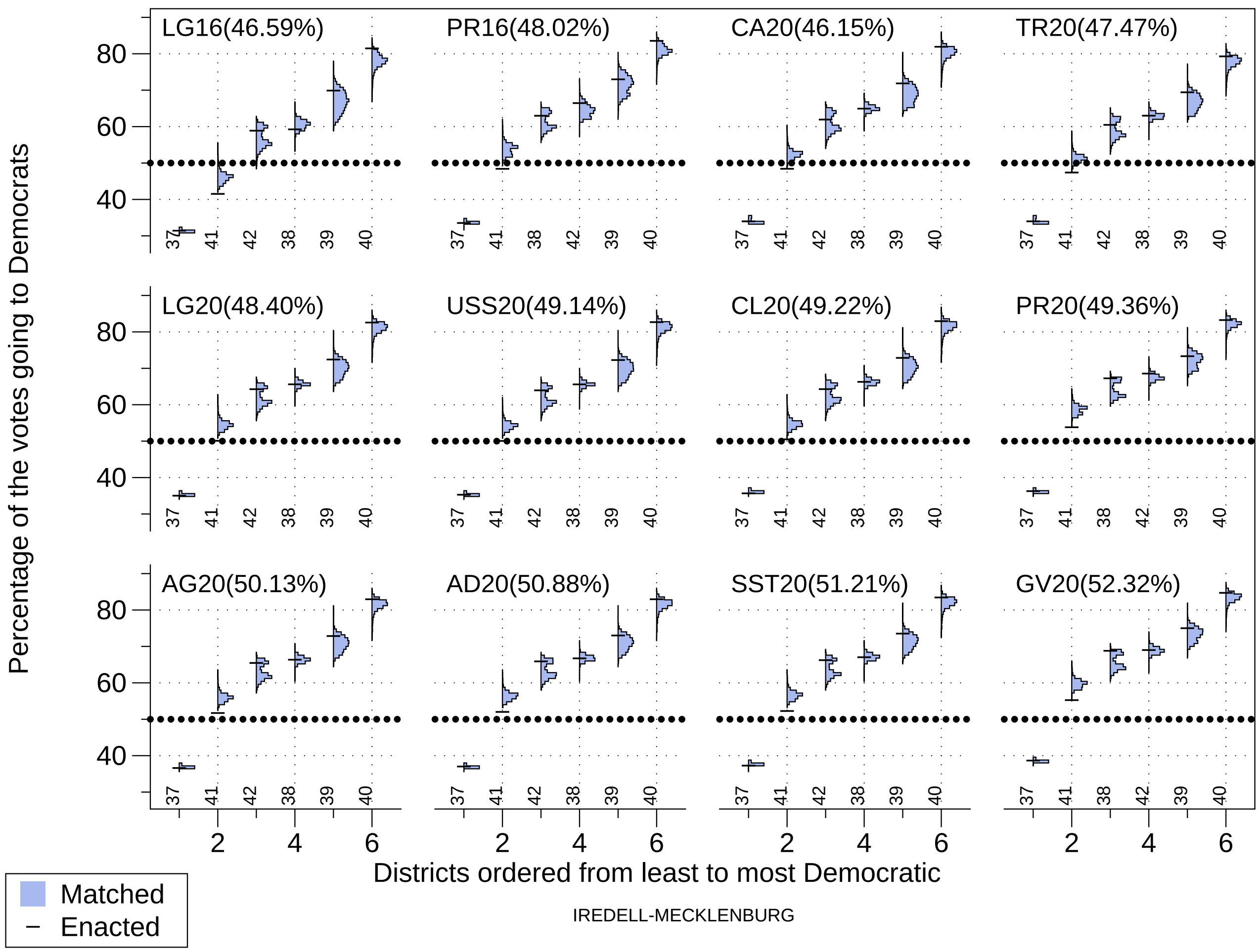


Figure 6.2.1: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

For each of the 2020 and 2016 elections we have consider, we found that none of approximately 80,000 plans in our ensemble had as low a fraction of Democrats in the two most Republican districts in the Iredell-Mecklenburg cluster as the enacted plan. Similarly, in the vast majority of the elections the ensemble had no plans with a higher fraction of democrats packed in the four most Democratic districts. In two elections 0.01% of the plans had a higher fraction of Democrats packed in the four most Democratic districts.

The effect discussed above is essentially the same when the municipality preservation is not prioritized. See Figure 6.2.2.

Municipal Splits and Ousted Population:

We see that in the Iredell-Mecklenburg cluster, the number of ousted people in the enacted plan is comparable the number of ousted people in the ensemble prioritizing municipalities. The enacted plan splits two municipalities which coincides with the most typical number split by the ensemble prioritizing municipalities. Though this ensemble sometimes splits a number more municipalities, it typically displaces a comparable number of people to the enacted plan.

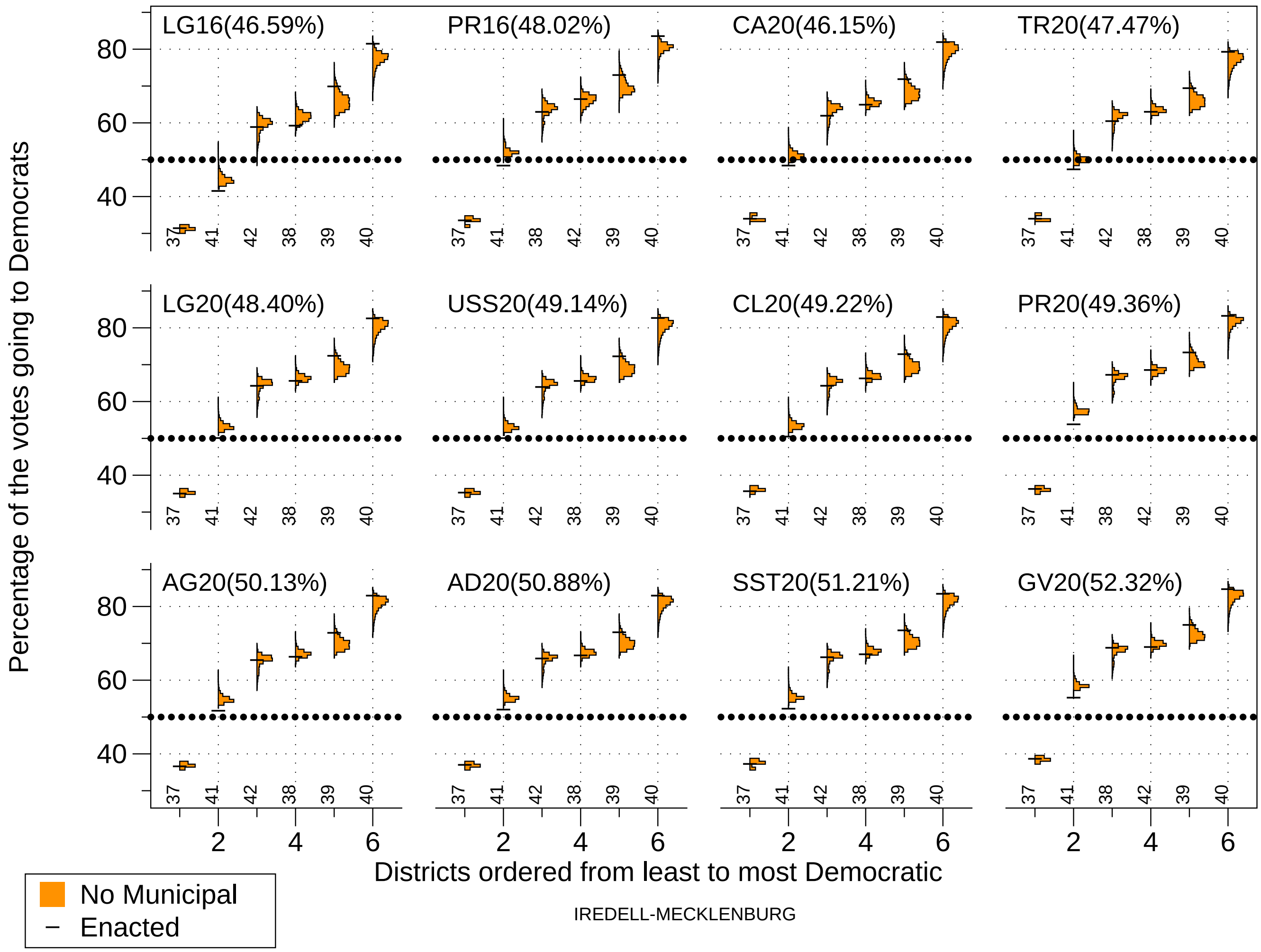


Figure 6.2.2: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the NC Senate Secondary ensemble which does not explicitly preserves municipalities.

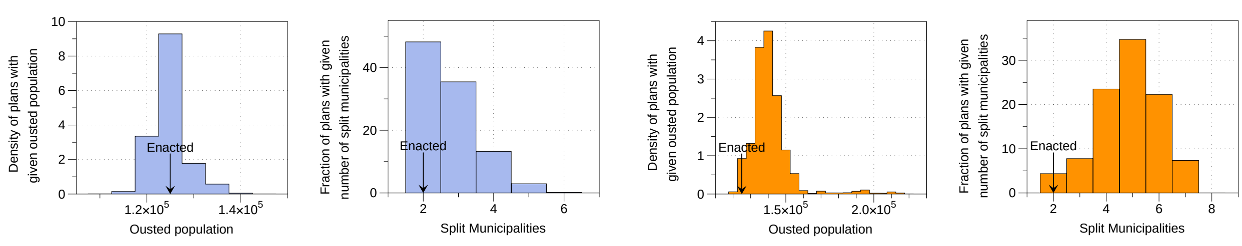


Figure 6.2.3: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.2.2 Granville-Wake

The enacted plan is chosen to be at the extreme edge of the ensemble. It maximizes the chance of the Republicans winning Districts 17 and 18 by packing a larger than typical number of Democrats in districts 14, 15, 16, and 18. The effect is shown in Figure 6.2.4 across the 12 elections. For each of the 2020 and 2016 elections we have consider, we found that none of

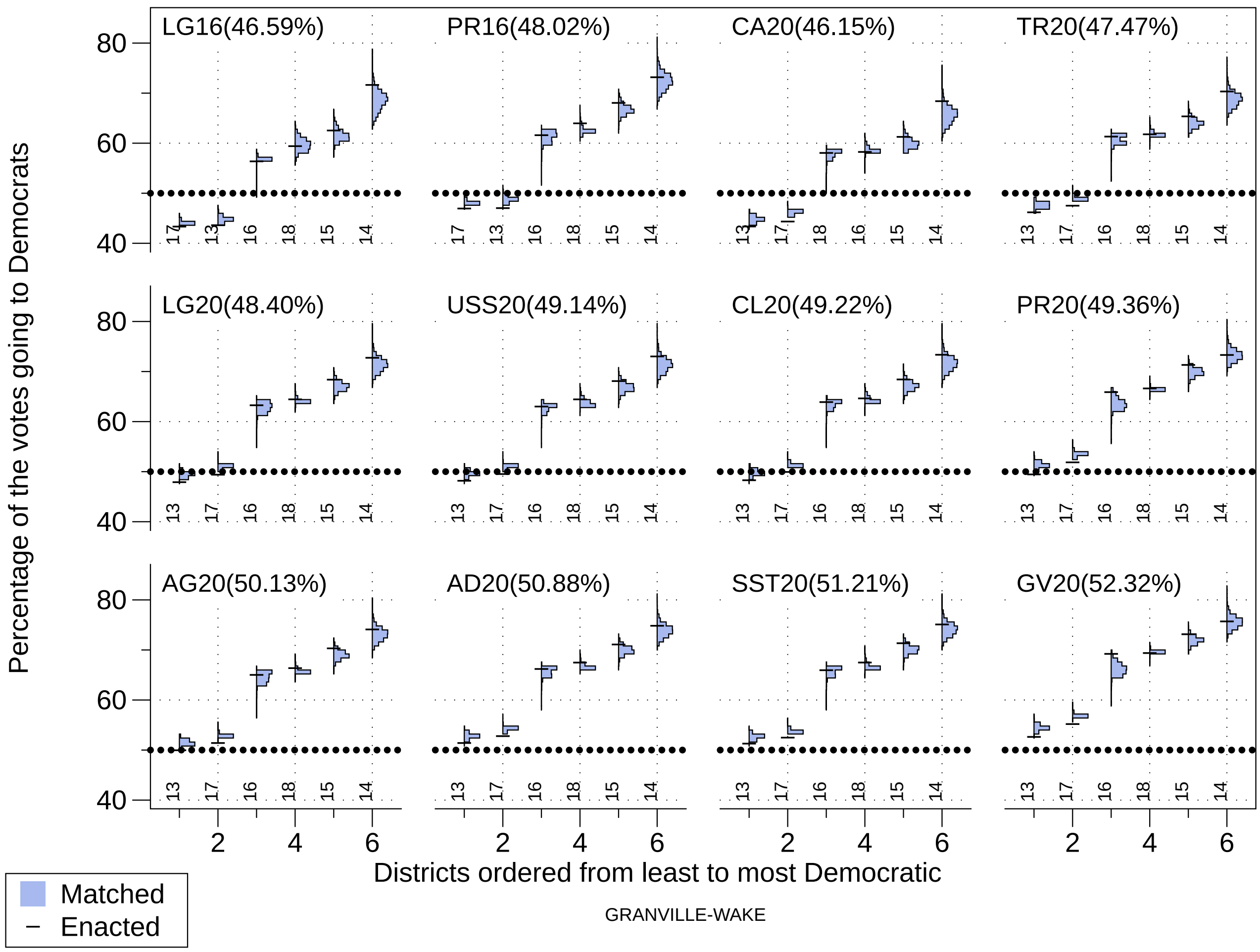


Figure 6.2.4: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

approximately 40,000 plans in our ensemble had as low a fraction of Democrats in the two most Republican districts in the Granville-Wake cluster as the enacted plan. Similarly, in six of the elections, the ensemble has no plans with more democrats packed in the four most Democratic districts. In six elections at most 0.022% of the plans had a higher fraction of Democrats packed in the four most Democratic districts than the enacted plan.

In this cluster, the prioritization of municipal preservation has a dramatic effect of packing Democrats in four districts and Republicans into two districts. The effect is show in Figure 6.2.5 across the 12 elections.

Municipal Splits and Ousted Population:

We see that in the Granville-Wake cluster, the number of ousted people in the enacted plan is significantly more than the number of ousted people in the ensemble prioritizing municipalities. The enacted plan splits three municipalities which coincides with the most typical number split by the ensemble prioritizing municipalities. Though this ensemble sometimes splits a number more municipalities, it typically displaces significantly fewer people than the enacted plan. From the perspective of the number of people ousted, the enacted plan is situated squarely between our ensemble prioritizing municipal preservation and that which does not.

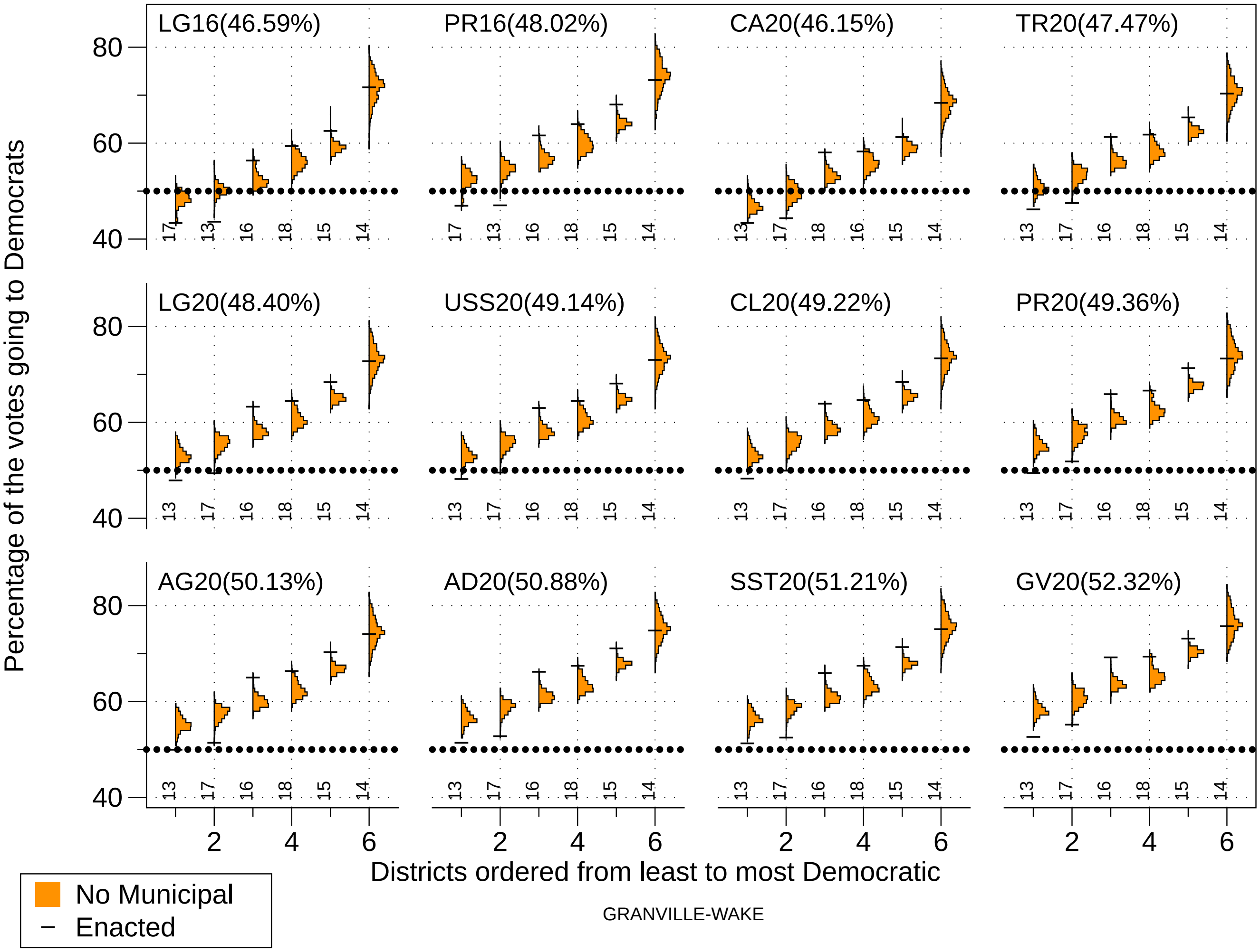


Figure 6.2.5: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The “ – ” on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the “ – ”. This plot uses the NC Senate Secondary ensemble which does not explicitly preserves municipalities.

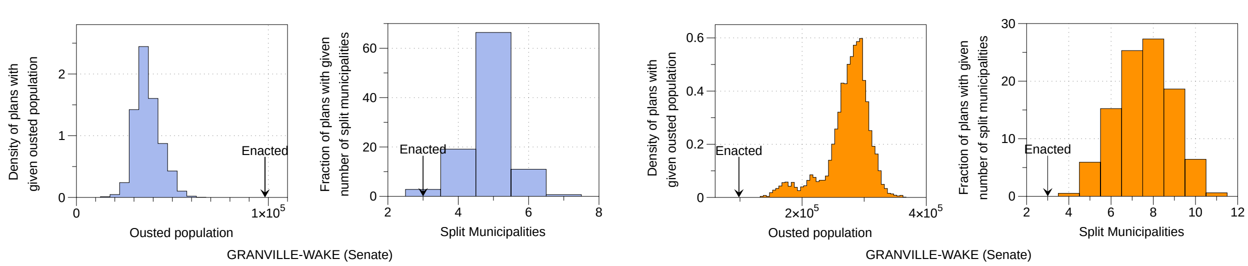


Figure 6.2.6: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.2.3 Forsyth-Stokes

There are only two districts in this cluster. The districts in the enacted plan are chosen to maximize the number of Democrats in the more democratic district and the number of republicans in the most Republican district. The map is an extreme outlier in both of these regards. The effect is a maximally non-responsive map. The effect is shown in Figure 6.2.7 across the 12 elections. Of the almost 80,000 maps in the ensemble, less than 1% had as low a fraction of Democrats in the most

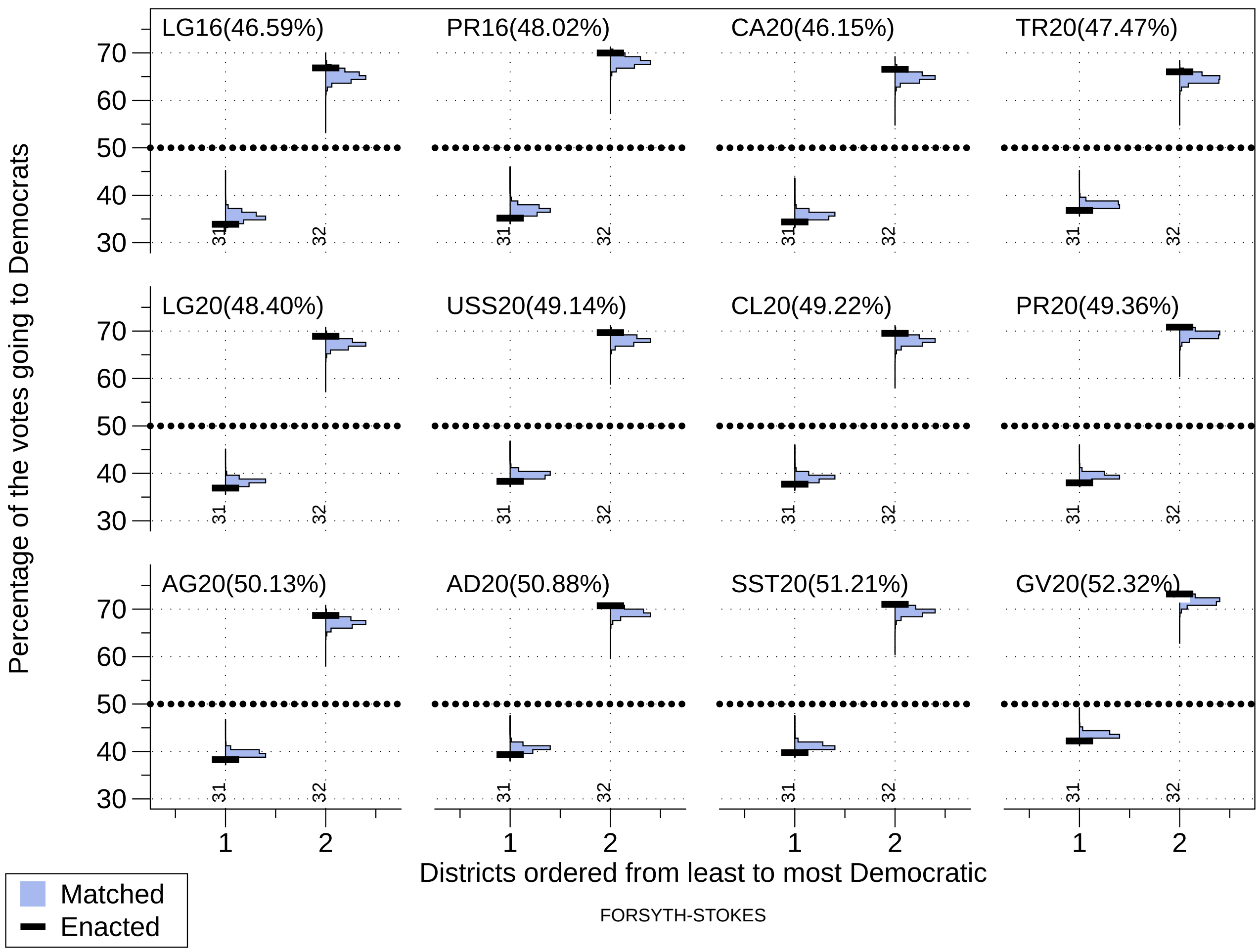


Figure 6.2.7: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

Republican district under the 2020 and 2016 elections considered. And between 1% and 5% of the plans had such a high Democratic fraction in the most Republican District.

When municipal preservation is not prioritized, the enacted map becomes an even more extreme outlier; showing an extreme level of packing of Democrats into one district and Republicans into the other. The effect is shown in Figure 6.2.8 across the 12 elections.

Municipal Splits and Ousted Population: In the Forsyth-Stokes Cluster we see that the number of people ousted from municipalities is comparable between the enacted plan and the municipality prioritizing ensemble. Additionally, the enacted plan splits one municipality which is the most common number of splits in the municipality prioritizing ensemble.

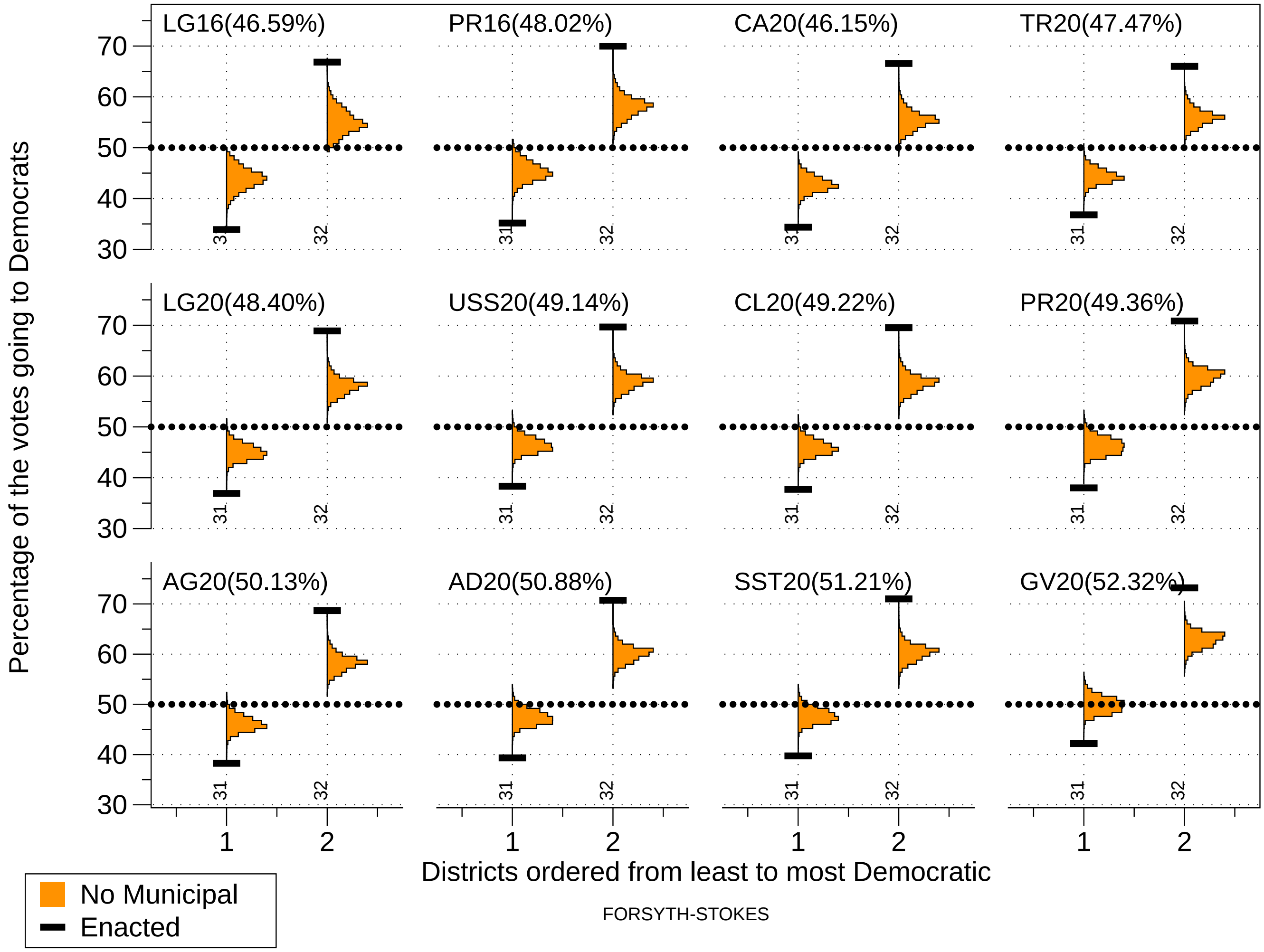


Figure 6.2.8: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the NC Senate Secondary ensemble which does not explicitly preserves municipalities.

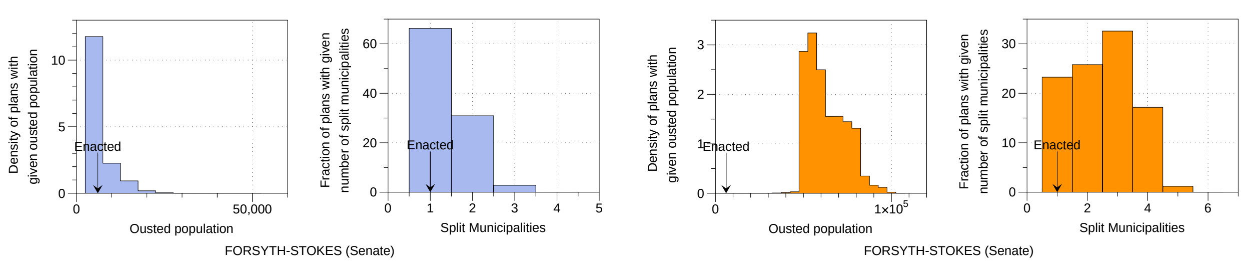


Figure 6.2.9: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.2.4 Cumberland-Moore

There are only two districts in this cluster. The districts in the enacted are chosen to maximize the number of Democrats in the more democratic district and the number of republicans in the most Republican district. The map is an extreme outlier in both of these regards. The effect is a maximally non-responsive map. The effect is shown in Figure 6.2.10 across the 12 elections. In each of the elections considered, no more than 0.06% of the ensemble plans have a lower fraction of Democrats

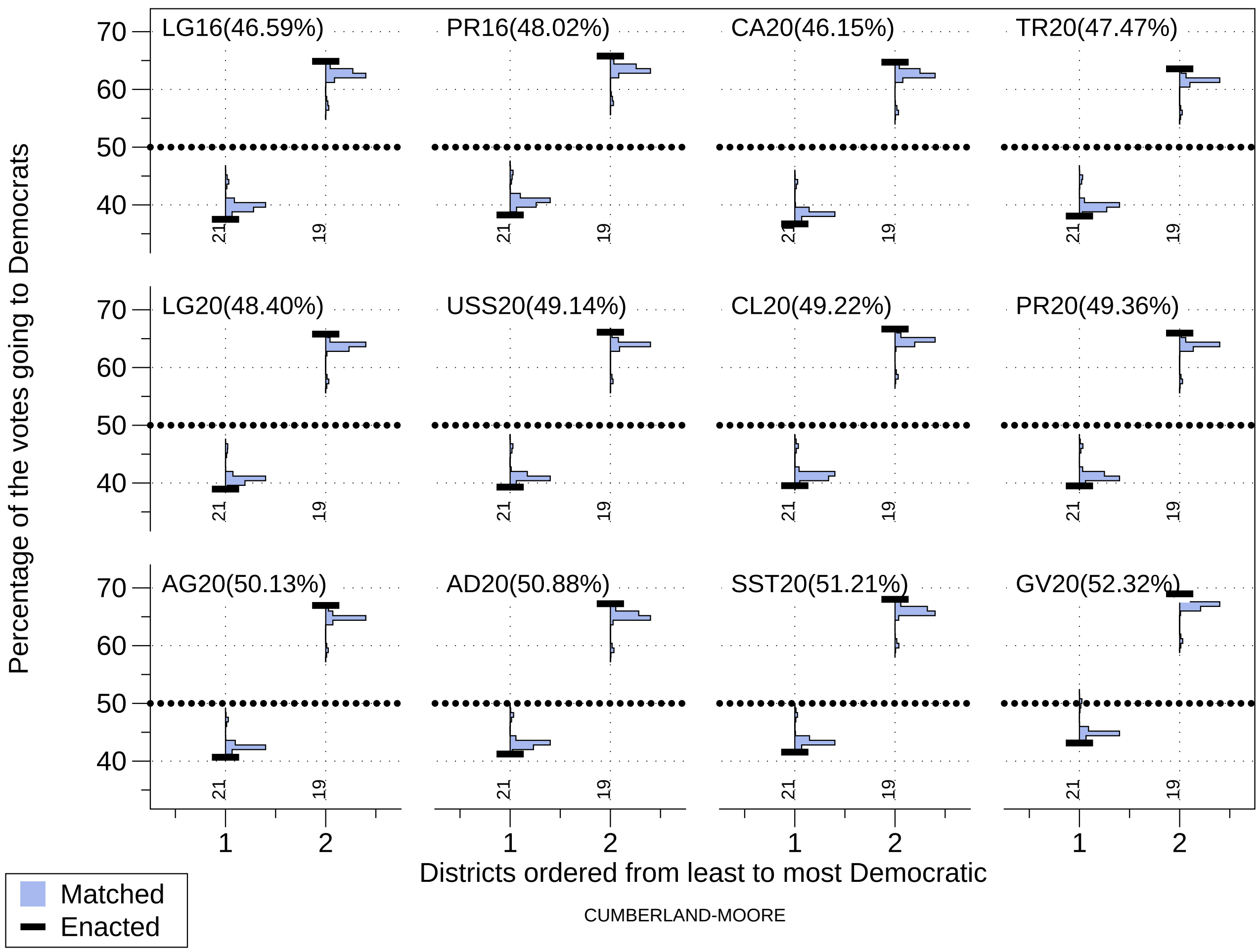


Figure 6.2.10: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

in the most Republican districts. Also no more than 0.06% of the ensemble plans have a higher fraction of Democrats in the most Democratic districts.

The prioritization of municipal preservation leads a dramatically less responsive pair of districts. When municipalities are less prioritized, both district have politically more centrist make up. Additionally, the more Republican district would regularly lean democratic without the prioritization of municipal preservation. The effect is show in Figure 6.2.11 across the 12 elections.

Municipal Splits and Ousted Population: In the Cumberland-Moore cluster, the enacted plan ousts a number of people close to the minimum number of ousted people seen in the ensemble prioritization municipal preservation. The enacted plan splits two municipalities which is the most common number of splits found in the ensemble prioritization municipal preservation.

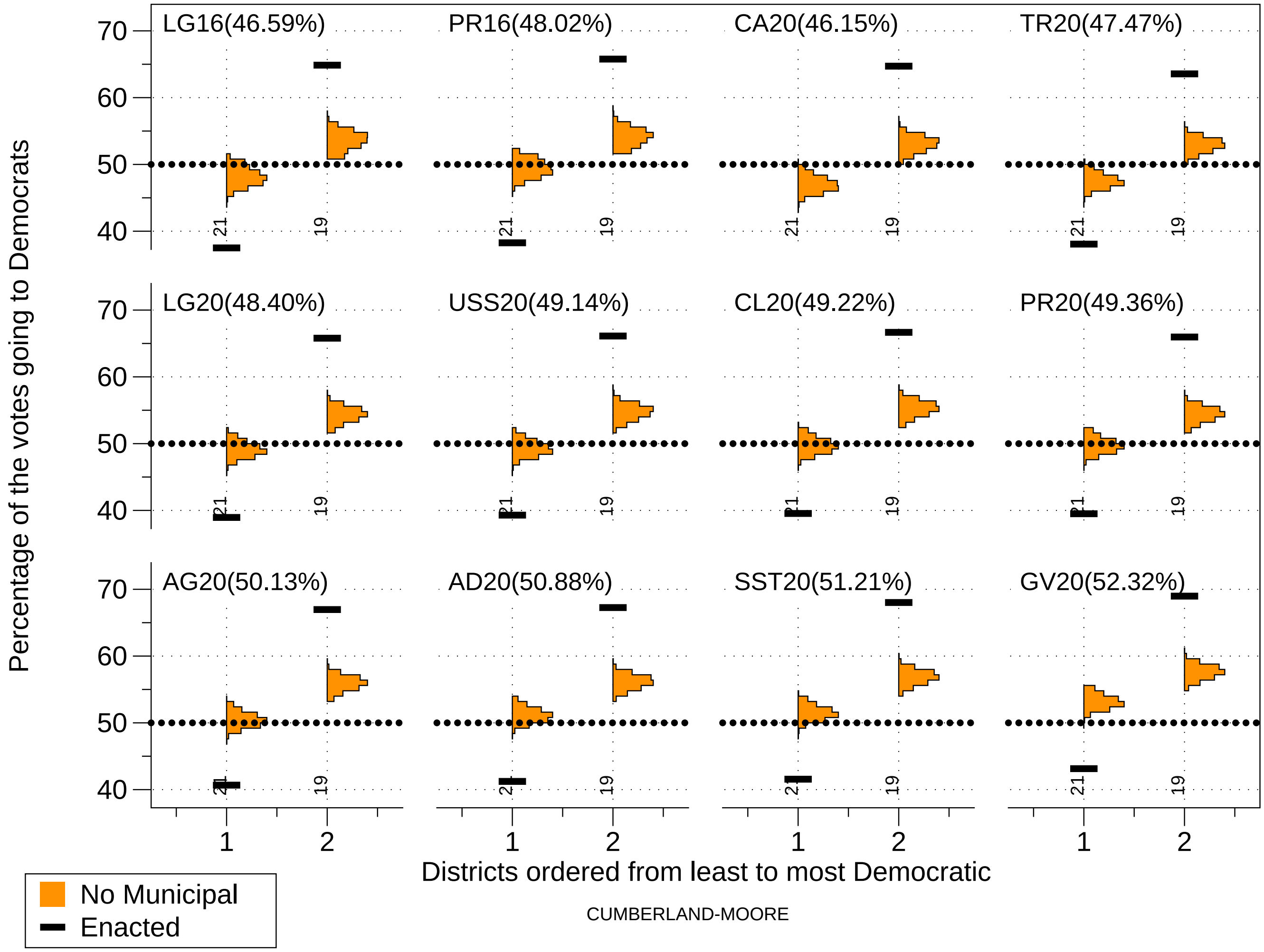


Figure 6.2.11: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The “ – ” on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the “ – ”. This plot uses the NC Senate Secondary ensemble which does not explicitly preserves municipalities.

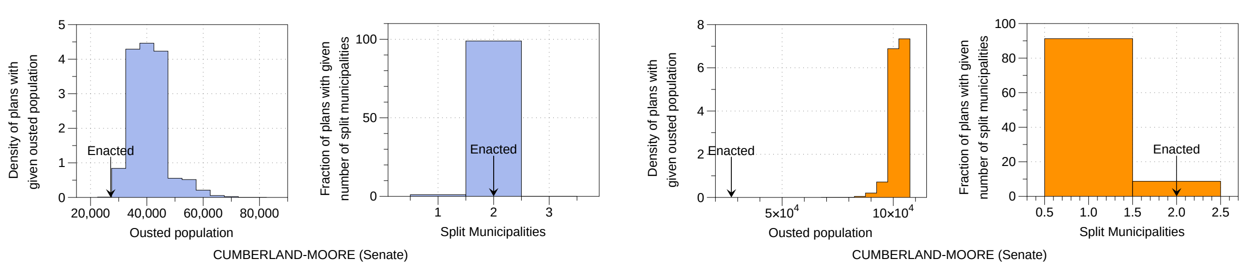


Figure 6.2.12: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.2.5 Guilford-Rockingham

The three districts in the Guilford-Rockingham cluster are constructed to pack an exceptional number of democrats in the most democratic district (district 28) and exceptionally few Democrats in the most Republican district (district 26). The effect is to ensure a Republican victory in the district 26, when in some elections the most republican district would be at risk of going to the Democratic Party. The effect is shown in Figure 6.2.13 across the 12 elections. In the Guilford-Rockingham

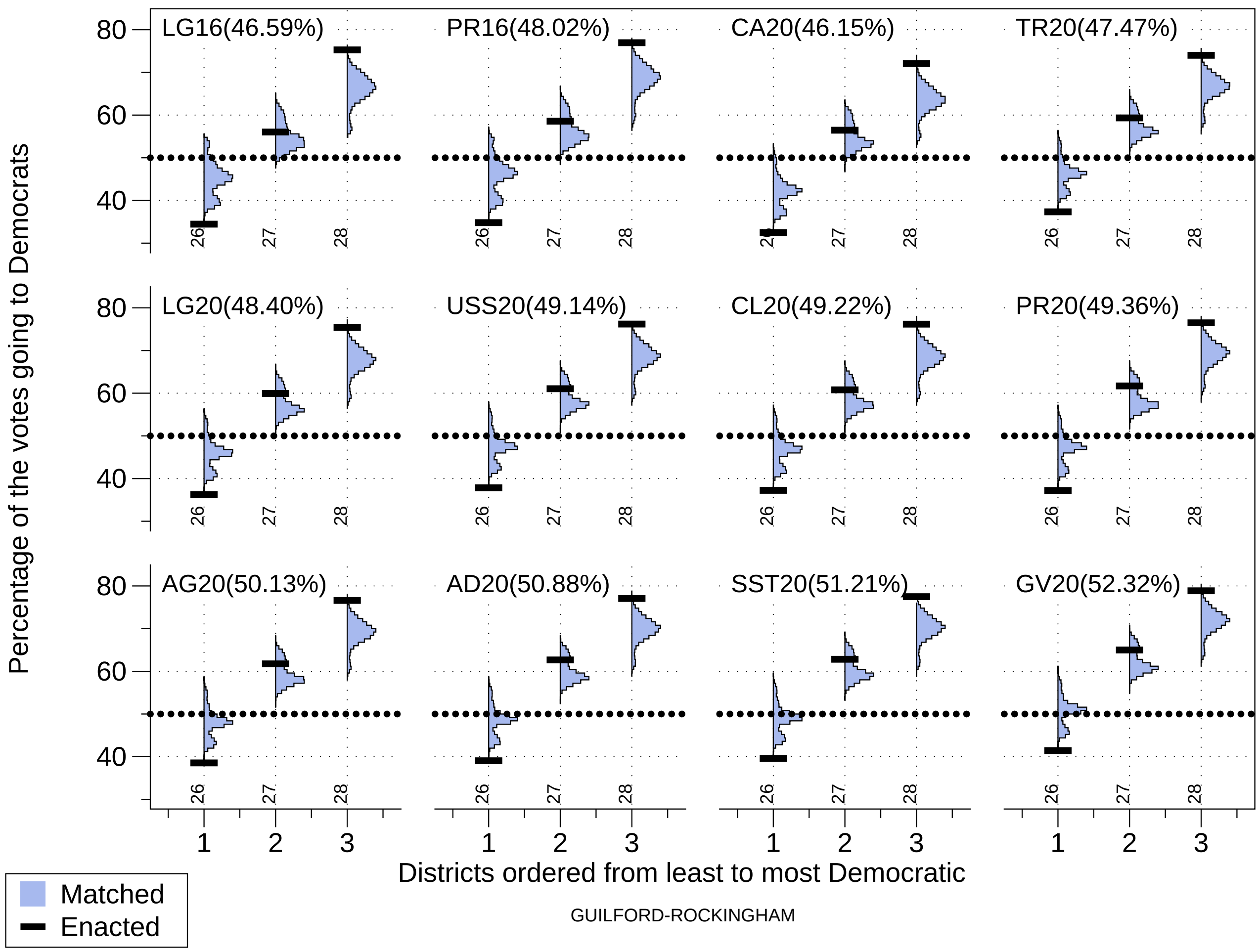


Figure 6.2.13: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

across all of the elections considered, none of the plans have lower fraction of Democrats in the most Republican district than the enacted plan. Conversely, in none of the elections considered do more than 0.08% of the plans have more Democrats packed in the most Democratic district than the enacted plan.

When municipalities are prioritized less, the effect is even more dramatic. In that setting, the extreme number of Democrats packed into the most democratic district and Republicans into the most Republican distinct is even more extreme. The effect is shown in Figure 6.2.14 across the 12 elections.

Municipal Splits and Ousted Population: In the Guilford-Rockingham cluster, the enacted plan splits one municipality and ousts a number of people which is typically found in the ensemble prioritizing municipality preservation which has an average ousted population which is slightly higher than the enacted plan.

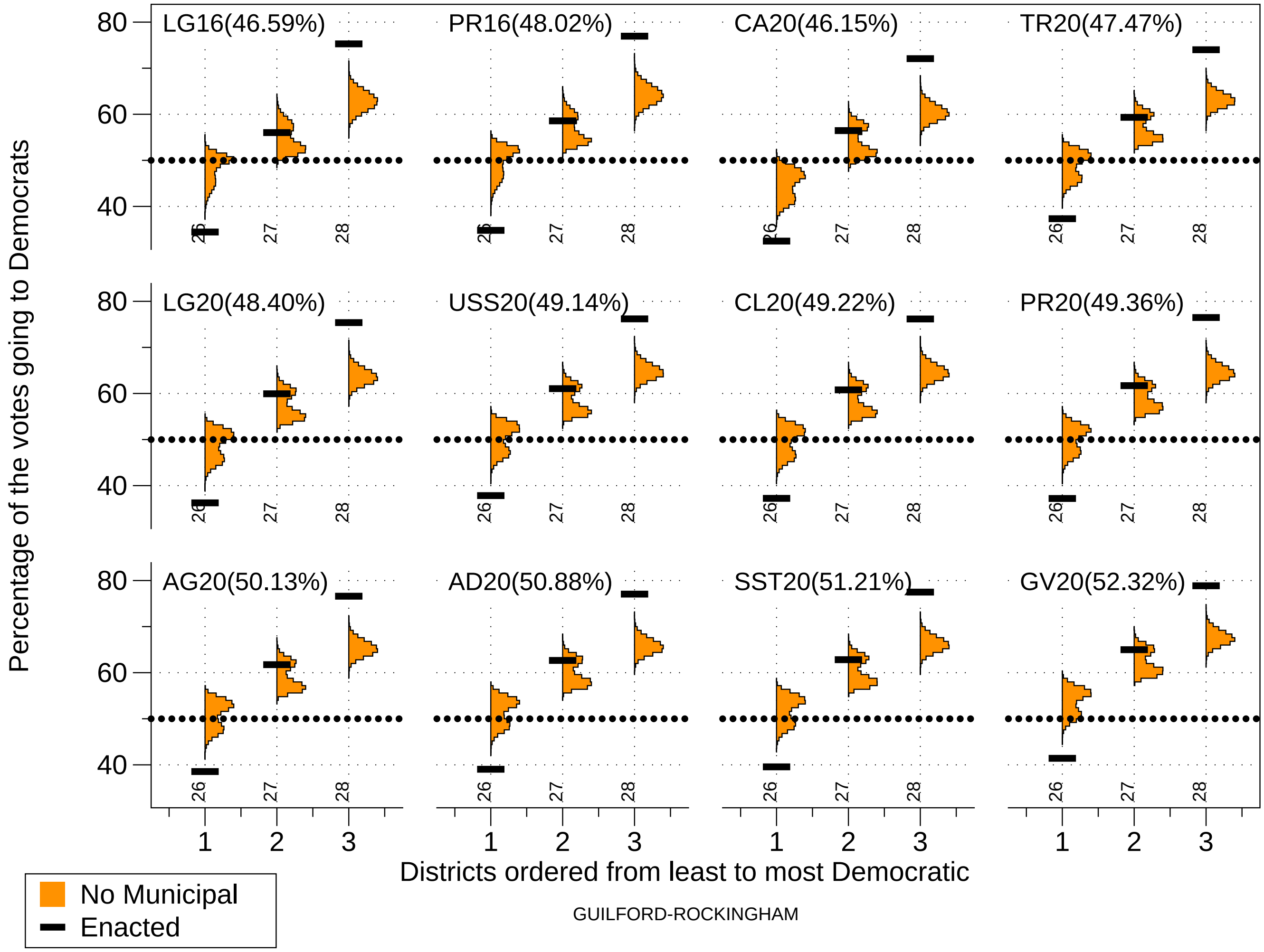


Figure 6.2.14: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the NC Senate Secondary ensemble which does not explicitly preserves municipalities.

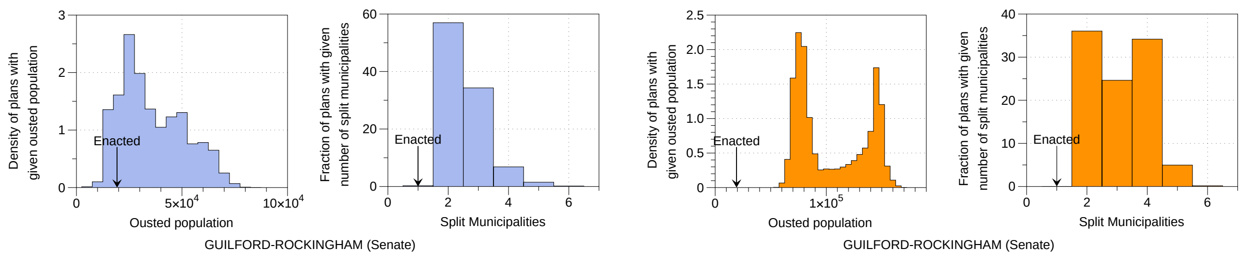


Figure 6.2.15: Plots showing the distribution of the number of people ousted from municipalities in this cluster under the primary and secondary ensemble. The amount of people ousted by the enacted map is also shown.

6.2.6 Northeastern County Cluster

In the NC Senate, there is more than one possible group of county clusters in the northeast corner of the state. As described in Figure 4.3.1 from Section 4.3, there is a choice between two different groups of county clusters. Each group consists of two different county clusters. Based on their population, each of these clusters has only one district. Thus, there is no choice on how to redistrict this region once the county grouping is set. We now explore partisan implications of choosing one county grouping over the other. As shown in the table below, under the enacted county groupings, Republicans win both districts in every election we consider. By contrast, under the alternative county grouping, each party won one of the two districts under every election we consider.

	Enacted Cluster 1	Enacted Cluster 2	Alternative Cluster 1	Alternative Cluster 2
County Clusters	Martin, Warren, Halifax, Hyde, Pamlico, Chowan, Washington, Carteret	Gates Currituck Pasquotank Dare Bertie Camden Perquimans Hertford Tyrrell Northampton	Pasquotank, Dare, Perquimans, Hyde, Pamlico, Chowan, Washington, Carteret	Gates, Currituck, Camden, Bertie, Warren, Halifax, Hertford, Tyrrell, Northampton, Martin
Democratic Vote %(LG16)	46.07%	47.74%	38.51%	55.42%
Democratic Vote % (PR16)	45.60%	46.70%	37.83%	54.59%
Democratic Vote % (CA20)	42.28%	44.47%	36.48%	50.75%
Democratic Vote % (USS20)	45.31%	45.36%	38.45%	52.75%
Democratic Vote % (TR20)	44.12%	44.58%	37.61%	51.59%
Democratic Vote % (GV20)	46.79%	47.56%	40.75%	54.12%
Democratic Vote % (AD20)	47.79%	47.72%	41.02%	54.99%
Democratic Vote % (SST20)	47.56%	47.85%	41.03%	54.89%
Democratic Vote % (AG20)	45.88%	46.11%	39.15%	53.40%
Democratic Vote % (PR20)	44.09%	45.54%	38.30%	51.84%
Democratic Vote % (LG20)	43.80%	45.12%	37.74%	51.69%
Democratic Vote % (CL20)	45.23%	46.42%	39.12%	52.00%

Table 3: Voting History for the two different choices of county grouping northeast corner in the NC Sente.

7 State Legislature: Additional Details

7.1 State Legislature: Details on the Sampling Method

To effectively generate a representative ensemble of maps from the desired non-partisan distributions, we use the well-established method of parallel tempering. It allows one to effectively sample from a possibly difficult to sample distribution by connecting it to an easy to sample distribution through a sequence of intermediate "interpolating" distributions.

We connect our desired distributions to a distribution on redistricting plans that favors plans with a larger number of spanning trees. This alternative distribution satisfies the same constraints, however, it does not consider compactness nor municipal preservation. We make this choice because it can be effectively sampled using a variation on the Metropolized Multiscale Forest RECOM sampling algorithm outlined in [1, 2] coupled with the Metropolis-Hasting algorithm. Using Parallel Tempering, we interpolate between the desired distribution on redistricting and a distribution which is chosen so that the Markov Chain Monte Carlo algorithm converges to its target distribution quickly.

In sampling the interpolating ladder of distributions between the easier-to-sample distribution and our target distribution with the needed policy considerations, we use parallel tempering with a classical Metropolis-Hasting sampling scheme to sample each level of the interpolating ladder of distributions. As proposals in the Metropolis-Hasting sampling scheme, we use a mixture of the Multiscale Forest RECOM proposals and single node flip proposals, depending on what is appropriate for the distribution associated with the given level in the interpolation. The Multiscale Forest RECOM has a number of advantages. Its multiscale nature seems to provide improvements in computational efficiency and the global moves of RECOM lead empirically to faster mixing. Additionally, it can efficiently preserve counties and other groupings. Lastly, it can be effectively combined with the Metropolis-Hasting algorithm to produce an algorithm that samples from the specified distribution.

To facilitate mixing and for computational practicality, we often split the interpolating groups of manageable size, typically between 10 and 30 interpolating levels. Each grouping is then run to produce an ensemble at the top level which approaches; which is closer to the desired ensemble. This ensemble is then used as an *independent sample reservoir* to generate independent samples for the next group of interpolating levels. This process is repeated until the desired level is reached. We typically use between 60 and 100 interpolating levels in our sampling schemes. The number of plans sampled differs from cluster to cluster. We also sometimes group clusters together for sampling. Usually the number of samples in around 80,000 but in all cases we have check various empirical measure to evaluate if the sampling has converged and is well mixed.

7.2 State Legislature: Mathematical Description of Ensemble Distribution

In designing our distributions, we have chosen to define explicit distributions and then use an implementation of the Metropolis-Hastings algorithm to generate the ensemble. We feel this choice promotes transparency because an explicit distribution can better be discussed and critiqued. It also allows us to more explicitly translate the policy considerations into the ensemble.

In order to formally define our distributions, we consider the labeling ξ of the precincts of the map of NC with the number $\{1, \ldots, d\}$, where d is the total number of districts. So for the i-th precinct, $\xi(i)$ gives the district to which the precinct belongs. If we let $A_j(\xi)$ and $B_j(\xi)$ be respectively the surface area and perimeter (or length of the boundary) of the j-district then our compactness score is

$$J_{\text{compact}}(\xi) = \sum_{j=1}^{d} \frac{A_j(\xi)}{B_j^2(\xi)}.$$

Then the probability of drawing the redistricting ξ is

$$\text{Prob}(\xi) = \begin{cases} \frac{1}{Z} e^{-w_{\text{compact}} J_{\text{compact}}(\xi)} & \text{for } \xi \text{ which is allowable} \\ 0 & \text{for } \xi \text{ which is not allowable} \end{cases}$$

Here Z is a number that makes the sum of Prob(ξ) over all redistricting plans are equal to one.

The collection of allowable redistricting plans ξ is defined to be all redistricting plans which satisfy the following conditions:

1. all districts are connected
2. the populations of each district is within %5 of the ideal district population unless the district in the wake county cluster in the senate or the Craven-Carteret county cluster in the house.[2]
3. The number of split counties is minimized.
4. We minimize the occurrence of districts traversing county boundaries.

The second distribution includes a municipality score, $J_{MCD}(\xi)$. This score describes the number of people who have been displaced from a district that could have preserved the voters within their municipality, and is defined as

$$J_{MCD}(\xi) = \sum_{m \in M} \text{pop}_{oust}(\xi, m),$$

where M is the set of all MCDs, and $\text{pop}_{\text{oust}}(\xi, m)$ is the number of displaced people from the municipality m under the redistricting plan ξ. We define pop_{oust} in one way if the population of the municipality is less than the size of a district and another if it is greater.

[2]In the two exceptional clusters, it is impossible to draw districts that preserve precincts and also achieve population balance within 5%. For Wake in the senate, we sample with a deviation of 6% and generate an associated ensemble; past experience has shown that this does not create a partisan effect and we will be confirming this in follow on analyses. In Craven-Carteret, precinct 02 in Craven is the only precinct that connects the bulk of Craven with Carteret and it must be split to achieve population balance between the two districts within this cluster. We have examined the voting patterns when assigning this precinct to the district with the bulk of Craven or with all of Carteret and found minimal effects on the outcome.

If m has a population that is less than the population of a district, we consider the district that holds the most people from the municipality m as the representative district for that municipality. Any person within municipality m, but not within the representative district is considered to have been displaced.

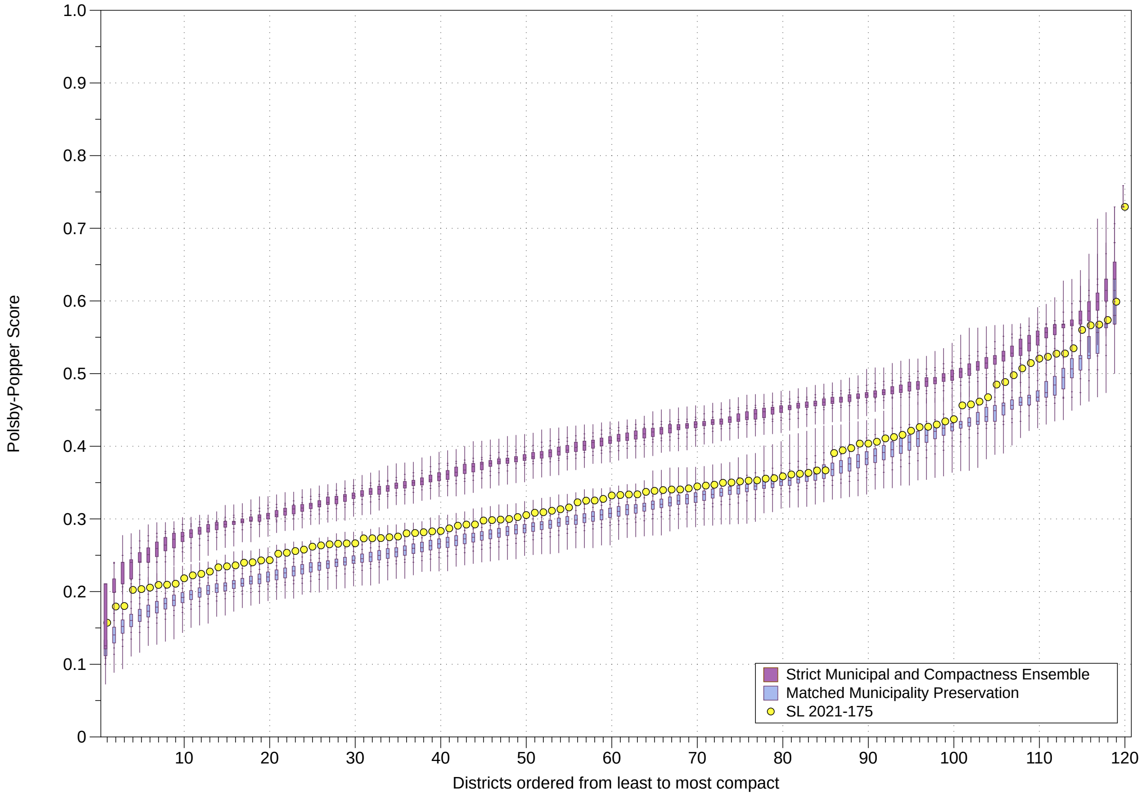
If m has a population that is greater than the population of a district, we consider the number of districts that could fit within m to be $d(m) = \lfloor \text{pop}(m)/\text{pop}_{\text{ideal}} \rfloor$, where $\text{pop}(m)$ is the population of the MCD m and $\text{pop}_{\text{ideal}}$ is the ideal district population. We also consider the remaining population in the municipality that cannot fit within a whole district to be $r(m) = \text{pop}(m) - d(m) \times \text{pop}_{\text{ideal}}$. To determine the displaced population, we look at the $d(m)$ districts that contain the largest populations from the municipality m. Hypothetically, everyone in these districts could live in the municipality m. Therefore, anyone who is in one of these districts and that does not live in the municipality m could be replaced by someone who does live in the municipality. Thus, we sum the number of people not in m in the $d(m)$ districts that contain the largest populations of m. We also note that the remaining population $r(m)$ could hypothetically be kept intact when drawing a $(d(m) + 1)$th district. We, therefore, look at the number of people in the municipality m who are living in the district with the $(d(m) + 1)$th most population of the municipality. If the number of people in m is less than $r(m)$, then we add this difference to the number of ousted people (since each of these people in the municipality could have conceivably been placed in the district).

Formally, we let the $|M| \times d$ matrix, $MCD(\xi)_{m,j}$ represent the number of people who are in the municipality m and the district ξ_j. Then

$$\text{pop}_{\text{oust}}(\xi, m) \begin{cases} \sum_j MCD(\xi)_{m,j} - \max_j(MCD(\xi)_{m,j}) & \text{pop}(m) < \text{pop}_{\text{ideal}} \\ \sum_{j \in D(m)} (\text{pop}(\xi_j) - MCD(\xi)_{m,j}(\xi)) & \text{pop}(m) \geq \text{pop}_{\text{ideal}} \\ \quad + \max(0, MCD(\xi)_{m,N(m)} - r(m)) & \end{cases},$$

where $\text{pop}(\xi_j)$ is the population of district ξ_j, $D(m)$ is the set of district indices that represent the $d(m)$ districts with the largest populations of municipality m, and $N(m)$ represents the district index with the $d(m) + 1$ most population of municipality m.

7.3 State Legislature: Additional Ensemble Statistics



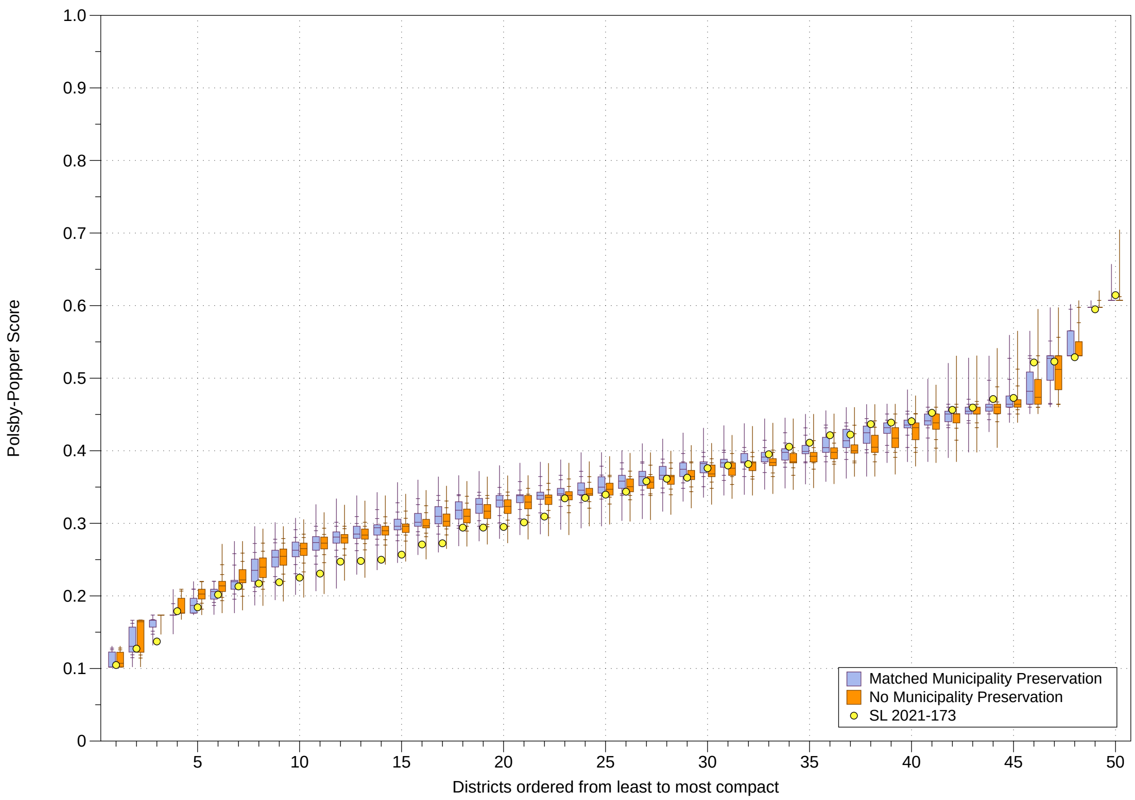


Figure 7.3.1: These plots compare the Polsby-Popper Score of the enacted maps (shown we the yellow dots) with the marginal histograms of the primary and secondary ensembles.

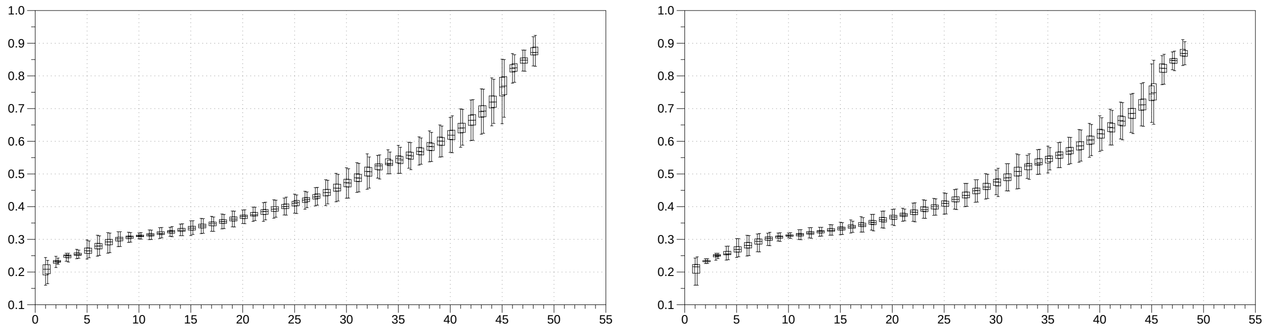


Figure 7.4.1: We compare a subset of the threads to the remaining threads. Each thread represents a different initial condition, and thus takes a different trajectory through the phase space. We compare our standard observables, such as the ranked ordered marginal distributions and confirm that they yield equivalent results. On the left we show an example of comparing one thread with all threads in a parallel tempering run; on the right we show an example of comparing half of the thread with the other half of the threads in a parallel tempering run.

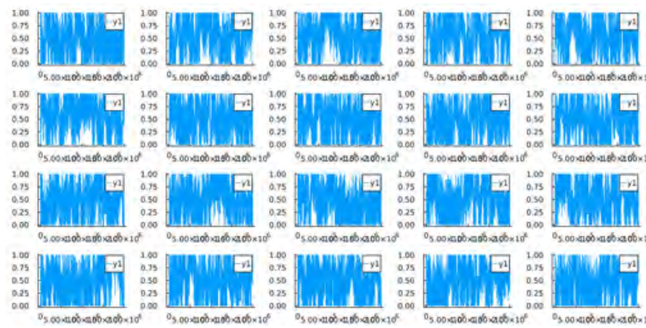


Figure 7.4.2: We examine how each of the parallel tempering threads swaps as a function of the proposal number. The vertical axis represents different measures and the horizontal axis represents the proposal in the Markov Chain. When the thread (or redistricting) is near the bottom of the vertical axis it mixes quickly when drawing from the reservoir; when it is at the top of the vertical axis it is at the desired measure which is either the desired measure we are sampling from or an intermediate measure that will act as a subsequent reservoir.

7.4 State Legislature: Convergence Tests

We performed a number of tests to assess if our sampling of the desired distribution was sufficient to provide an accurate representation of the desired distribution. Sometimes many samples are needed, yet in other cases a much smaller number is sufficient. We use a number of different methods to assess convergence.

Many of our runs were generated with an implementation of the *parallel tempering algorithm* with an *independent sample reservoir*. The use of parallel tempering provides a number of different threads that can be grouped and then compared against each other. As each thread starts from a different initial condition, if the distributions look similar then there is evidence that the system is mixing. Similarly, if a subset of the threads has a similar distribution to all of the threads, then there is evidence that enough samples were used.

The following plots show representative ranked ordered histograms for some NC House and NC Senate runs where different threads in a parallel tempering run are compared.

Each time a thread exchanges its state with the independent sample reservoir, it receives a new configuration that is independent of the previous state of the system. Additionally, if the thread then progresses up to the parameter level of interest, then we have strong evidence that we are producing decorated samples. The following plots show the current level of each for the different threads in a parallel tempering run. Switching regularly from the highest level (the desired sample distribution) to the lowest level (the level with the independent sample reservoir) is a strong indication that the system will be well mixed and converged.

In some cases, we run two or more complete sampling runs for the same target distribution. If the ensembles generated are close then we have strong evidence that the ensembles are converged as each run started from different initial conditions and used different randomness.

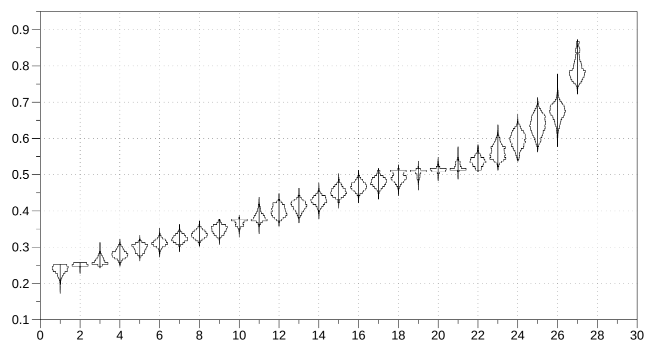


Figure 7.4.3: We compare the ranked ordered marginals on two independent parallel tempering runs.

8 Congressional Plan

As with the NC House and NC Senate plans, we place a probability distribution on Congressional plans for North Carolina. The distributions embody different policy choices. With each distribution, we produce representative ensembles of maps to serve as benchmarks against which to compare specific maps. The ensembles are generated by using the Metropolis-Hasting Markov Chain Monte Carlo Algorithm in a parallel tempering framework which employs the proposal from the Multiscale Forest RECOM algorithm [2, 1].

This analysis parallels the analysis already presented for the NC House and NC Senate with the simplification that we no longer need to consider County Clusters and that some of the criteria are modified. The details are given in Sections 8.1 and 7.2.

8.1 Congressional: Ensemble Overview

Similarly to the distribution placed on the NC Legislative redistricting plans in Section 4.4, we consider a distribution (and hence an ensemble) satisfying the following constraints:

- The maps split no more than 14 counties.
- The maps split no county into more than two districts.
- Districts traverse counties as few times as possible.
- All districts are required to consist of one contiguous region.
- The deviation of the total population in any district is within 1% of the ideal district population. We have verified in previous work in related settings that the small changes needed to make the districting plan have perfectly balanced populations do not change the results. (See [7] and the expert report in Common Cause v. Rucho).
- **Compactness**: The distributions on redistricting plans are constructed so that a plan with a larger total isoperimetric ratio is less likely than those with a lower total isoperimetric ratio. The total isoperimetric ratio of a redistricting plan is simply the sum of the isoperimetric ratios over each district. The isoperimetric ratio is the reciprocal of the Polsby-Poper score; hence, smaller isoperimetric ratio corresponds to larger Polsby-Poper scores. As the General Assembly stated in its guidance that the plans should be compact according to the Polsby-Popper score [9], we tuned the distribution so that it yields plans of a similar compactness to those of the legislature. (See Figure 10.2.1 in Section 10.2.) We further limited our distribution only to include those with an Isoparametric score less than 80.

  The legislature also listed the Reock score as another measure of compactness which one could consider. However, we have found Polsby-Popper/isoperimetric score to be a better measure when generating districts computationally. In our previous work, we have seen that this choice did not qualitatively change our conclusions (see [7] and the expert report in Common Cause v. Rucho).

8.2 Congressional Plan: Sampling Method

We have chosen the distribution from which to draw our ensemble to comply with the desired policy and legal considerations. It is well accepted that not all distributions on possible redistricting plans are equally easy to sample from.

As discussed in Section 7.1 to effectively generate a representative ensemble of maps from these distributions, we use the well-established method of parallel tempering. It allows one to effectively sample from a possibly difficult to sample distribution by connecting it to an easy to sample distribution through a sequence of intermediate "interpolating" distributions.

We connect our desired distributions, which includes a compactness score, to a measure on redistricting plans which is uniform on spanning forests which satisfy the population and county constants. Furthermore, the enacted plan can be effectively sampled using a variation on the Metropolized Multiscale Forest RECOM sampling algorithm outlined in [1, 2].

In sampling the interpolating ladder of distributions between the easier-to-sample measure and our target measure which includes a compactness score, we use parallel tempering with a classical Metropolis-Hasting sampling scheme to sample each level of the interpolating ladder of distributions. As proposals in the Metropolis-Hasting sampling scheme, we use Multiscale Forest RECOM proposals. We sample around 80,000 plans have confirmed that the distribution seems well mixed and than it has been sufficiently sampled to provide stable statistics.

8.3 Election Data Used in Analysis

The same historic elections and abbreviations were use to analyze the congressional plan and ensemble as were used for the NC legislative maps and ensemble. See Section 4.6.

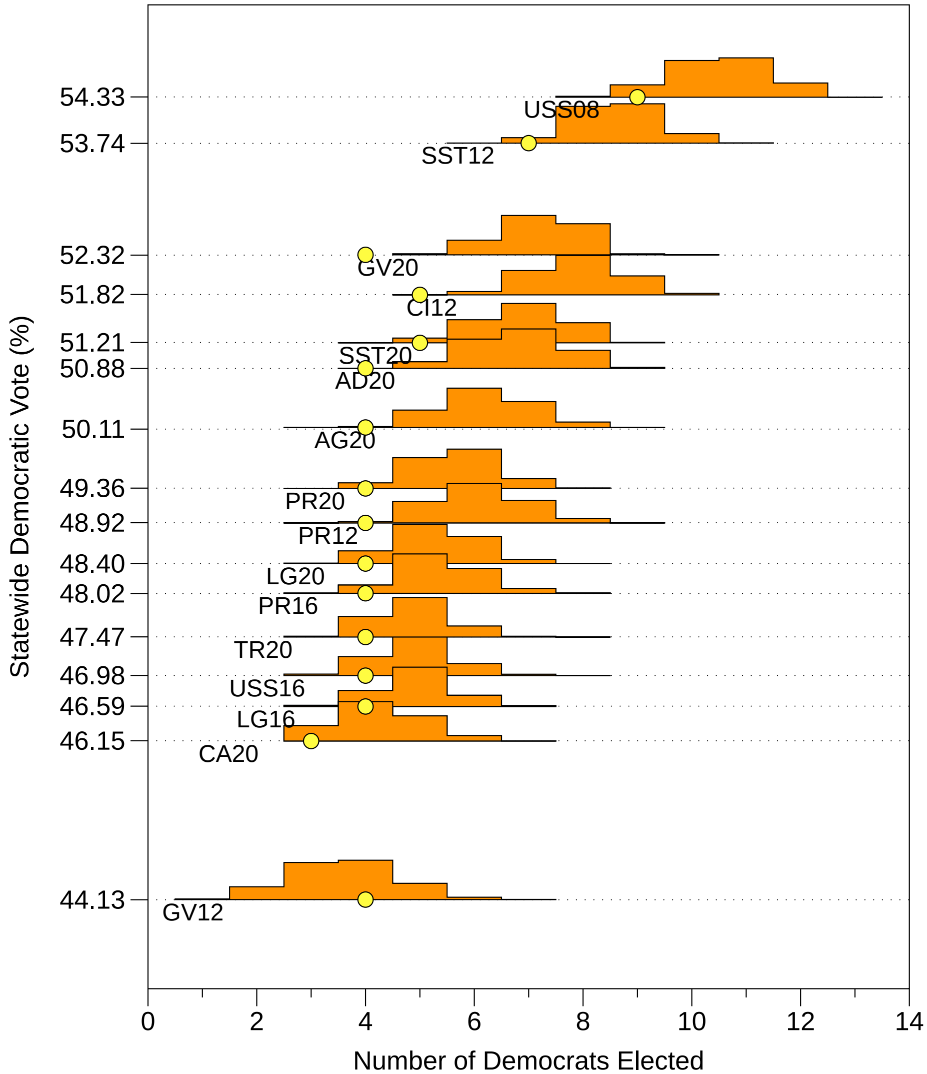


Figure 9.0.1: Each histogram represents the range and distribution of possible Democratic seats won in the ensemble of plans; the height is the relative probability of observing the result. The yellow dots represent the results from the enacted congressional plan under the various historic votes.

9 Congressional Plan: Main Analysis

Figure 9.0.1 gives the *Collected Seat Histograms* for the ensemble sampled from the distribution. This figure also shows how many Democrats the enacted congressional plan would have elected under the votes from a variety of historic elections.

Without reference to a particular ensemble, a primary message of this plot is that the enacted congressional plan is largely stuck electing 4 of 14 Democrats despite large shifts in the statewide vote fraction and across a variety of election structures. Over the statewide vote Democratic partisan vote range of 46.59% to 52.32%, the enacted map only twice changes the number of Republicans elected. The outcome of the election is largely stuck at 4 Democrats. This shows the enacted map to be highly non-responsive to the changing opinion of the electorate. Without holding the election one largely knows that the result will be 10 Republicans and 4 Democrats.

This non-responsiveness is not observed in the ensemble. The ensemle shows that a typical map drawn without political considerations gradually shift from 4-5 Democrats typically being elected at one end of this regime to 7-8 being elected at the other end. Hence, under historic elections in which Democrats win 46% to 53% of the statewide vote, a typical map would gradually shift from around 4 Democrats in the NC congressional delegation to around 8 Democrats as the electorate changed is vote. This does not happen under the enacted plan with the elections considered. Instead, as described above, the enacted map sticks at only 4 Democrats in North Carolina's congressional delegation under nearly all of these elections.

To better illuminate the structure responsible for making the enacted map an extreme outlier, we turn to the Rank Ordered Box plots already discussed in general in Section 3.4 and in the context of the state legislative maps in the previous sections. The plots show extreme packing of Democrats in the three most Democratic districts and depletion of Democrats from the

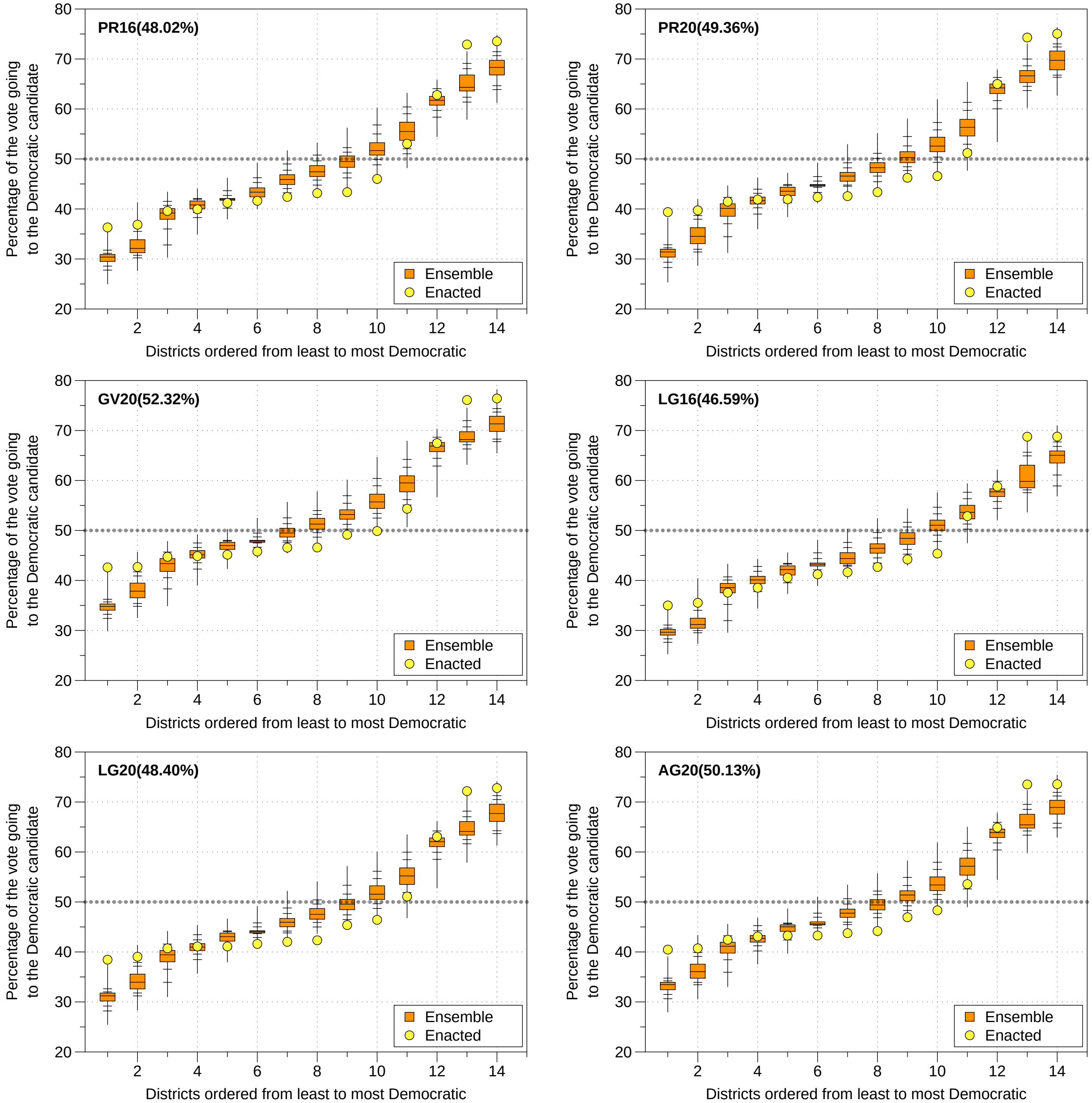


Figure 9.0.2: The Ranked Marginal Box-plots for the NC Congressional Plan. The ranked ordered marginals for the enacted map are shown in yellow. 50% of the ensemble is contained within the box. Inside the first pair of tick marks is 80% of the data and inside the second set is 95% of the points.

next 7 to 9 most Democratic districts. The effect of this cracking and packing is the non-responsiveness seen in Figure 9.0.1.

Motivated by the cracking and packing of Democrats shown in Figure 9.0.1, we ask how common is such a highly polarized districts in our non-partisan ensemble of maps. The results are summarized in Table 4. They show that the Congressional map is not only non-responsive to the changing preferences of the electorate but it is also an extreme partisan gerrymander. Maps which lock in such an extreme partisan outcome do not occur in our ensemble.

Election	Plans with the same or more Dem (1-2)	Plans with the same or more Rep (5-11)	Plans with the same or more Dem (12-14)	Total Plans
LG16	18	0	0	79997
PR16	0	0	0	79997
CA20	0	0	0	79997
TR20	0	0	0	79997
LG20	0	0	0	79997
USS20	0	0	0	79997
CL20	0	0	0	79997
PR20	0	0	0	79997
AG20	0	0	0	79997
AD20	0	0	0	79997
SST20	0	0	0	79997
GV20	0	0	0	79997
CI20	0	0	0	79997
USS16	0	0	0	79997
GV16	1	0	0	79997
AG16	15	0	0	79997

Table 4: Over the approximately 80,000 plans in our ensemble, we ask how many plans have (1) as high Democratic fraction in the two most Republican districts, (2) as small a fraction of Democrats in the 5th through 11th most Republican districts, and (3) have as high a Democratic fraction in the 12th through 14th most Republican districts. The answer is given in this table along with the total number of plans in our ensemble.

10 Congressional: Additional Details

10.1 Congressional Plan: Mathematical Description of Ensemble Distribution

In specifying our distribution, we have chosen to define explicit distributions and then use an implementation of the Metropolis-Hastings algorithm to generate the ensemble. We feel this choice promotes transparency because an explicit distribution can better be discussed and critiqued. It also allows us to more explicitly translate the policy considerations into the ensemble.

In order to formally define our distributions, the partition of the precinct adjacency graph into a spanning forest $\mathcal{T}$ with 14 district trees $\{\mathcal{T}_1, \cdots, \mathcal{T}_{14}\}$ corresponding to each district. Hence $\mathcal{T} = \{\mathcal{T}_1, \cdots, \mathcal{T}_{14}\}$ completely specifies the redistricting.

If we let $A_j(\mathcal{T})$ and $B_j(\mathcal{T})$ be respectively the surface area and perimeter (or length of the boundary) of the j-district then our compactness score is

$$J_{\text{compact}}(\mathcal{T}) = \sum_{j=1}^{14} \frac{A_j(\mathcal{T})}{B_j^2(\mathcal{T})}.$$

Then the probability of drawing the spanning forest $\mathcal{T}$ is

$$\text{Prob}(\mathcal{T}) = \begin{cases} \frac{1}{Z} e^{-w_{\text{compact}} J_{\text{compact}}(\mathcal{T})} & \text{for } \mathcal{T} \text{ which is allowable} \\ 0 & \text{for } \mathcal{T} \text{ which is not allowable} \end{cases}$$

Here Z is a number which makes the sum of Prob($\mathcal{T}$) over all spanning forests with 14 trees equal to one.

The collection of allowable spanning forests $\mathcal{T}$ is defined as those which produce redistricting plans which satisfy the following conditions:

1. all districts are connected
2. the populations of each district is within %1 of the ideal district population.
3. No more than 14 counties are split with no county split more once.
4. We minimize the occurrence of districts traversing county boundaries.

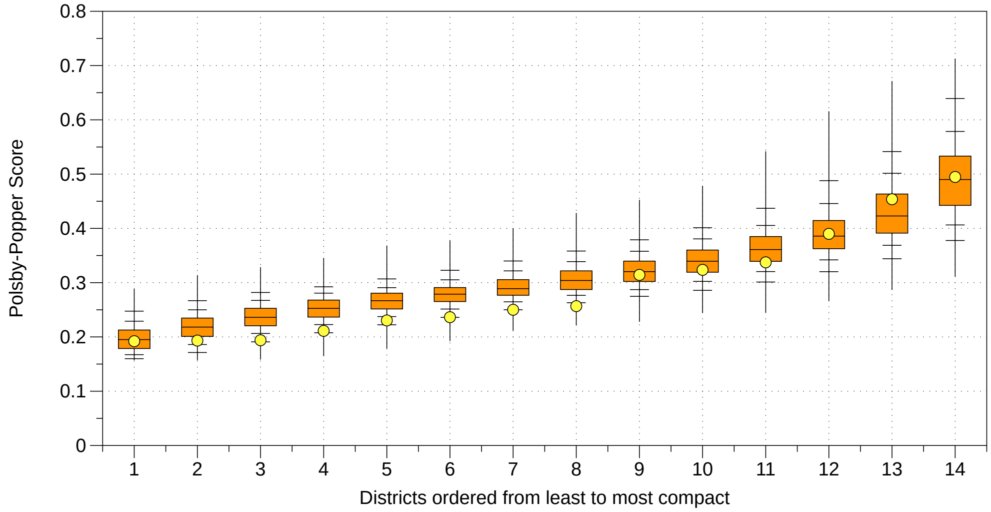


Figure 10.2.1: The yellow dots display the ordered Polsby-Popper score of the 14 districts in the enacted plan.

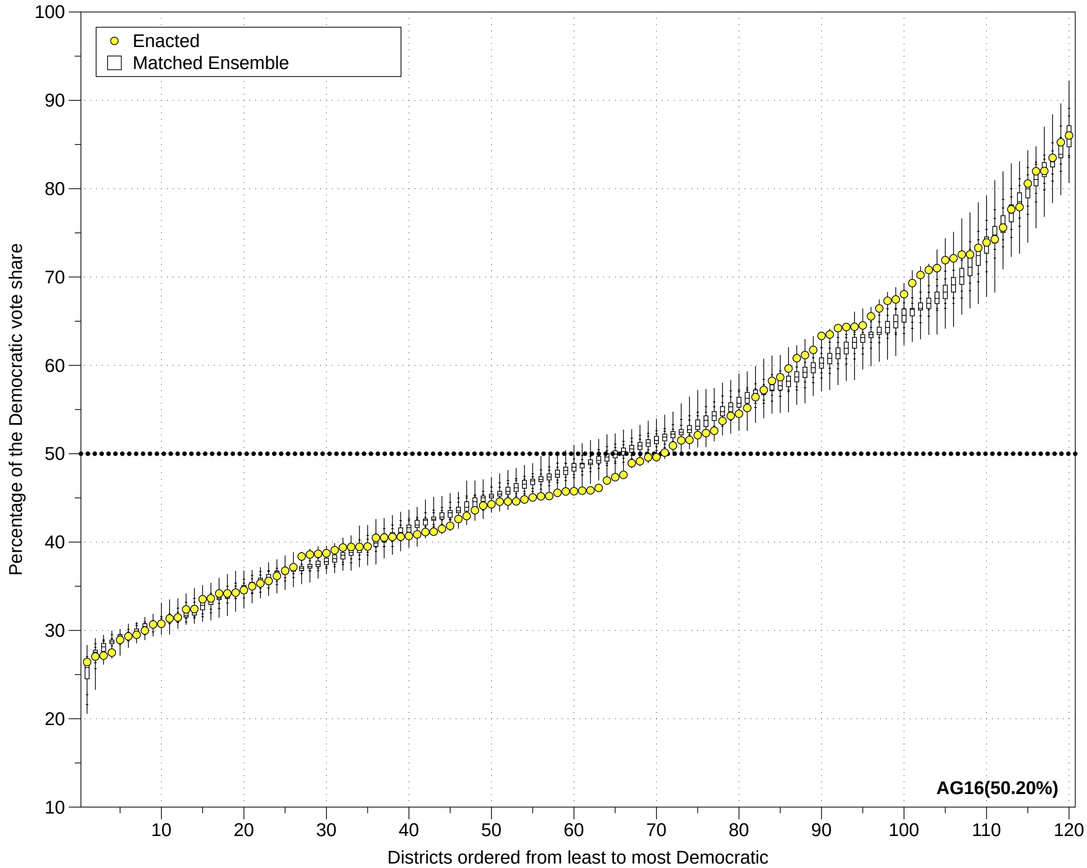
10.2 Congressional Plan: Additional Ensemble Statistics

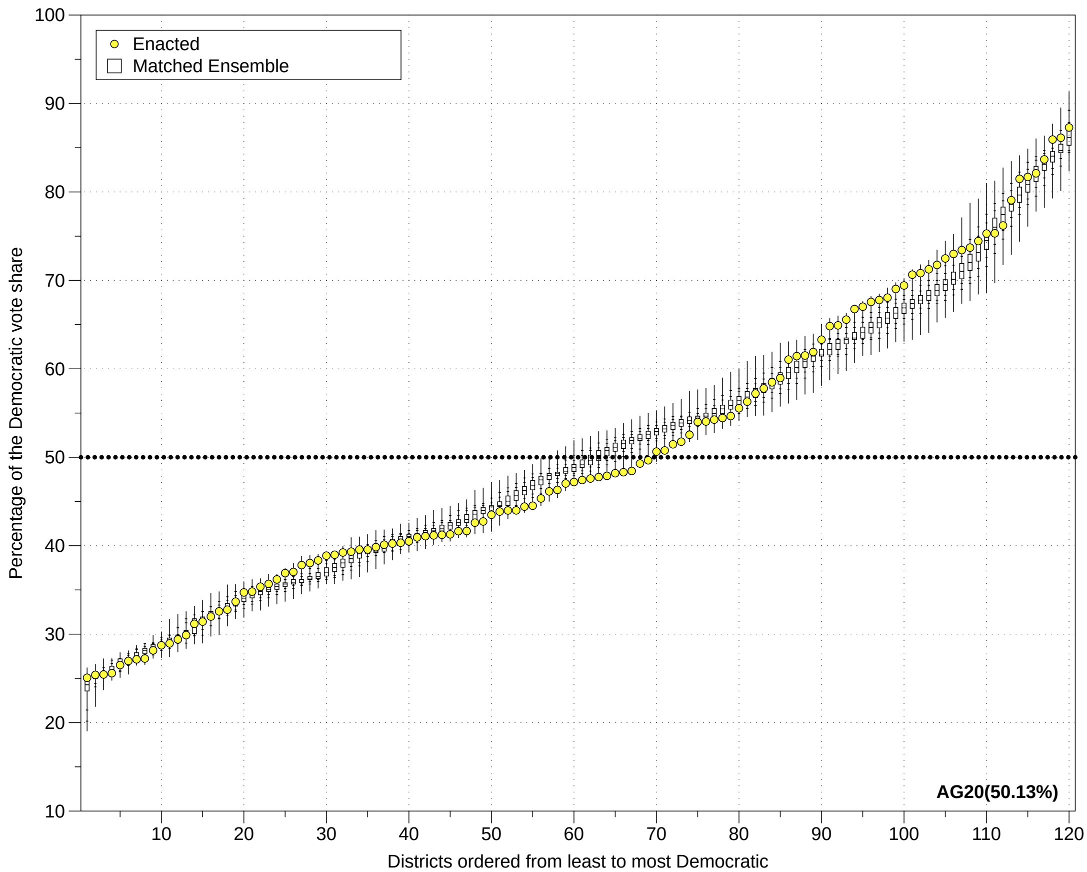
In Figure 10.2.1, we give the box-plots for the ranked ordered marginal distribution for the compactness score, namely the Polsby-Popper score (see companion methods document). We compare the ensemble of plans with the enacted plan.

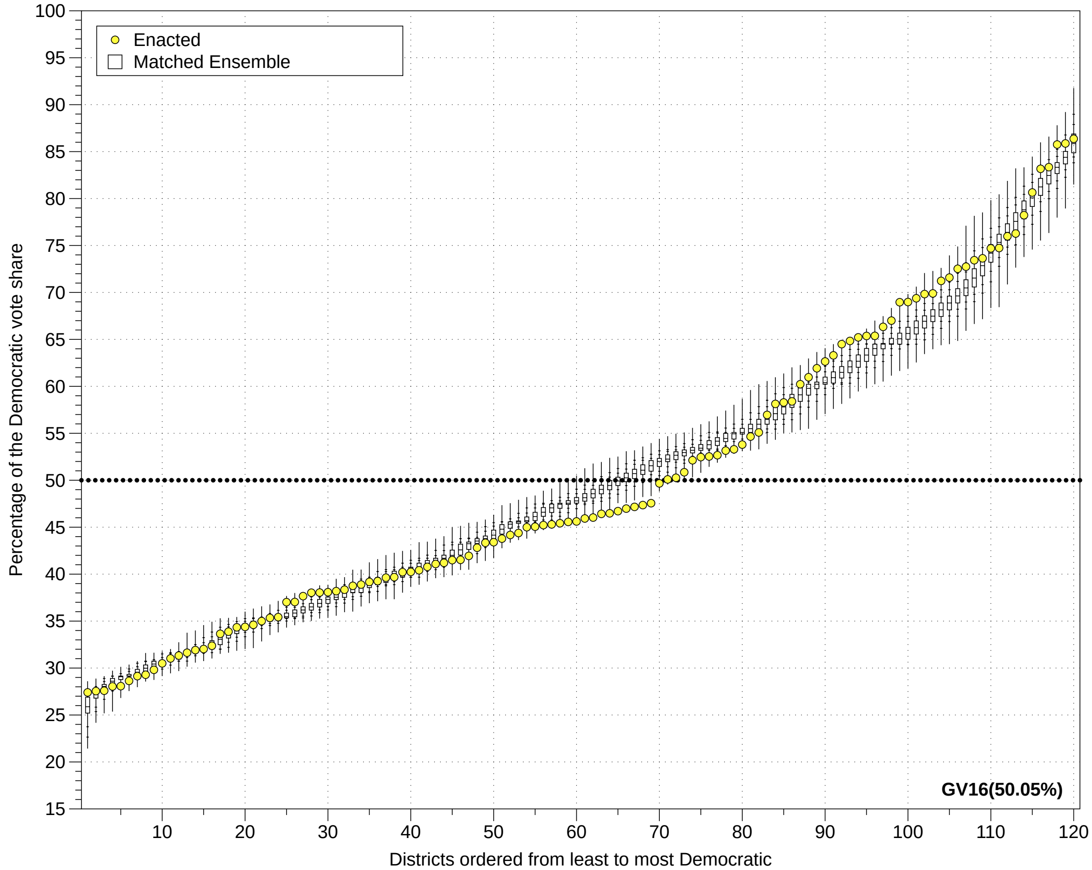
10.3 Congressional Plan: Convergence Tests

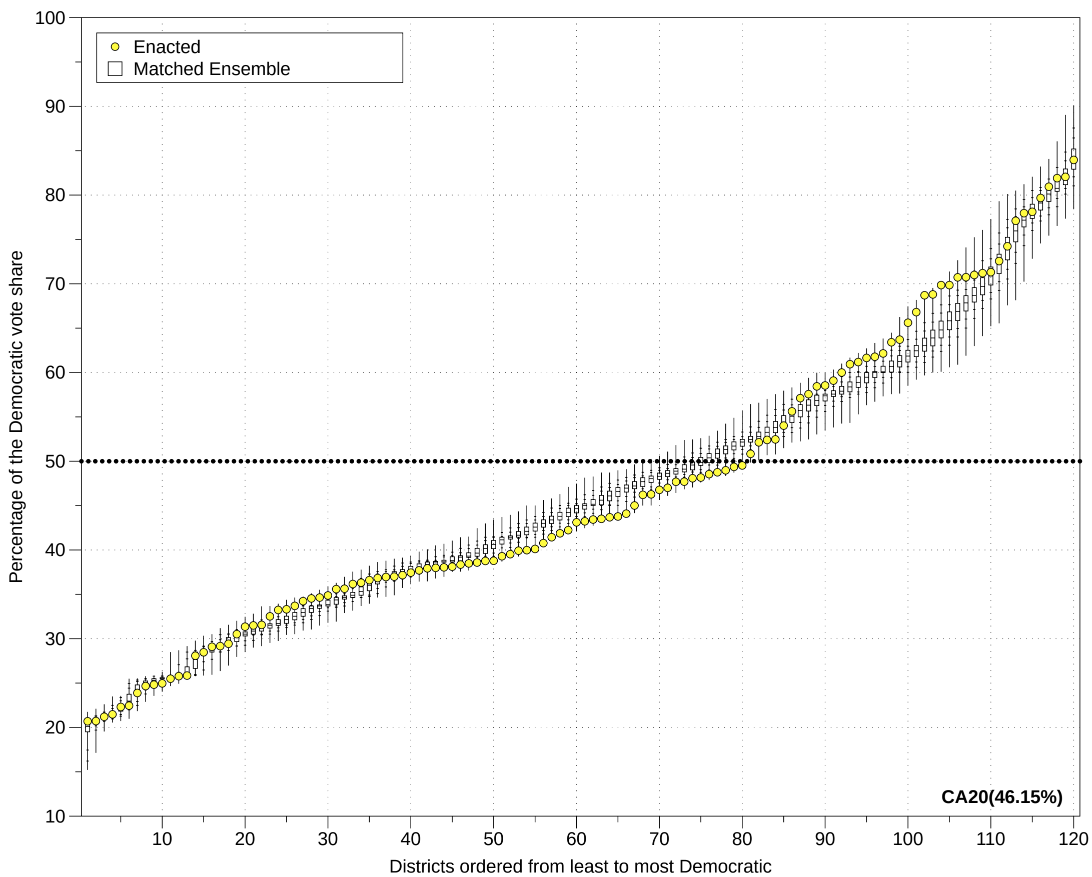
,

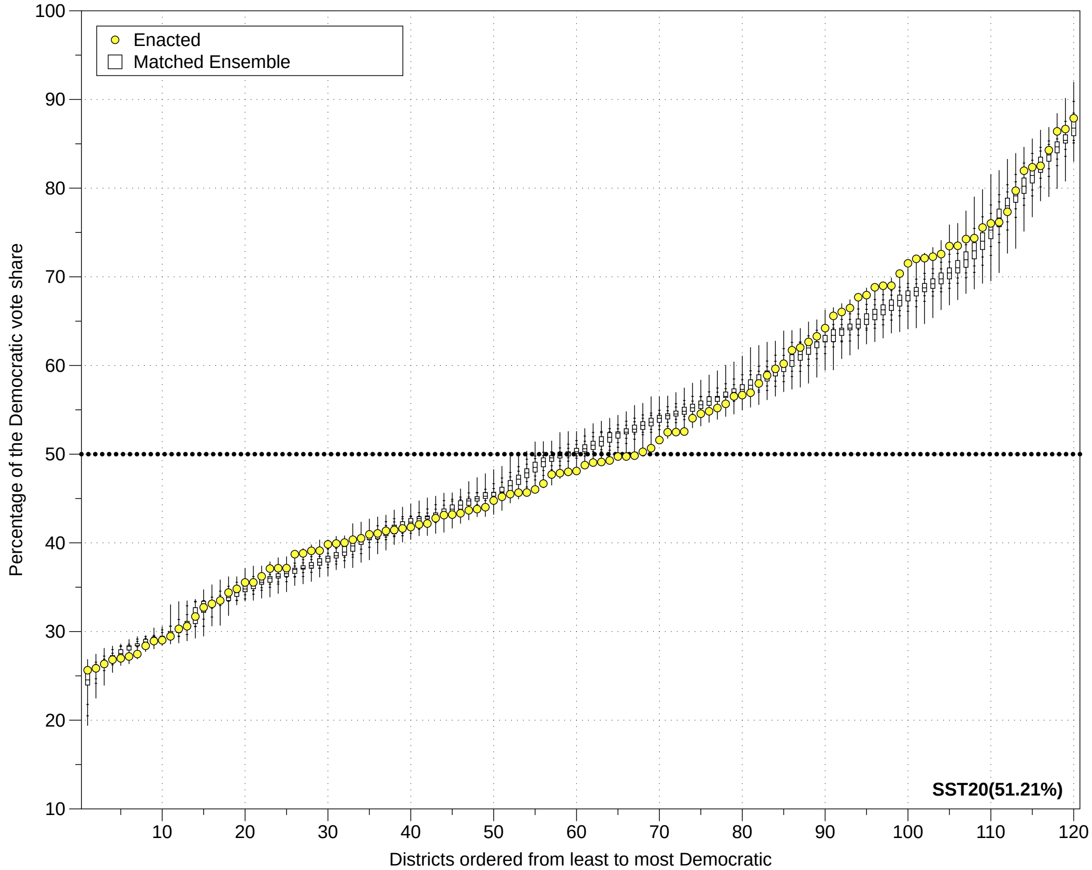
A NC House: Ranked-Ordered Marginal Boxplots

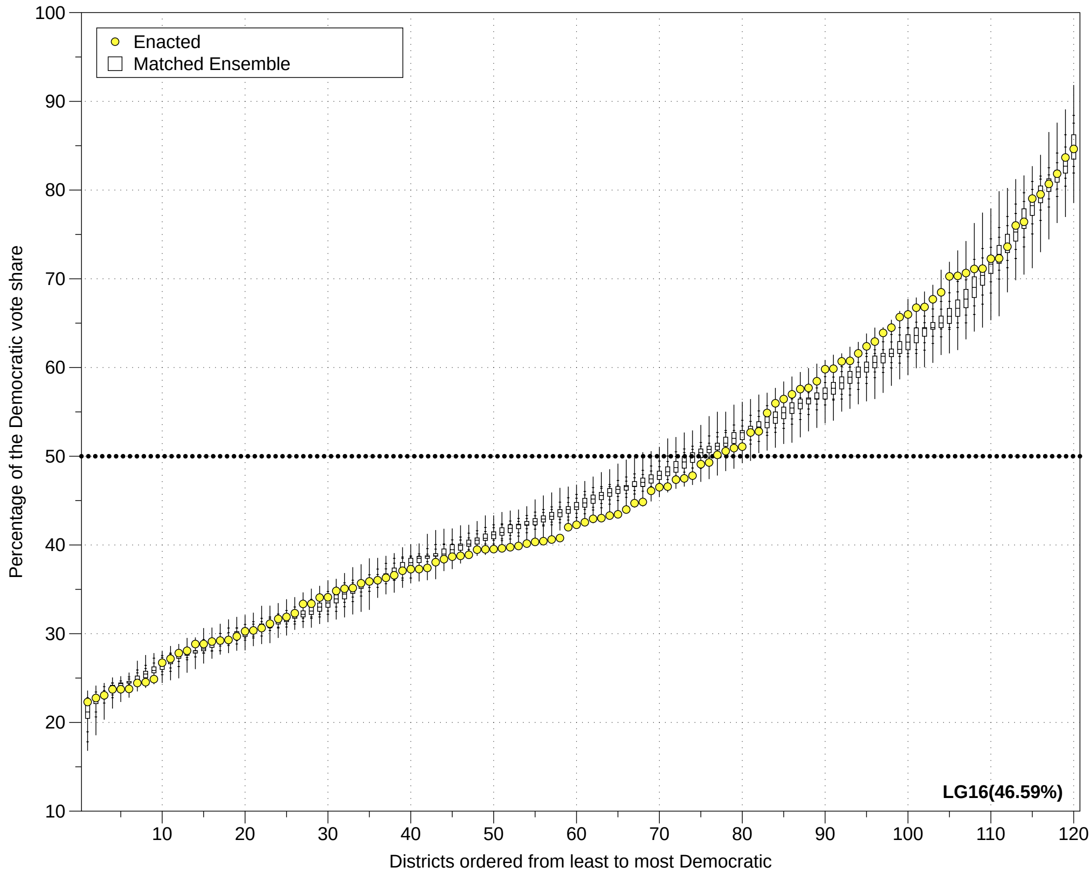


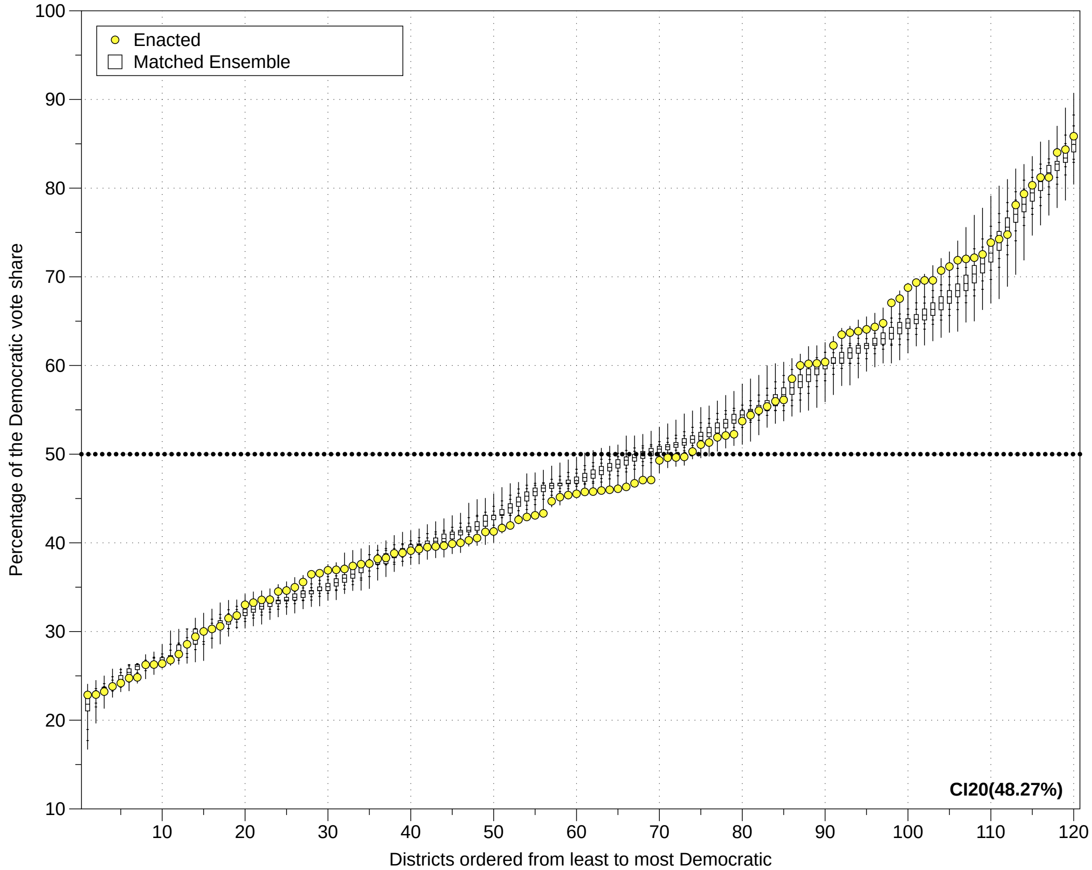


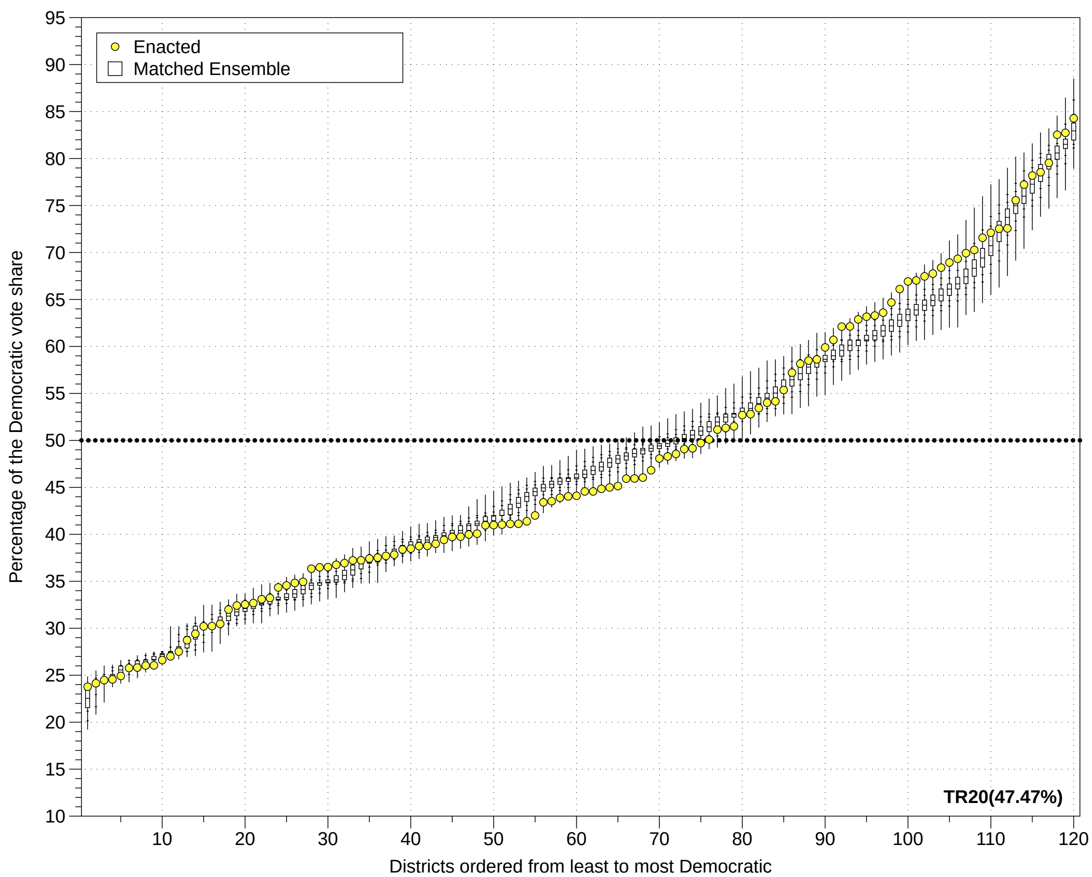

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
GV16(50.05%)

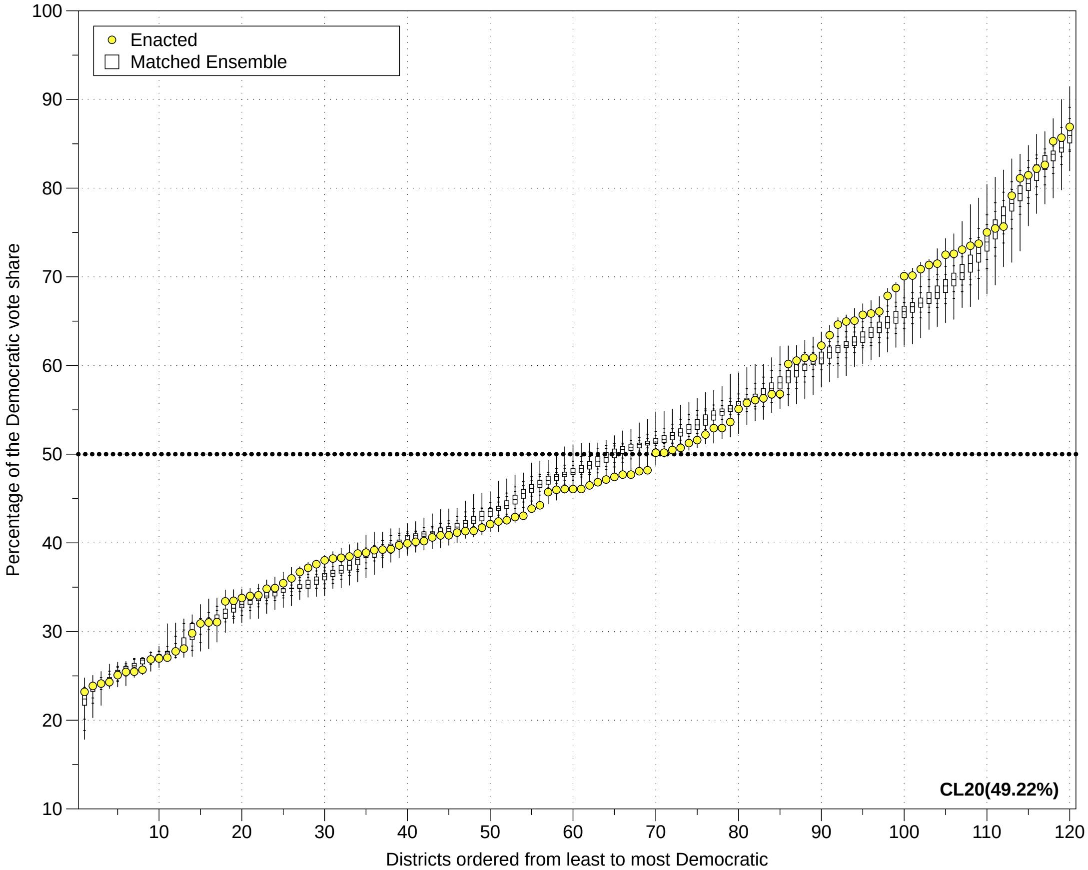

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
CA20(46.15%)

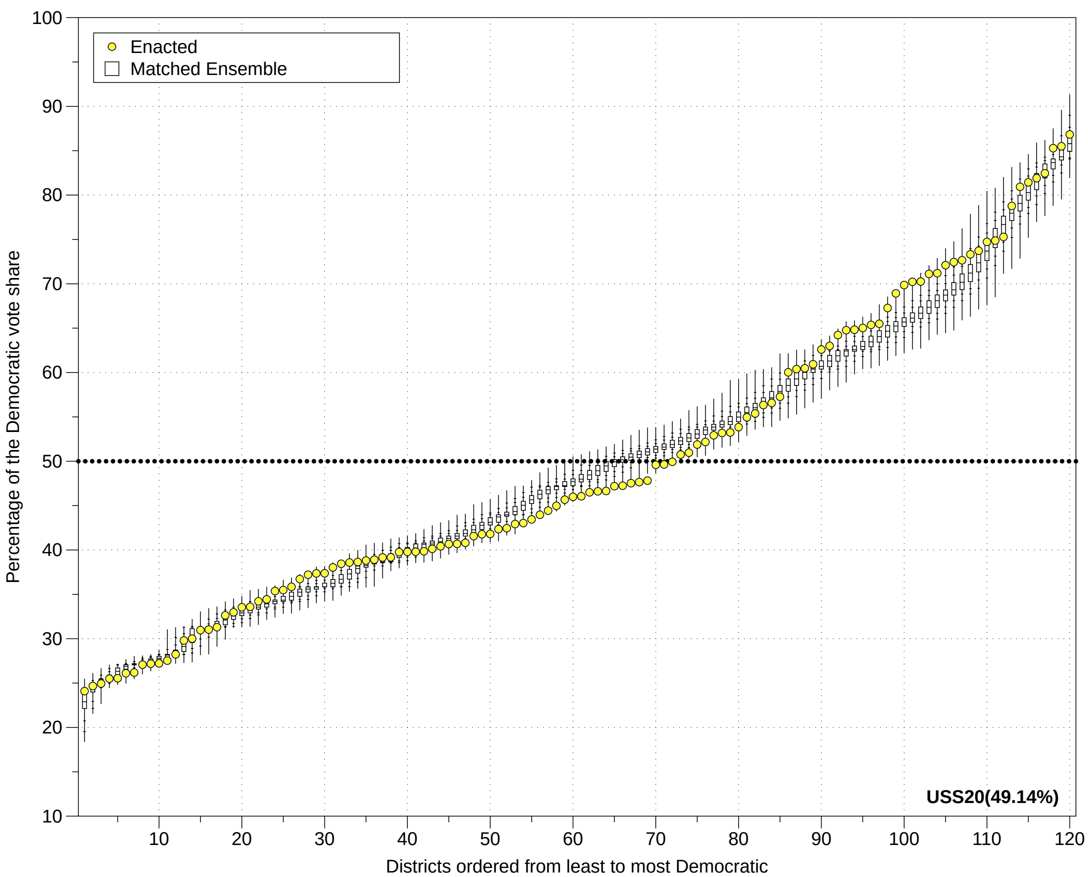

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
SST20(51.21%)
100
90
80
70
60
50
40
30
20
10
120

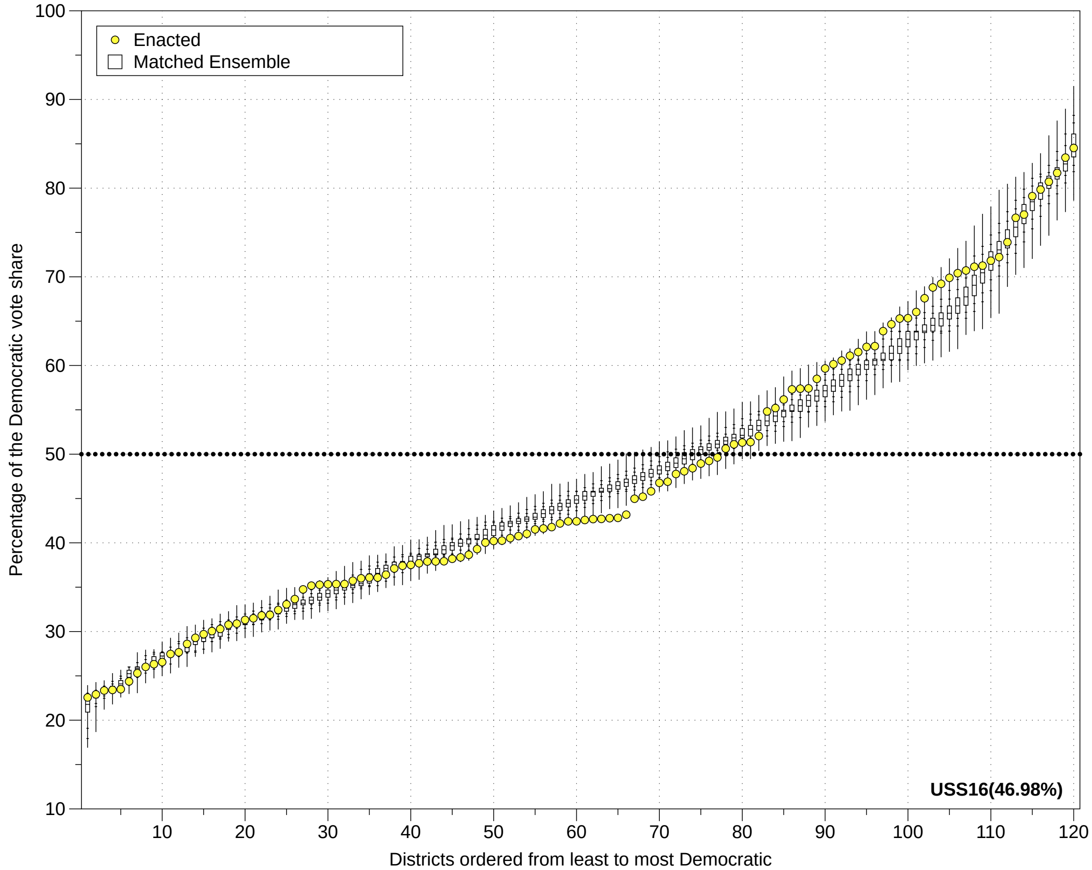

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
LG16(46.59%)
100
90
80
70
60
50
40
30
20
10
120

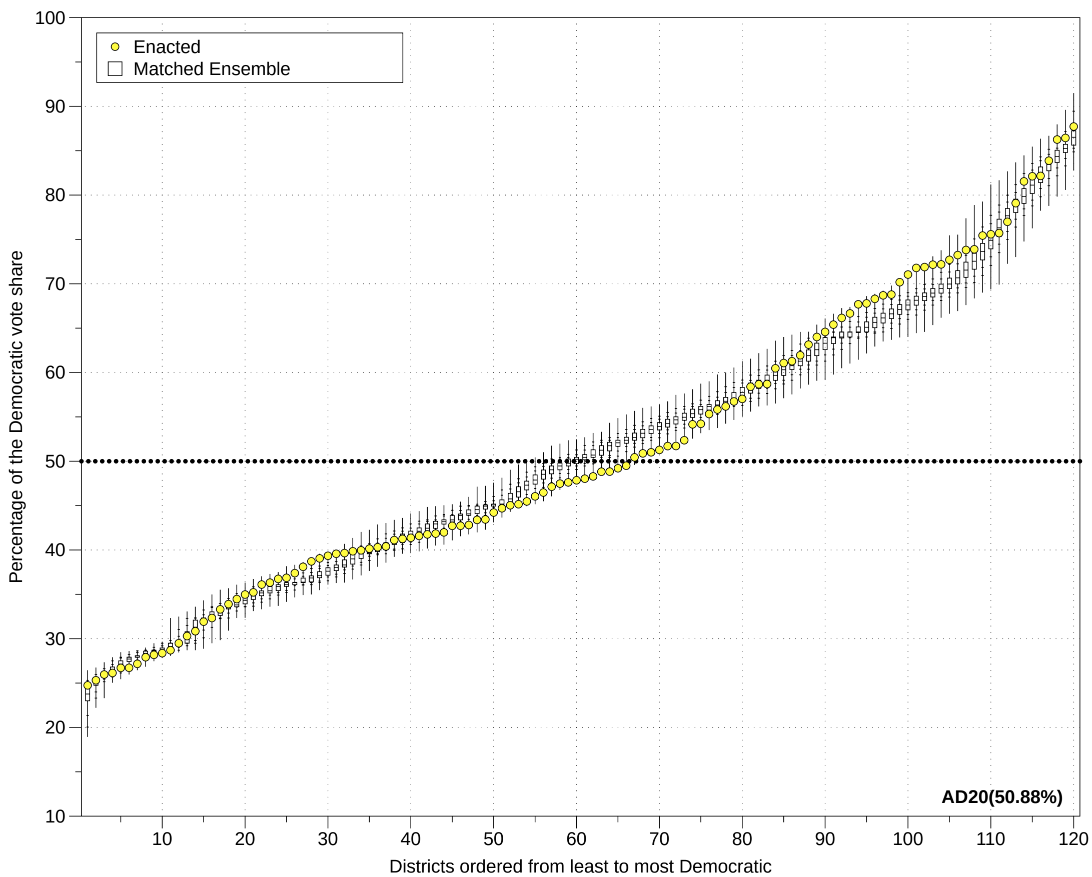

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
CI20(48.27%)
100
90
80
70
60
50
40
30
20
10
10
20
30
40
50
60
70
80
90
100
110
120

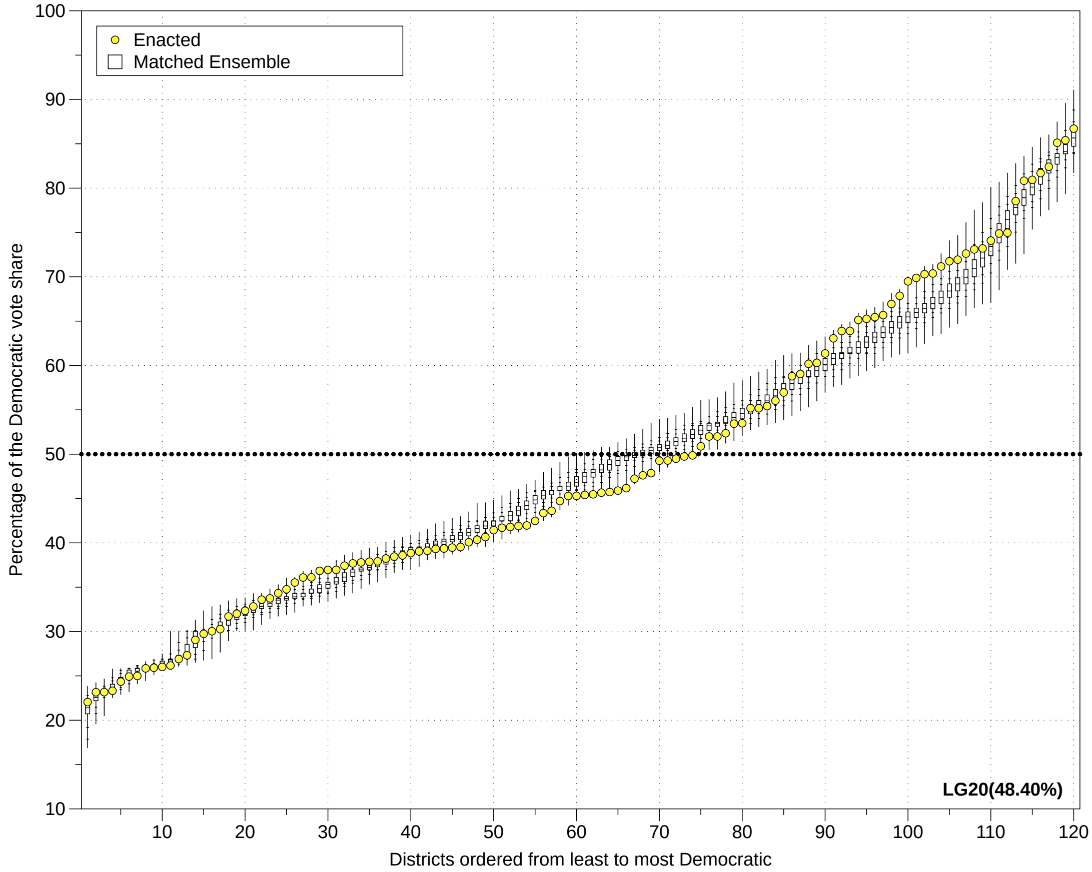

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
TR20(47.47%)
95
90
85
80
75
70
65
60
55
50
45
40
35
30
25
20
15
10
10
20
30
40
50
60
70
80
90
100
110
120


Enacted
Matched Ensemble
CL20(49.22%)
Percentage of the Democratic vote share
Districts ordered from least to most Democratic

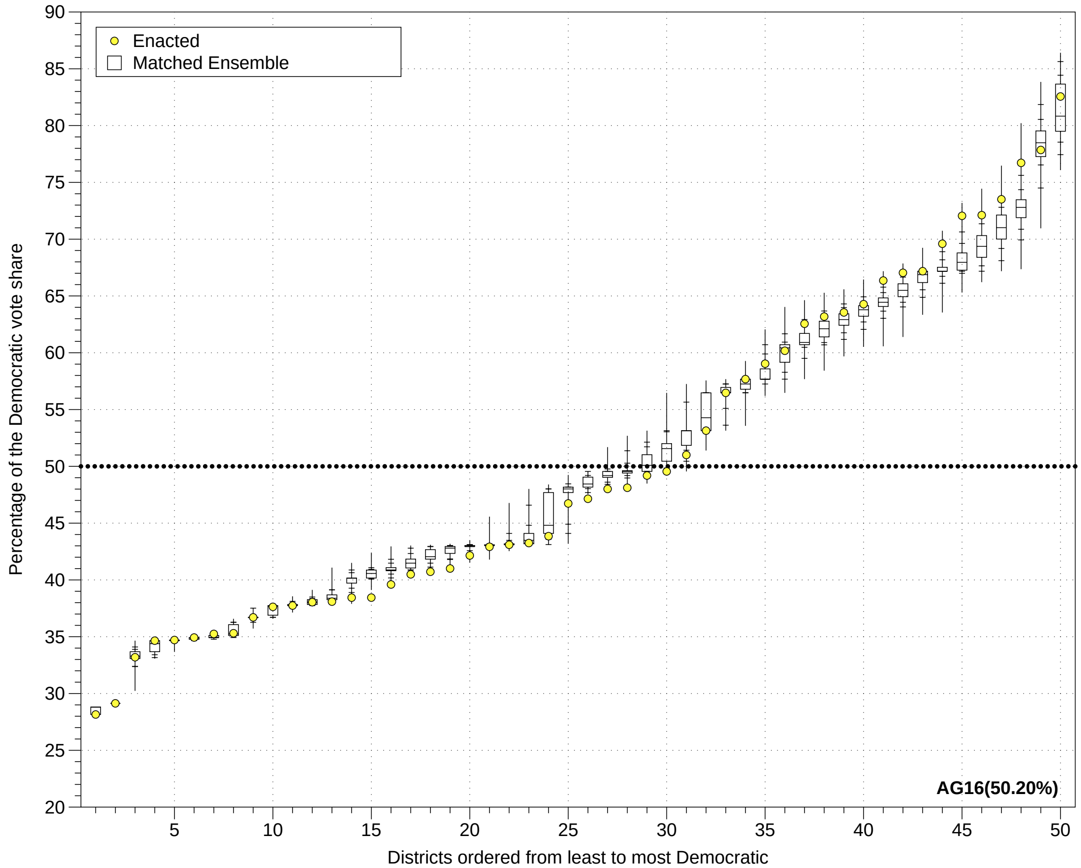

Enacted
Matched Ensemble
USS20(49.14%)
Percentage of the Democratic vote share
Districts ordered from least to most Democratic

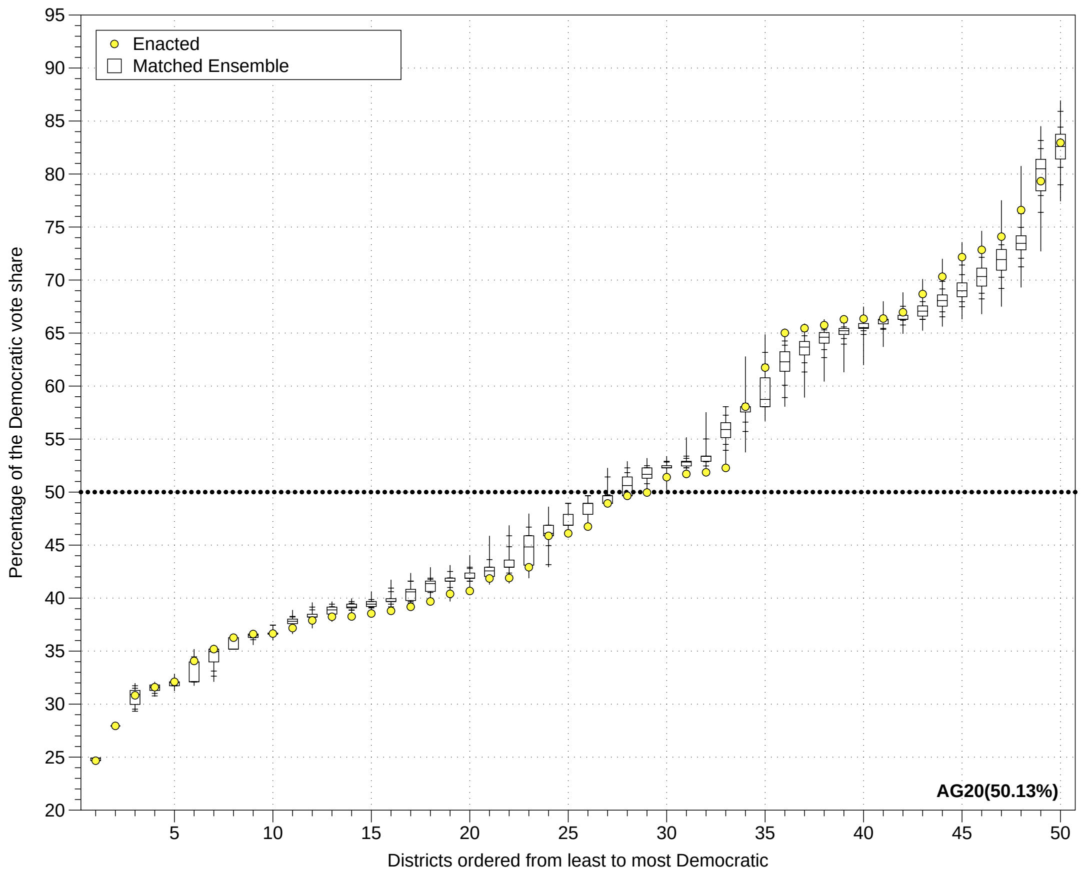

Enacted
Matched Ensemble
Percentage of the Democratic vote share
100
90
80
70
60
50
40
30
20
10
10
20
30
40
50
60
70
80
90
100
110
120
Districts ordered from least to most Democratic
USS16(46.98%)

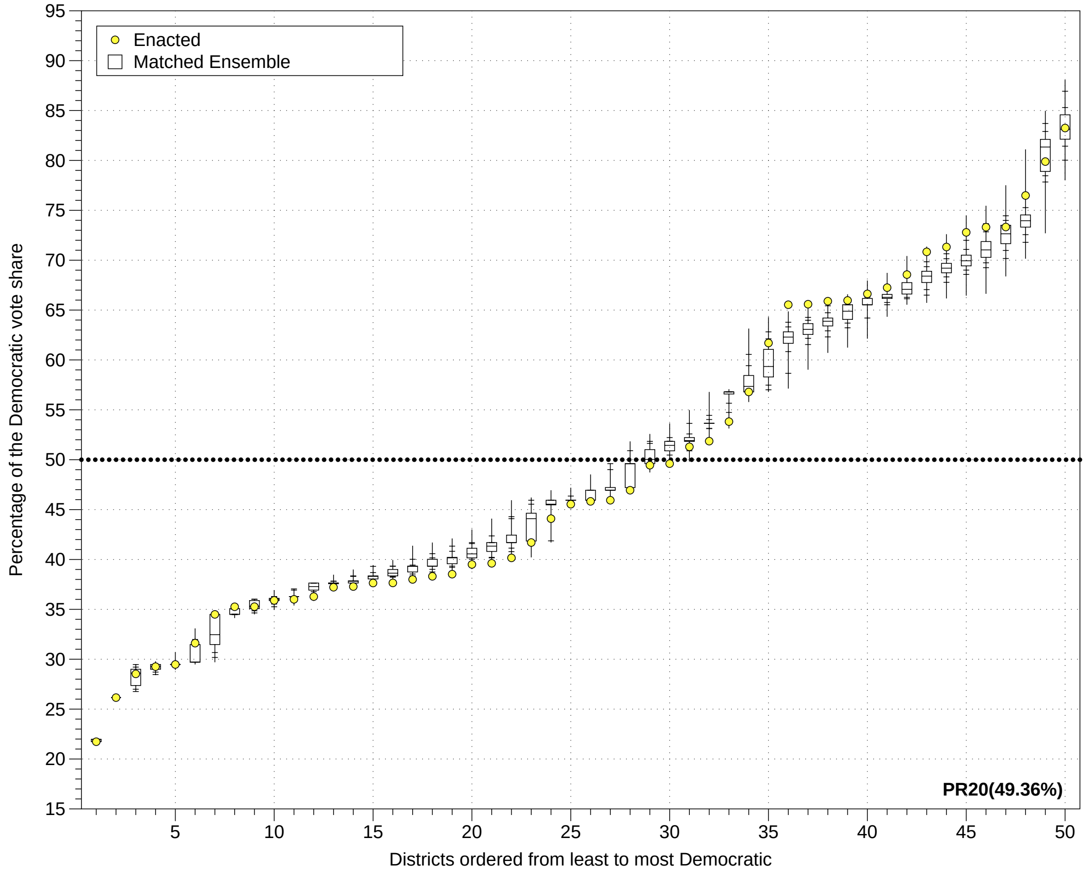

Enacted
Matched Ensemble
Percentage of the Democratic vote share
100
90
80
70
60
50
40
30
20
10
10
20
30
40
50
60
70
80
90
100
110
120
Districts ordered from least to most Democratic
AD20(50.88%)

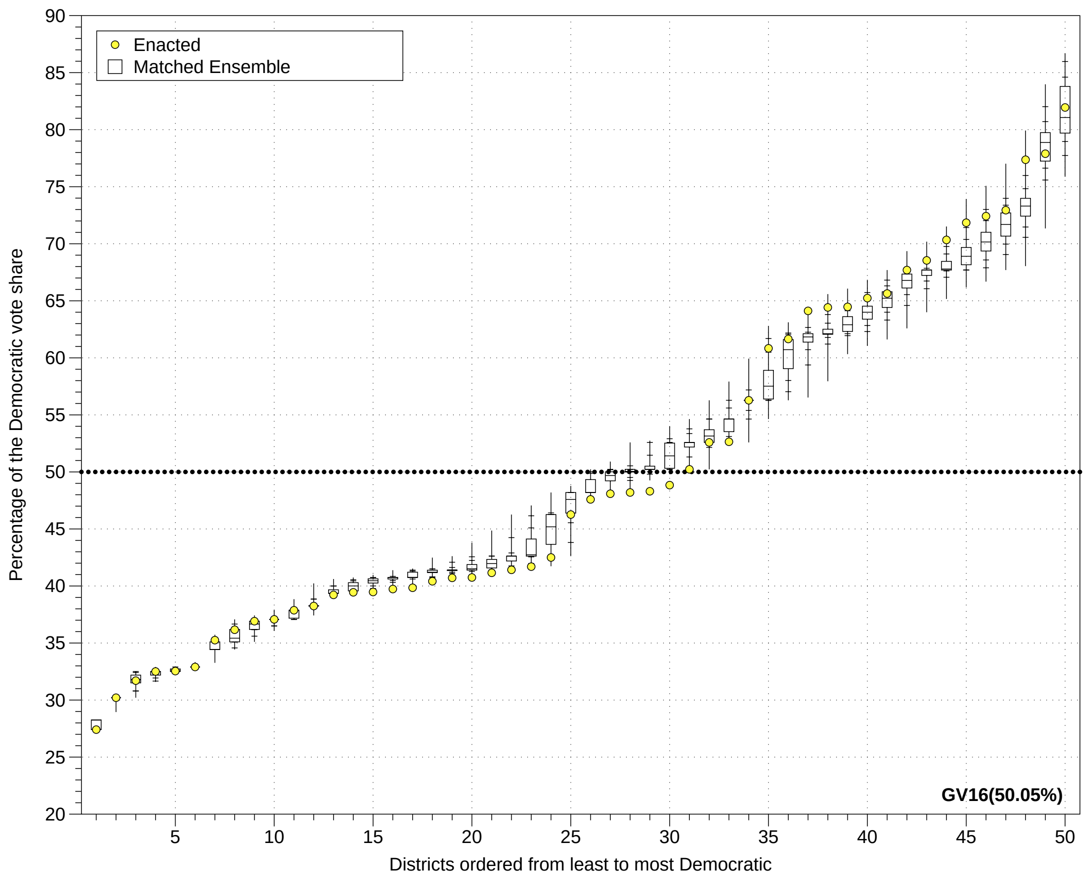

Enacted
Matched Ensemble
LG20(48.40%)
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
100
90
80
70
60
50
40
30
20
10
10
20
30
40
50
60
70
80
90
100
110
120

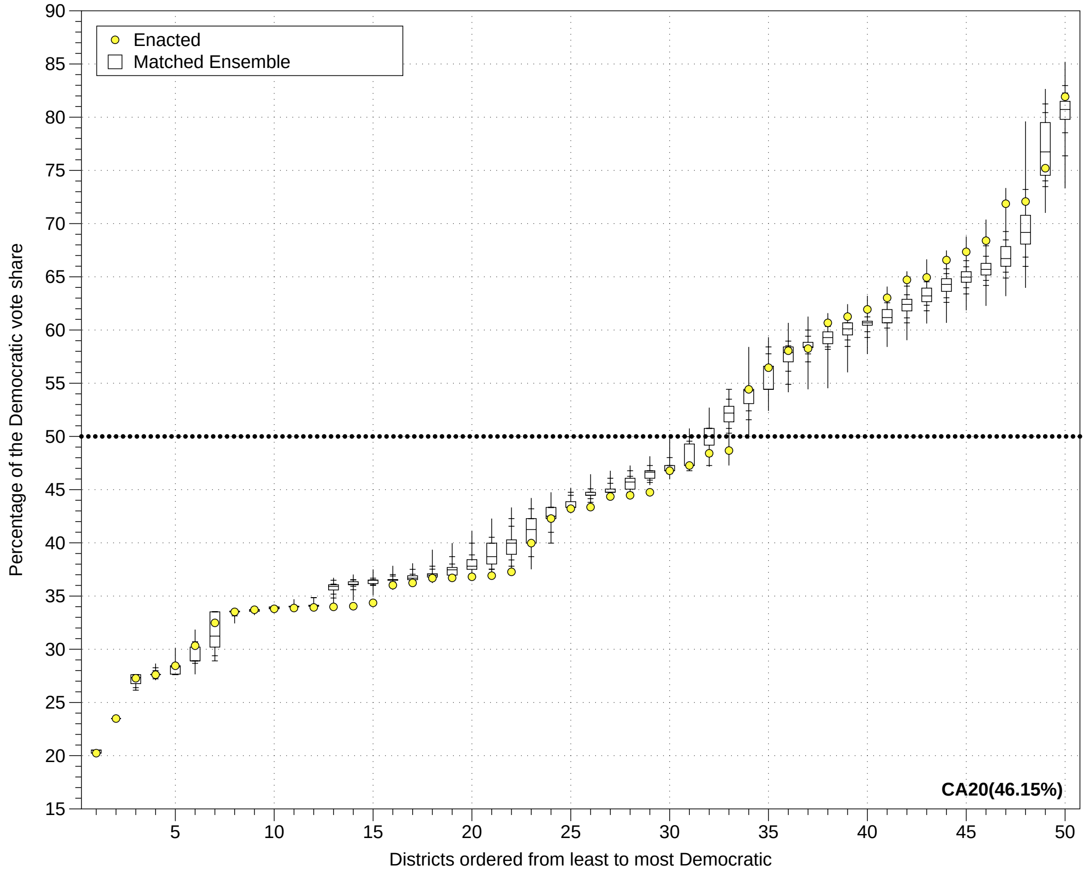
B NC Senate: Ranked-Ordered Marginal Boxplots

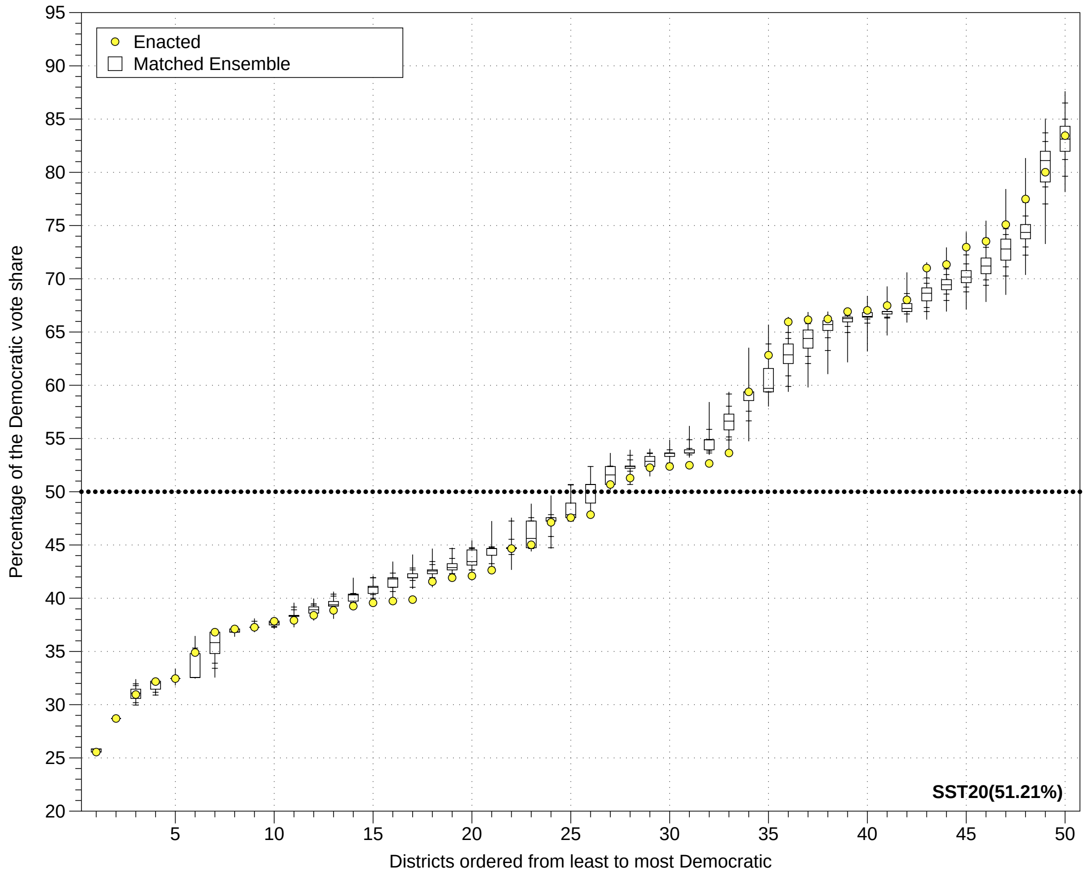


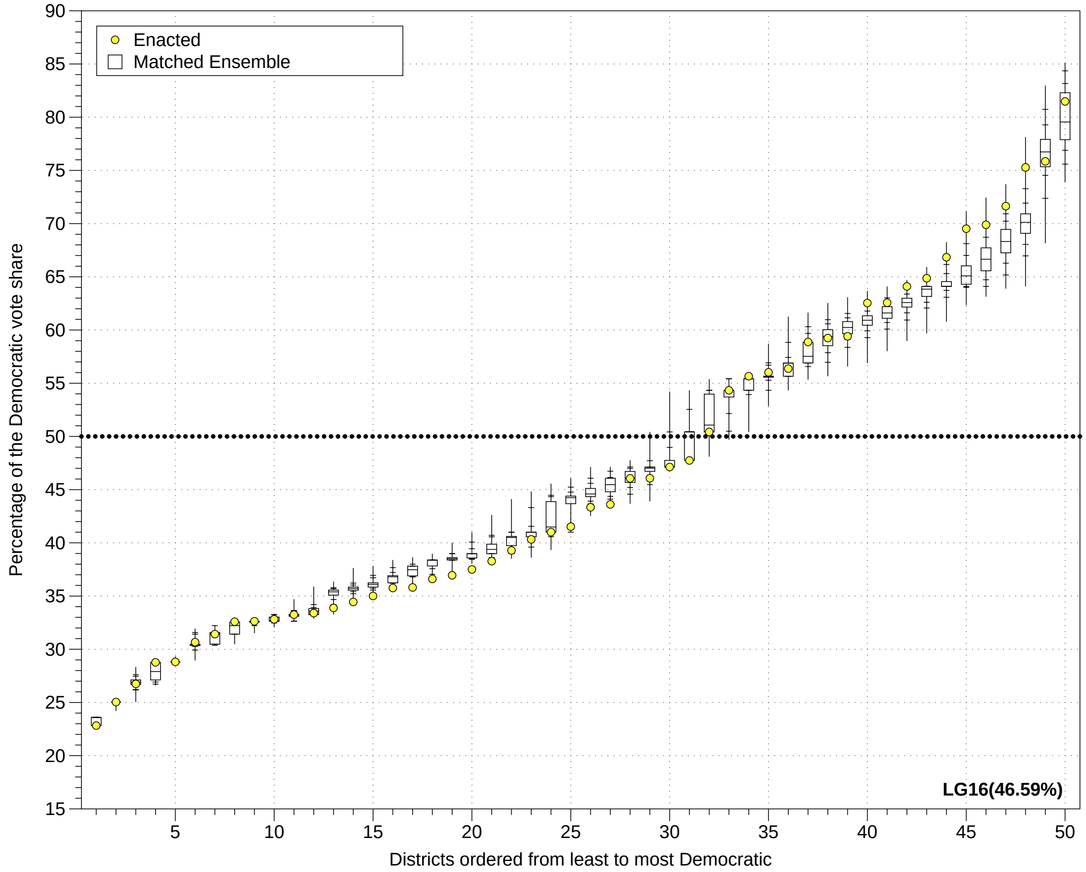


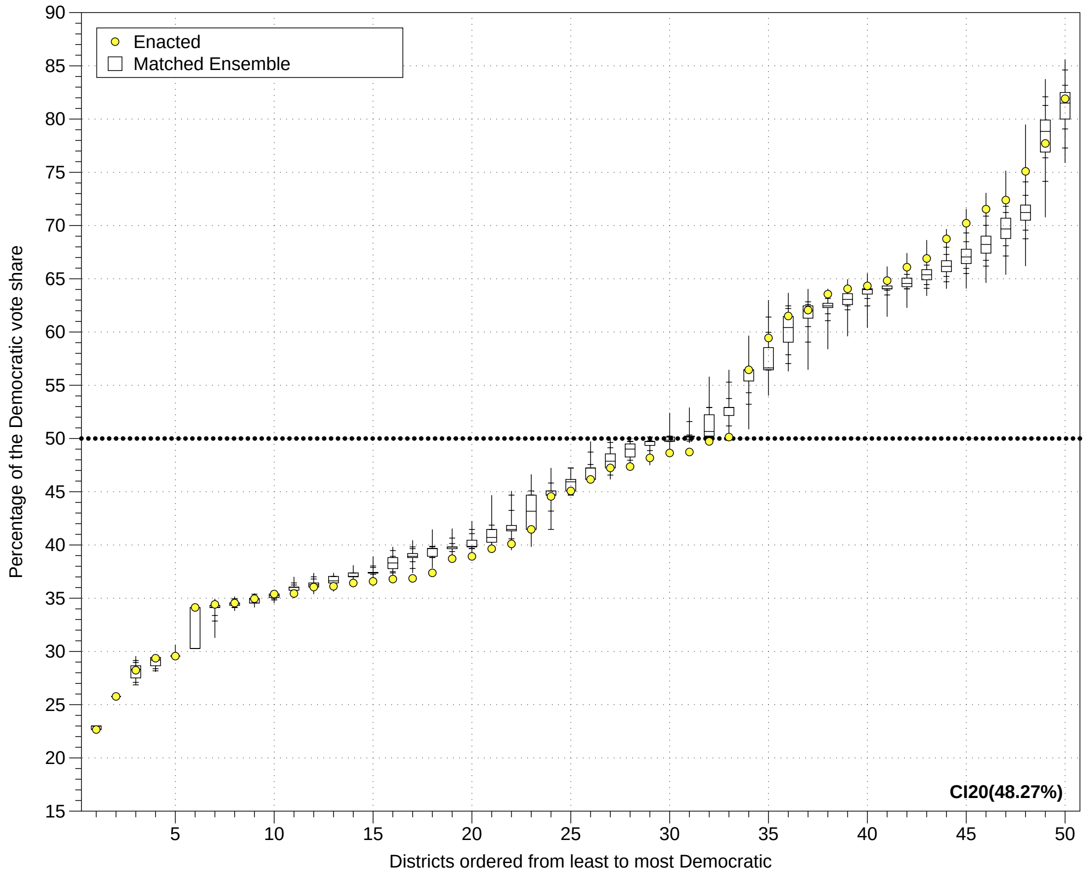

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
PR20(49.36%)

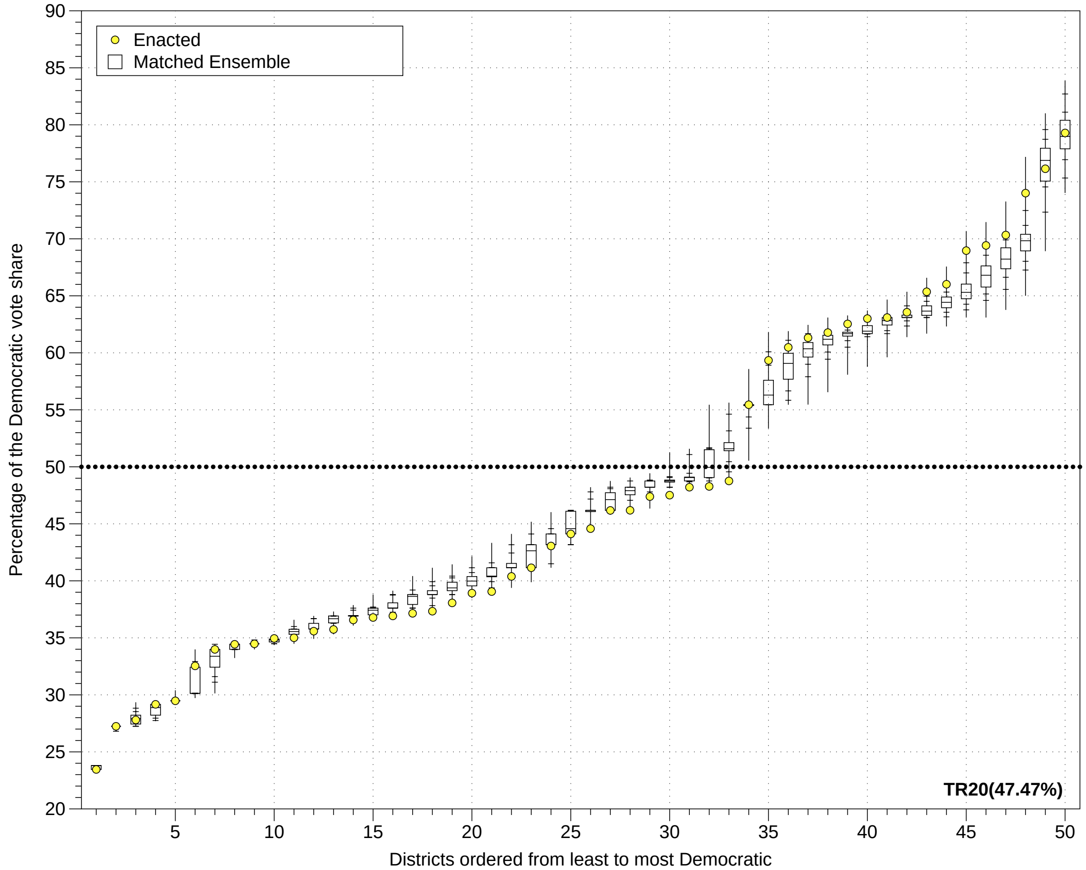

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
GV16(50.05%)


Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
CA20(46.15%)

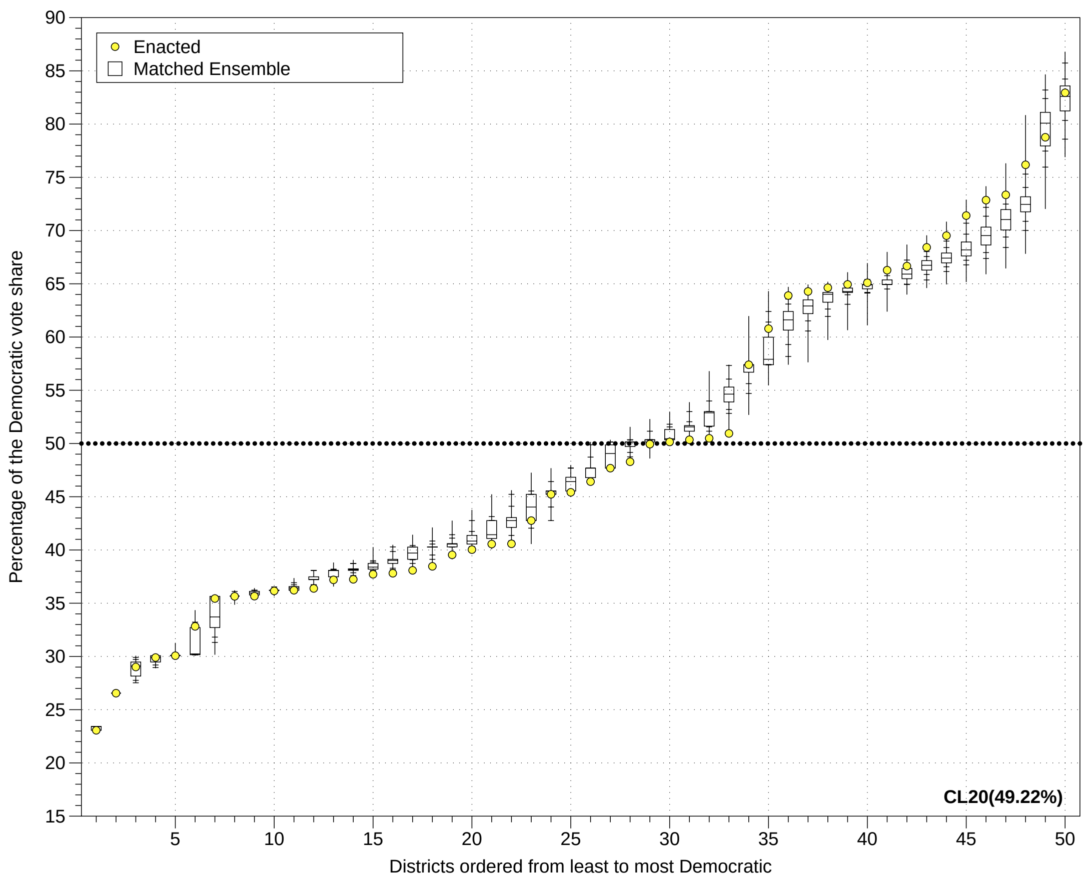

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
SST20(51.21%)


Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
LG16(46.59%)

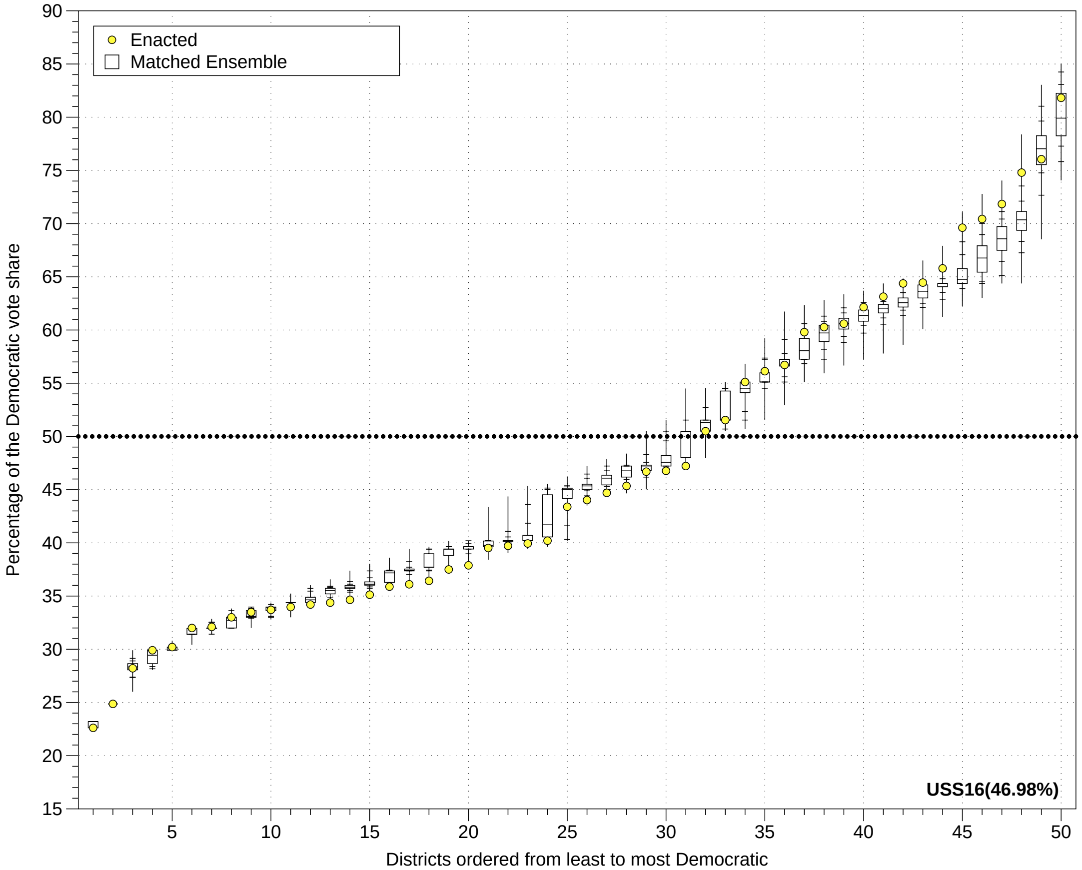

Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
CI20(48.27%)

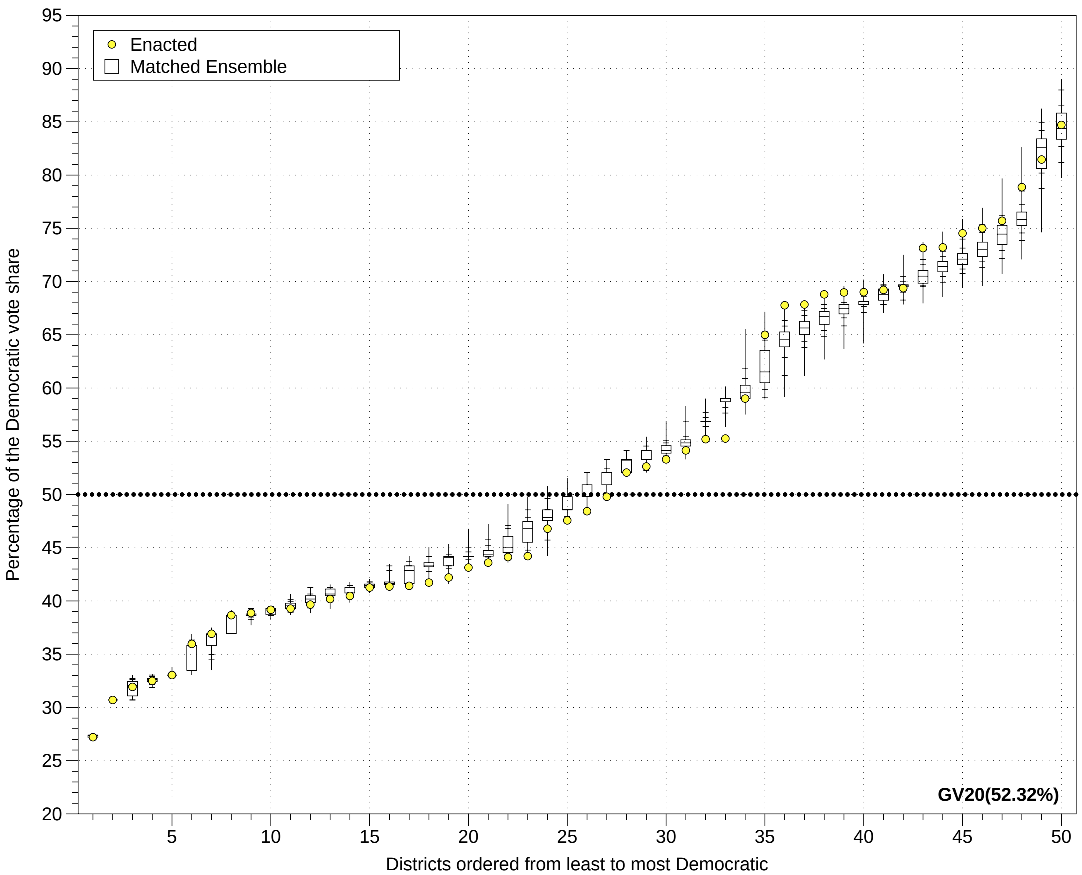


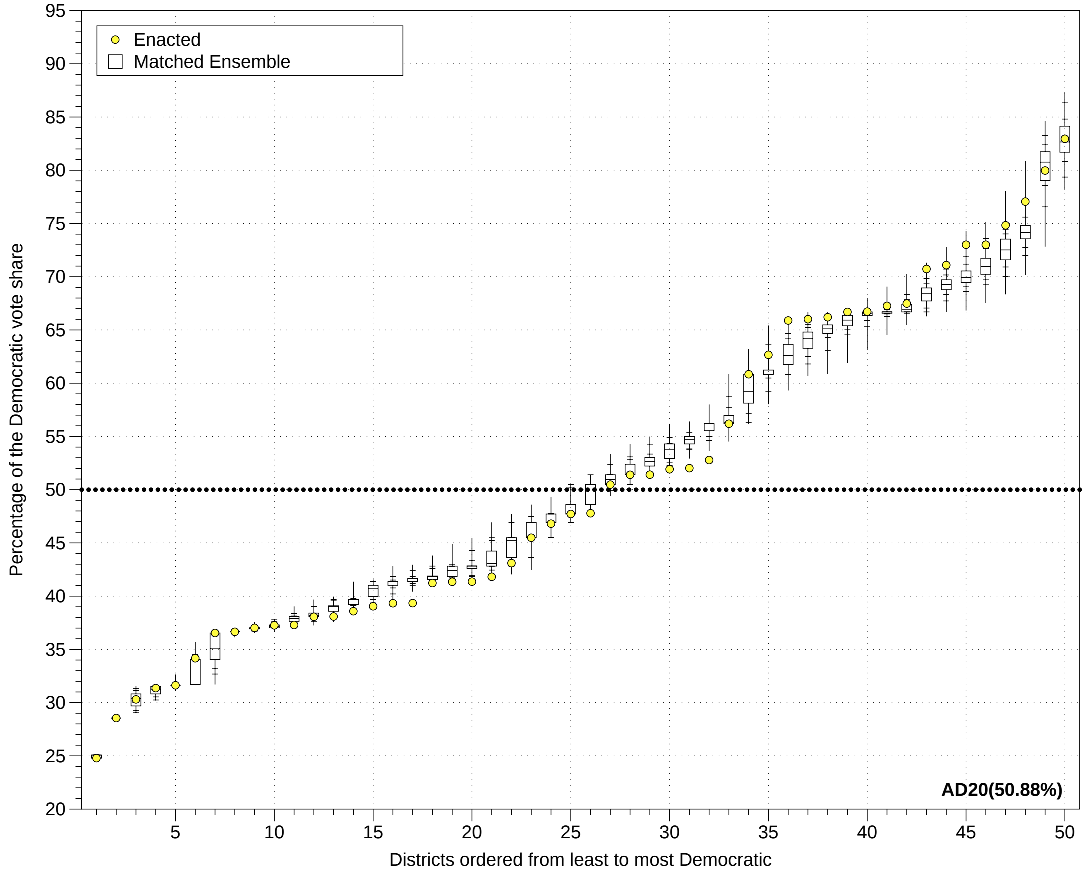
Enacted
Matched Ensemble
TR20(47.47%)
Percentage of the Democratic vote share
Districts ordered from least to most Democratic



Enacted
Matched Ensemble
CL20(49.22%)
Percentage of the Democratic vote share
Districts ordered from least to most Democratic


Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
USS16(46.98%)


Enacted
Matched Ensemble
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
GV20(52.32%)



Enacted
Matched Ensemble
AD20(50.88%)
Percentage of the Democratic vote share
Districts ordered from least to most Democratic
95
90
85
80
75
70
65
60
55
50
45
40
35
30
25
20
5
10
15
20
25
30
35
40
45
50

C NC House: Additional Plots

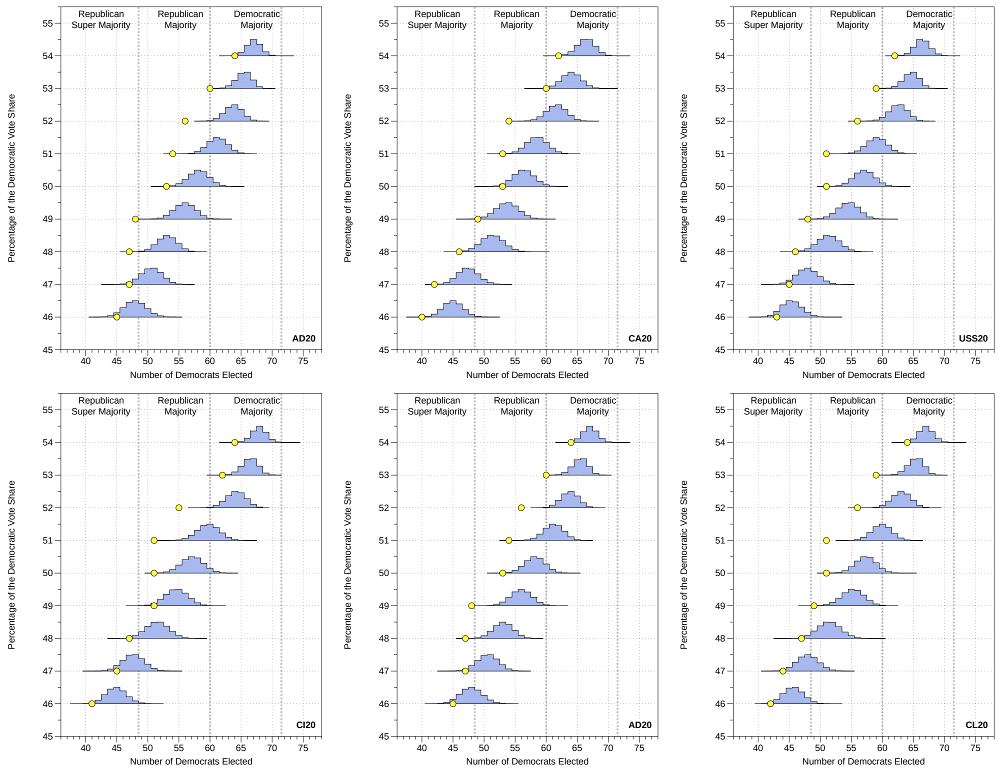


Figure C.0.1: The Collected Seat Histograms for the Primary Ensemble on the NC House built from a collection of voting data generated via uniform swing.

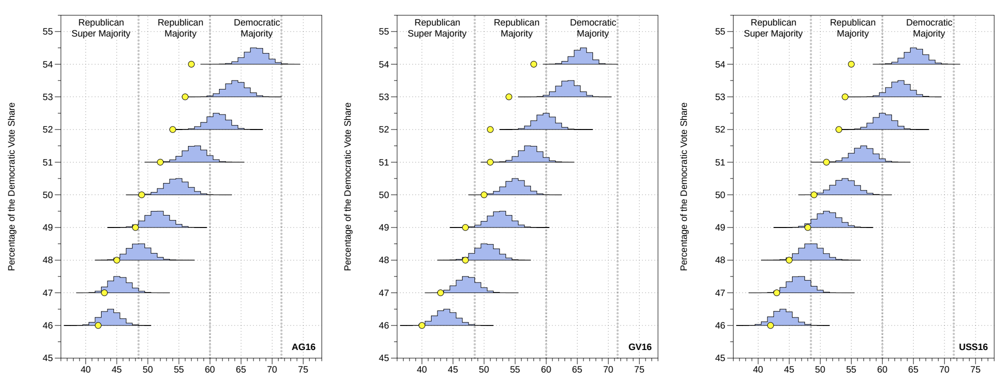


Figure C.0.2: The Collected Seat Histograms for the Primary Ensemble on the NC House built from a collection of voting data generated via uniform swing.

D NC Senate: Additional Plots

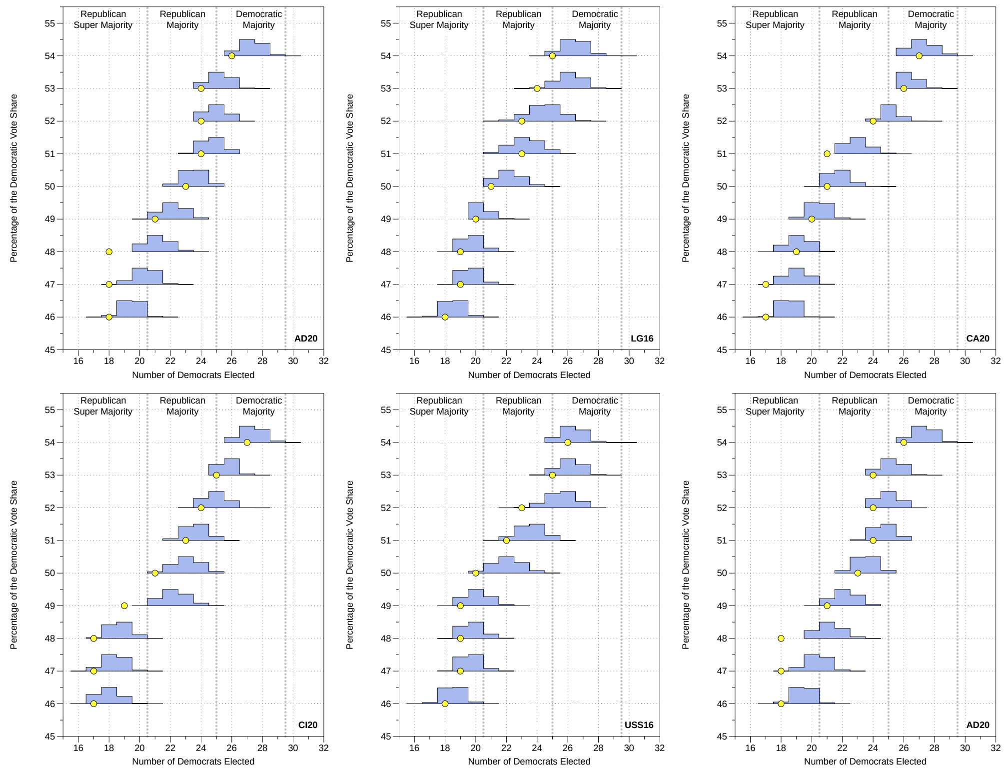


Figure D.0.1: The Collected Seat Histograms for the Primary Ensemble on the NC Senate built from a collection of voting data generated via uniform swing.

E NC Congressional: Ranked-Ordered Marginal Boxplots

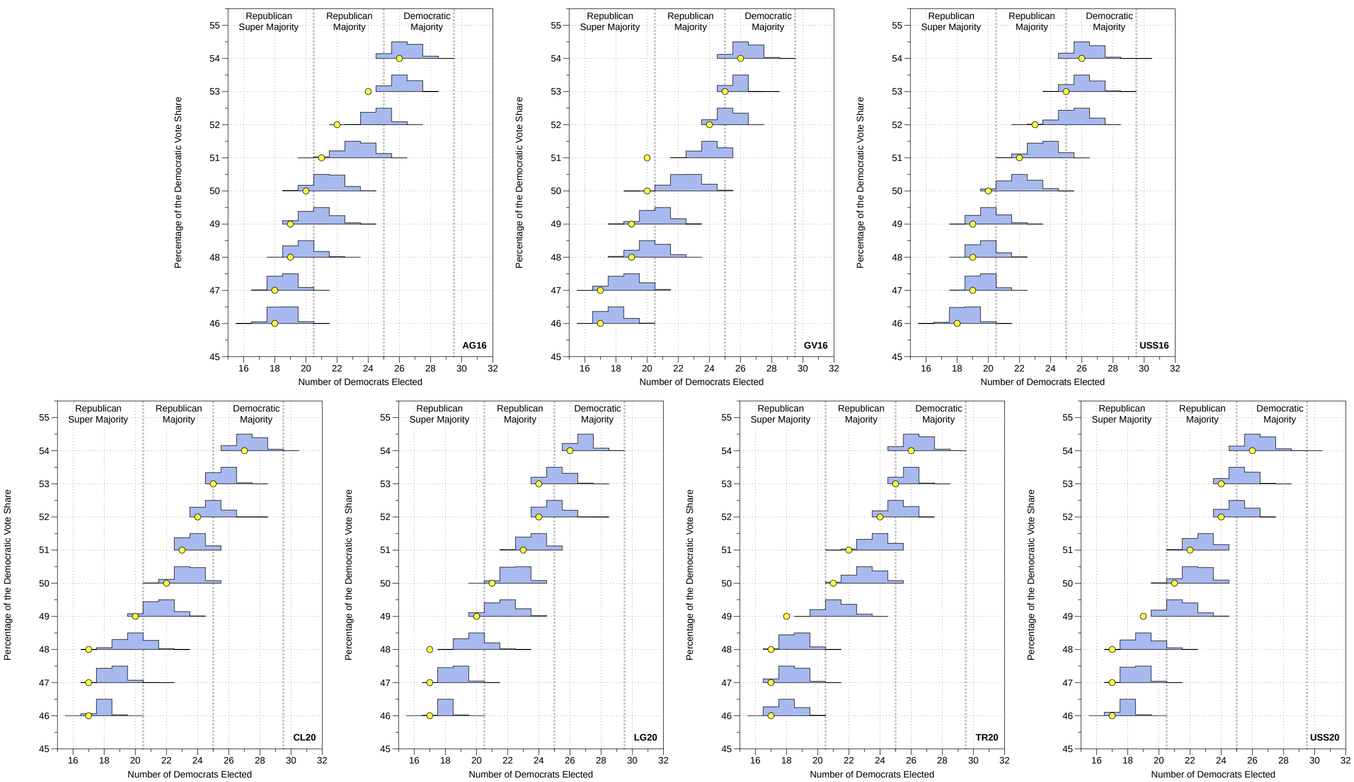


Figure D.0.2: The Collected Seat Histograms for the Primary Ensemble on the NC Senate built from a collection of voting data generated via uniform swing.

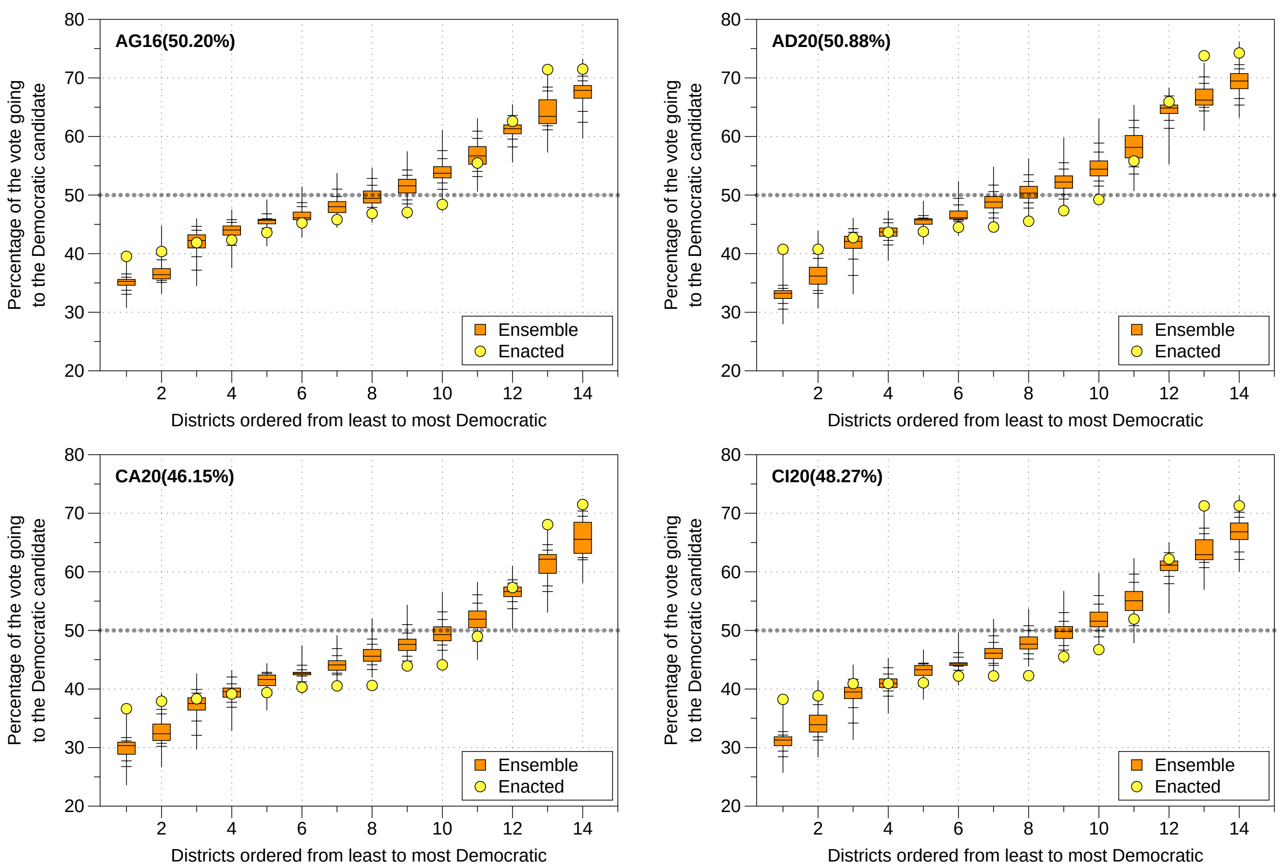


Figure E.0.1: something

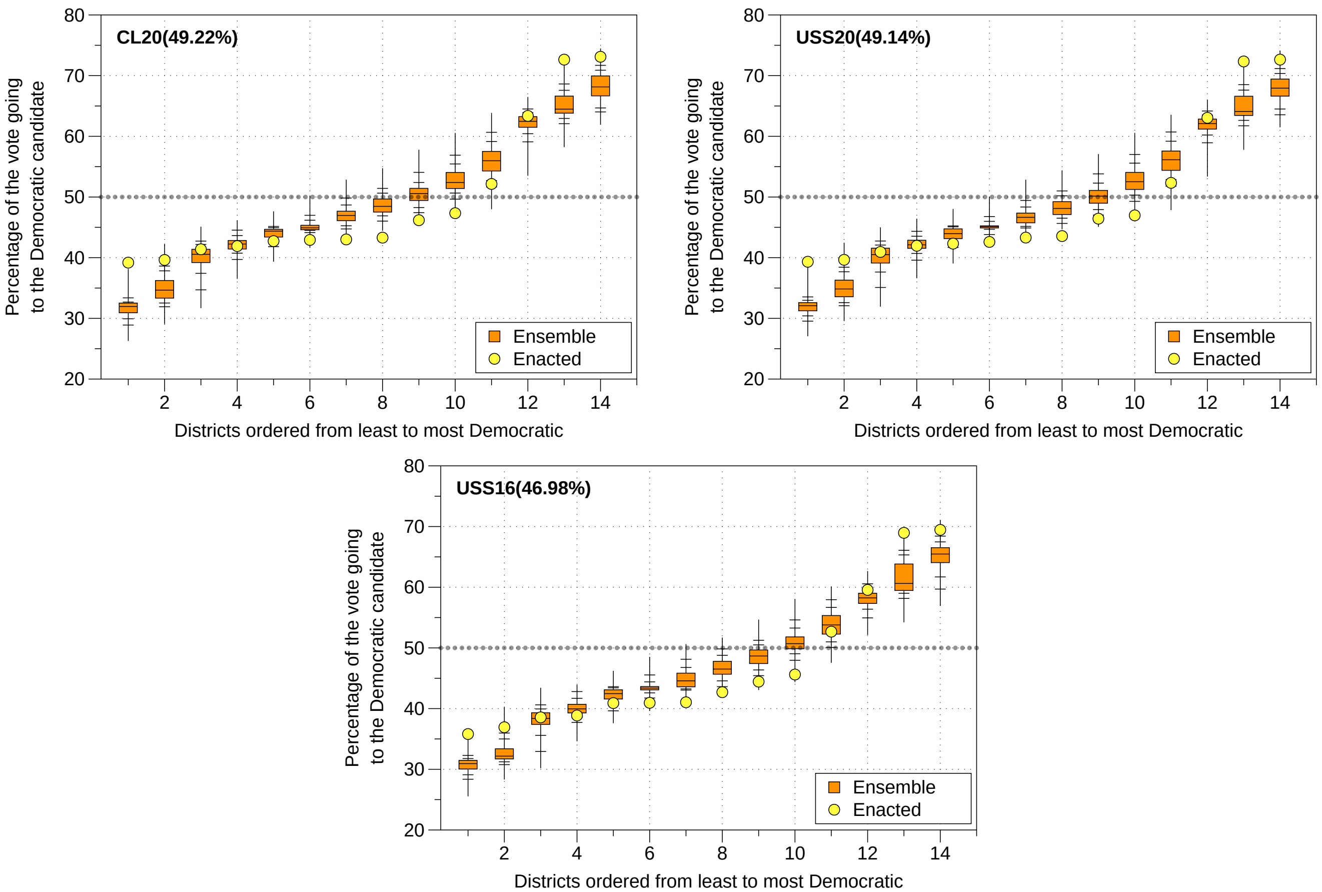


Figure E.0.2: something

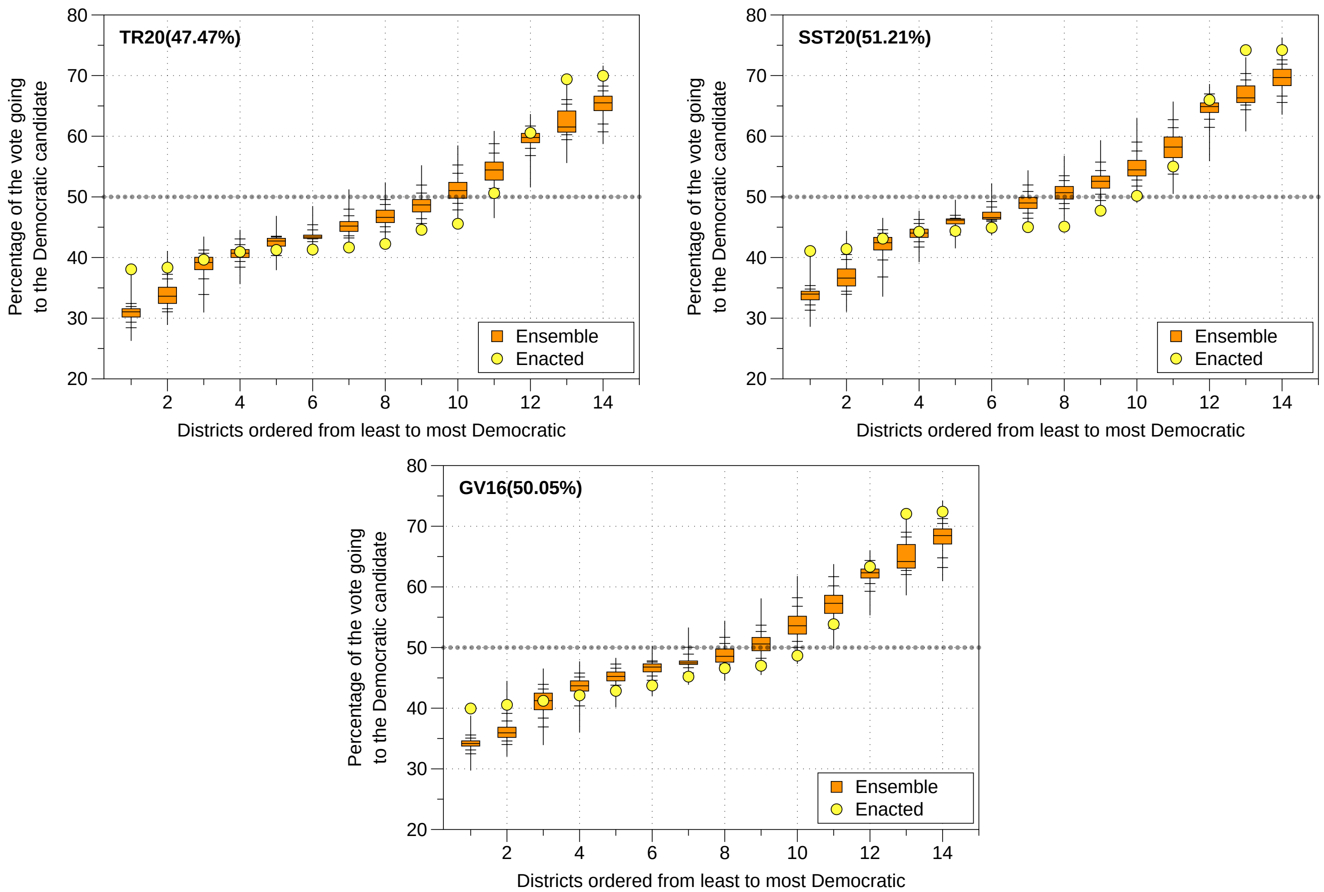


Figure E.0.3: something

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	13507	16.9	16380	20.5	79997	1	2
PR16	23688	29.6	25268	31.6	79997	1	2
AD20	7579	9.47	13561	17.0	79997	1	2
AG20	8831	11.0	14968	18.7	79997	1	2
CA20	7818	9.77	12779	16.0	79997	1	2
CL20	8308	10.4	14272	17.8	79997	1	2
GV20	14684	18.4	19730	24.7	79997	1	2
LG20	10040	12.6	15902	19.9	79997	1	2
PR20	15099	18.9	19674	24.6	79997	1	2
SST20	9265	11.6	15681	19.6	79997	1	2
TR20	10164	12.7	16049	20.1	79997	1	2
USS20	11197	14.0	16428	20.5	79997	1	2

Table 5: Alamance; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	384	0.48	2281	2.85	79997	2 3	4
PR16	288	0.36	4743	5.93	79997	2 3	4
AD20	72	0.09	5122	6.4	79997	2 3	4
AG20	64	0.08	5154	6.44	79997	2 3	4
CA20	48	0.06	4227	5.28	79997	2 3	4
CL20	56	0.07	4995	6.24	79997	2 3	4
GV20	200	0.25	6254	7.82	79997	2 3	4
LG20	80	0.1	5107	6.38	79997	2 3	4
PR20	128	0.16	5842	7.3	79997	2 3	4
SST20	72	0.09	5418	6.77	79997	2 3	4
TR20	80	0.1	4755	5.94	79997	2 3	4
USS20	56	0.07	4334	5.42	79997	2 3	4

Table 6: Brunswick-New Hanover; house

F Cluster-by-cluster outlier analysis

We quantify the visual trends seen in the cluster-by-cluster ordered marginal vote distributions. Similar to the analysis in Table 4, we group ranked districts and inquire how many plans in the ensemble have an average Democratic vote fraction that is more toward the extremes than the enacted plan. In general, lower numbers in the tables below signify more atypical clusters.

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	288	0.36	2406	3.01	79997	1	3
PR16	848	1.06	3910	4.89	79997	1	3
AD20	578	0.723	3738	4.67	79997	1	3
AG20	657	0.821	3711	4.64	79997	1	3
CA20	506	0.633	3072	3.84	79997	1	3
CL20	573	0.716	3578	4.47	79997	1	3
GV20	892	1.12	4803	6.0	79997	1	3
LG20	642	0.803	3699	4.62	79997	1	3
PR20	960	1.2	4790	5.99	79997	1	3
SST20	546	0.683	3305	4.13	79997	1	3
TR20	555	0.694	3295	4.12	79997	1	3
USS20	541	0.676	3404	4.26	79997	1	3

Table 7: Buncombe; house

Election	No. plans w/ ≥ Dems (First Cluster)	% of plans w/ ≥ Dems (First Cluster)	No. plans w/ ≤ Dems (Second Cluster)	% of plans w/ ≤ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	12935	16.2	12183	15.2	79997	3 4	5
PR16	13057	16.3	5371	6.71	79997	3 4	5
AD20	12585	15.7	1657	2.07	79997	3 4	5
AG20	12230	15.3	2081	2.6	79997	3 4	5
CA20	12445	15.6	1573	1.97	79997	3 4	5
CL20	12411	15.5	1785	2.23	79997	3 4	5
GV20	12167	15.2	1489	1.86	79997	3 4	5
LG20	12312	15.4	1789	2.24	79997	3 4	5
PR20	12320	15.4	921	1.15	79997	3 4	5
SST20	12059	15.1	1709	2.14	79997	3 4	5
TR20	12102	15.1	1537	1.92	79997	3 4	5
USS20	11901	14.9	1669	2.09	79997	3 4	5

Table 8: Cabarrus-Davie-Rowan-Yadkin; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	3767	4.71	13593	17.0	79997	2	3 4
PR16	5414	6.77	13064	16.3	79997	2	3 4
AD20	970	1.21	11880	14.9	79997	2	3 4
AG20	899	1.12	11149	13.9	79997	2	3 4
CA20	833	1.04	11167	14.0	79997	2	3 4
CL20	341	0.426	10790	13.5	79997	2	3 4
GV20	517	0.646	11339	14.2	79997	2	3 4
LG20	346	0.433	10829	13.5	79997	2	3 4
PR20	579	0.724	11315	14.1	79997	2	3 4
SST20	1206	1.51	12333	15.4	79997	2	3 4
TR20	587	0.734	10981	13.7	79997	2	3 4
USS20	360	0.45	10674	13.3	79997	2	3 4

Table 9: Cumberland; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	46063	57.6	46238	57.8	79997	1	2
PR16	43010	53.8	43894	54.9	79997	1	2
AD20	41097	51.4	41193	51.5	79997	1	2
AG20	38601	48.3	38516	48.1	79997	1	2
CA20	39051	48.8	39158	48.9	79997	1	2
CL20	38891	48.6	39038	48.8	79997	1	2
GV20	38179	47.7	38073	47.6	79997	1	2
LG20	38313	47.9	38392	48.0	79997	1	2
PR20	38660	48.3	38492	48.1	79997	1	2
SST20	41059	51.3	40686	50.9	79997	1	2
TR20	38891	48.6	39342	49.2	79997	1	2
USS20	38430	48.0	38734	48.4	79997	1	2

Table 10: Duplin-Wayne; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	0	0.0	2768	3.46	79997	1	3 4
PR16	0	0.0	409	0.511	79997	1	3 4
AD20	0	0.0	274	0.343	79997	1	3 4
AG20	0	0.0	312	0.39	79997	1	3 4
CA20	0	0.0	929	1.16	79997	1	3 4
CL20	0	0.0	417	0.521	79997	1	3 4
GV20	0	0.0	232	0.29	79997	1	3 4
LG20	0	0.0	328	0.41	79997	1	3 4
PR20	0	0.0	96	0.12	79997	1	3 4
SST20	0	0.0	296	0.37	79997	1	3 4
TR20	0	0.0	280	0.35	79997	1	3 4
USS20	0	0.0	497	0.621	79997	1	3 4

Table 11: Durham-Person; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	1	0.00125	659	0.824	79997	1 2 3	4 5
PR16	0	0.0	543	0.679	79997	1 2 3	4 5
AD20	8	0.01	952	1.19	79997	1 2 3	4 5
AG20	11	0.0138	1025	1.28	79997	1 2 3	4 5
CA20	11	0.0138	1032	1.29	79997	1 2 3	4 5
CL20	9	0.0113	995	1.24	79997	1 2 3	4 5
GV20	8	0.01	982	1.23	79997	1 2 3	4 5
LG20	8	0.01	980	1.23	79997	1 2 3	4 5
PR20	8	0.01	893	1.12	79997	1 2 3	4 5
SST20	0	0.0	912	1.14	79997	1 2 3	4 5
TR20	9	0.0113	944	1.18	79997	1 2 3	4 5
USS20	16	0.02	1106	1.38	79997	1 2 3	4 5

Table 12: Forsyth-Stokes; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	0	0.0	0	0.0	79997	1 2	3 4 5 6
PR16	0	0.0	0	0.0	79997	1 2	3 4 5 6
AD20	0	0.0	0	0.0	79997	1 2	3 4 5 6
AG20	0	0.0	0	0.0	79997	1 2	3 4 5 6
CA20	0	0.0	0	0.0	79997	1 2	3 4 5 6
CL20	0	0.0	0	0.0	79997	1 2	3 4 5 6
GV20	0	0.0	0	0.0	79997	1 2	3 4 5 6
LG20	0	0.0	0	0.0	79997	1 2	3 4 5 6
PR20	0	0.0	0	0.0	79997	1 2	3 4 5 6
SST20	0	0.0	0	0.0	79997	1 2	3 4 5 6
TR20	0	0.0	0	0.0	79997	1 2	3 4 5 6
USS20	0	0.0	0	0.0	79997	1 2	3 4 5 6

Table 13: Guilford; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	661	0.826	2	0.0025	79997	1 2 3 4	5 6 7 8
PR16	168	0.21	6	0.0075	79997	1 2 3 4	5 6 7 8
AD20	569	0.711	32	0.04	79997	1 2 3 4	5 6 7 8
AG20	763	0.954	35	0.0438	79997	1 2 3 4	5 6 7 8
CA20	1363	1.7	84	0.105	79997	1 2 3 4	5 6 7 8
CL20	1146	1.43	72	0.09	79997	1 2 3 4	5 6 7 8
GV20	396	0.495	40	0.05	79997	1 2 3 4	5 6 7 8
LG20	700	0.875	36	0.045	79997	1 2 3 4	5 6 7 8
PR20	202	0.253	19	0.0238	79997	1 2 3 4	5 6 7 8
SST20	496	0.62	29	0.0363	79997	1 2 3 4	5 6 7 8
TR20	975	1.22	88	0.11	79997	1 2 3 4	5 6 7 8
USS20	1082	1.35	69	0.0863	79997	1 2 3 4	5 6 7 8

Table 14: Mecklenburg; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	1194	1.49	899	1.12	79997	1	2
PR16	2115	2.64	1829	2.29	79997	1	2
AD20	8230	10.3	4317	5.4	79997	1	2
AG20	4434	5.54	2326	2.91	79997	1	2
CA20	2295	2.87	1334	1.67	79997	1	2
CL20	4069	5.09	2163	2.7	79997	1	2
GV20	6311	7.89	3379	4.22	79997	1	2
LG20	4123	5.15	2222	2.78	79997	1	2
PR20	6573	8.22	3564	4.46	79997	1	2
SST20	5386	6.73	2656	3.32	79997	1	2
TR20	4243	5.3	2177	2.72	79997	1	2
USS20	3799	4.75	2074	2.59	79997	1	2

Table 15: Pitt; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	209	0.261	6107	7.63	79997	1 2	3 4 5 6 7 8 9
PR16	160	0.2	4317	5.4	79997	1 2	3 4 5 6 7 8 9
AD20	240	0.3	4968	6.21	79997	1 2	3 4 5 6 7 8 9
AG20	230	0.288	4728	5.91	79997	1 2	3 4 5 6 7 8 9
CA20	1151	1.44	15113	18.9	79997	1 2	3 4 5 6 7 8 9
CL20	337	0.421	6643	8.3	79997	1 2	3 4 5 6 7 8 9
GV20	225	0.281	3777	4.72	79997	1 2	3 4 5 6 7 8 9
LG20	298	0.373	5552	6.94	79997	1 2	3 4 5 6 7 8 9
PR20	241	0.301	4462	5.58	79997	1 2	3 4 5 6 7 8 9
SST20	291	0.364	4572	5.72	79997	1 2	3 4 5 6 7 8 9
TR20	377	0.471	7229	9.04	79997	1 2	3 4 5 6 7 8 9
USS20	354	0.443	6912	8.64	79997	1 2	3 4 5 6 7 8 9

Table 16: Wake; house

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	48	0.06	0	0.0	79997	1	2
PR16	48	0.06	48	0.06	79997	1	2
AD20	48	0.06	48	0.06	79997	1	2
AG20	48	0.06	48	0.06	79997	1	2
CA20	48	0.06	48	0.06	79997	1	2
CL20	48	0.06	48	0.06	79997	1	2
GV20	48	0.06	48	0.06	79997	1	2
LG20	48	0.06	48	0.06	79997	1	2
PR20	48	0.06	48	0.06	79997	1	2
SST20	48	0.06	48	0.06	79997	1	2
TR20	48	0.06	48	0.06	79997	1	2
USS20	48	0.06	48	0.06	79997	1	2

Table 17: Cumberland-Moore; senate

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	855	1.07	3472	4.34	79997	1	2
PR16	600	0.75	1822	2.28	79997	1	2
AD20	506	0.633	1745	2.18	79997	1	2
AG20	595	0.744	2455	3.07	79997	1	2
CA20	570	0.713	2521	3.15	79997	1	2
CL20	550	0.688	2191	2.74	79997	1	2
GV20	471	0.589	1496	1.87	79997	1	2
LG20	485	0.606	1967	2.46	79997	1	2
PR20	447	0.559	1392	1.74	79997	1	2
SST20	515	0.644	1827	2.28	79997	1	2
TR20	646	0.808	2696	3.37	79997	1	2
USS20	498	0.623	2174	2.72	79997	1	2

Table 18: Forsyth-Stokes; senate

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	0	0.0	6	0.015	39991	1 2	3 4 5 6
PR16	0	0.0	3	0.0075	39991	1 2	3 4 5 6
AD20	0	0.0	0	0.0	39991	1 2	3 4 5 6
AG20	0	0.0	0	0.0	39991	1 2	3 4 5 6
CA20	0	0.0	9	0.0225	39991	1 2	3 4 5 6
CL20	0	0.0	4	0.01	39991	1 2	3 4 5 6
GV20	0	0.0	0	0.0	39991	1 2	3 4 5 6
LG20	0	0.0	0	0.0	39991	1 2	3 4 5 6
PR20	0	0.0	0	0.0	39991	1 2	3 4 5 6
SST20	0	0.0	0	0.0	39991	1 2	3 4 5 6
TR20	0	0.0	5	0.0125	39991	1 2	3 4 5 6
USS20	0	0.0	4	0.01	39991	1 2	3 4 5 6

Table 19: Granville-Wake; senate

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	0	0.0	13	0.0163	79997	1	3
PR16	0	0.0	13	0.0163	79997	1	3
AD20	0	0.0	54	0.0675	79997	1	3
AG20	0	0.0	33	0.0413	79997	1	3
CA20	0	0.0	15	0.0188	79997	1	3
CL20	0	0.0	23	0.0288	79997	1	3
GV20	0	0.0	56	0.07	79997	1	3
LG20	0	0.0	22	0.0275	79997	1	3
PR20	0	0.0	59	0.0738	79997	1	3
SST20	0	0.0	32	0.04	79997	1	3
TR20	0	0.0	20	0.025	79997	1	3
USS20	0	0.0	23	0.0288	79997	1	3

Table 20: Guilford-Rockingham; senate

Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
LG16	0	0.0	0	0.0	79997	1 2	3 4 5 6
PR16	0	0.0	0	0.0	79997	1 2	3 4 5 6
AD20	0	0.0	0	0.0	79997	1 2	3 4 5 6
AG20	0	0.0	0	0.0	79997	1 2	3 4 5 6
CA20	0	0.0	8	0.01	79997	1 2	3 4 5 6
CL20	0	0.0	0	0.0	79997	1 2	3 4 5 6
GV20	0	0.0	0	0.0	79997	1 2	3 4 5 6
LG20	0	0.0	0	0.0	79997	1 2	3 4 5 6
PR20	0	0.0	0	0.0	79997	1 2	3 4 5 6
SST20	0	0.0	0	0.0	79997	1 2	3 4 5 6
TR20	0	0.0	8	0.01	79997	1 2	3 4 5 6
USS20	0	0.0	0	0.0	79997	1 2	3 4 5 6

Table 21: Iredell-Mecklenburg; senate

References

[1] Eric Autrey, Daniel Carter, Gregory Herschlag, Zach Hunter, and Jonathan C. Mattingly. Metropolized forest recombination for monte carlo sampling of graph partitions, 2021.

[2] Eric A. Autry, Daniel Carter, Gregory Herschlag, Zach Hunter, and Jonathan C. Mattingly. Multi-scale merge-split markov chain monte carlo for redistricting, 2020.

[3] Daniel Carter, Gregory Herschlag, Zach Hunter, and Jonathan Mattingly. A merge-split proposal for reversible monte carlo markov chain sampling of redistricting plans. *CoRR*, abs/1911.01503, 2019.

[4] Daniel Carter, Zach Hunter, Dan Teague, Gregory Herschlag, and Jonathan Mattingly. Optimal legislative county clustering in north carolina. *Statistics and Public Policy*, 7(1):19–29, 2020.

[5] Christopher Cooper, Blake Esselstyn, Gregory Herschlag, Jonathan Mattingly, and Rebecca Tippett. NC general assembly county clusterings from the 2020 census. Technical report, Duke University, August 2020. Quantyfing Gerrymandering Blog.

[6] Christopher Cooper, Blake Esselstyn, Gregory Herschlag, Jonathan Mattingly, and Rebecca Tippett. Nc general assembly county clusterings from the 2020 census. https://sites.duke.edu/quantifyinggerrymandering/files/2021/08/countyClusters2020.pdf, 2021.

[7] Gregory Herschlag, Han Sung Kang, Justin Luo, Christy Vaughn Graves, Sachet Bangia, Robert Ravier, and Jonathan C. Mattingly. Quantifying gerrymandering in north carolina. *Statistics and Public Policy*, 7(1):30–38, 2020.

[8] Gregory Herschlag, Robert Ravier, and Jonathan C. Mattingly. Evaluating Partisan Gerrymandering in Wisconsin. *arXiv e-prints*, page arXiv:1709.01596, Sep 2017.

[9] NC Legislature. 2021 joint redistricting committee adopted criteria. https://www.ncleg.gov/documentsites/committees/Senate2021-154/2021/08-12-2021/Criteria.adopted.8.12.pdf, 2021.

[10] J. C. Mattingly and C. Vaughn. Redistricting and the Will of the People. *ArXiv e-prints*, October 2014.

I declare under penalty of perjury under the laws of the state of North Carolina that the foregoing is true and correct to the best of my knowledge.

Jonathan Mattingly, 12/23/2021

CURRICULUM VITAE

Jonathan Christopher Mattingly

December 2021

120 Science Drive, I, Durham, NC 27708
297 Physics Building, Box 90320, Durham, NC 27708
jonathan.mattingly@duke.edu

CURRENT APPOINTMENTS AND AFFILIATIONS

Professor of Mathematics
James B. Duke Distinguished Professor
Professor of Statistical Science

EDUCATION, TRAINING AND CERTIFICATIONS

Ph.D., Princeton University 1998
B.S., Yale University 1992
High School, NC School of Science and Math, Durham NC 1988

APPOINTMENT HISTORY

Chair of the Department of Mathematics, Mathematics 2016 - 2020
Professor in the Department of Statistical Science, Statistical Science 2012 - 2015
Associate Professor, Statistical Science 2008 - 2011
Associate Professor of Mathematics, Mathematics 2006 - 2012
Assistant Professor of Mathematics, Mathematics 2002 – 2005
Member special year in SPDE/Tubulence, Institute for Advance Study, Princeton. 2002 - 2003
NSF Post-Doctoral Fellow, Stanford University. 1999 - 2002
Szego Assistant Professor of Mathematics, Stanford University. 1998 – 2002
Post-Doctoral Member, MSRI, UC Berkeley. 1998 - 1998
Contractor, AT&T Shannon Labs. 1999 – 1999 Summer Intern, Bell Labs, Lucent. 1996 - 1996

OTHER ACADEMIC POSITIONS

Member , Institute for Advance Study, Princeton, 2021
Simons Professor , MSRI, UC Berkeley. 2015 – 2015
Visiting Professor, Berlin Summer School, TU Berlin. 2009 - 2009
Visiting Member, Centro De Giorgi, SNS Pisa, Italy. 2006 - 2006
Visiting Professor, University de Marseilles. 2002 - 2002
Visiting Professor, MSRI, UC Berkeley. 2007 - 2007
Visiting Professor, University de Marseilles. 2010
Visiting Professor, University de Nice. 2012
Principle Lecturer, Saint Flour Summer school in Probability. 2007
Visiting Professor, University de Paris VI. 2008
Visiting Member, Institut Universitaire de France. 2003
Visiting Scholar, Mathematics Institute, Warwick University. 2000

PUBLICATIONS

Gao, Y., et al. "LIMITING BEHAVIORS OF HIGH DIMENSIONAL STOCHASTIC SPIN ENSEMBLES*." Communications in Mathematical Sciences, vol. 19, no. 2, Jan. 2021, pp. 453–94. Scopus, doi:10.4310/CMS.2021.v19.n2.a7.

Li, L., et al. "Numerical Methods For Stochastic Differential Equations Based On Gaussian Mixture." Communications in Mathematical Sciences, vol. 19, no. 6, Jan. 2021, pp. 1549–77. Scopus, doi:10.4310/CMS.2021.v19.n6.a5.

Bakhtin, Y., et al. "Singularities of invariant densities for random switching between two linear ODEs in 2D." Siam Journal on Applied Dynamical Systems, vol. 20, no. 4, Jan. 2021, pp. 1917–58. Scopus, doi:10.1137/20M1364345.

Autry, Eric A., et al. "Metropolized Multiscale Forest Recombination for Redistricting." Multiscale Modeling & Simulation, vol. 19, no. 4, Society for Industrial & Applied Mathematics (SIAM), Jan. 2021, pp. 1885–914. Crossref, doi:10.1137/21m1406854.

Gao, Y., et al. "Nonlocal stochastic-partial-differential-equation limits of spatially correlated noise-driven spin systems derived to sample a canonical distribution." Physical Review E, vol. 102, no. 5, Nov. 2020. Scopus, doi:10.1103/PhysRevE.102.052112.

Gao, Yuan, et al. "Nonlocal stochastic-partial-differential-equation limits of spatially correlated noise-driven spin systems derived to sample a canonical distribution." Physical Review. E, vol. 102, no. 5–1, Nov. 2020, p. 052112. Epmc, doi:10.1103/physreve.102.052112. PMID: 33327182.

Herschlag, G., et al. "Quantifying Gerrymandering in North Carolina." Statistics and Public Policy, vol. 7, no. 1, Jan. 2020, pp. 30–38. Scopus, doi:10.1080/2330443X.2020.1796400.

AGAZZI, A., and J. C. MATTINGLY. "SEEMINGLY STABLE CHEMICAL KINETICS CAN BE STABLE, MARGINALLY STABLE, OR UNSTABLE." Communications in Mathematical Sciences, vol. 18, no. 6, Jan. 2020, pp. 1605–42. Scopus, doi:10.4310/CMS.2020.v18.n6.a5.

Lu, Y., and J. C. Mattingly. "Geometric ergodicity of Langevin dynamics with Coulomb interactions." Nonlinearity, vol. 33, no. 2, Jan. 2020, pp. 675–99. Scopus, doi:10.1088/1361-6544/ab514a.

Chikina, M., et al. "Separating Effect From Significance in Markov Chain Tests." Statistics and Public Policy, vol. 7, no. 1, Jan. 2020, pp. 101–14. Scopus, doi:10.1080/2330443X.2020.1806763.

Carter, D., et al. "Optimal Legislative County Clustering in North Carolina." Statistics and Public Policy, vol. 7, no. 1, Jan. 2020, pp. 19–29. Scopus, doi:10.1080/2330443X.2020.1748552.

Herzog, David P., and Jonathan C. Mattingly. "Ergodicity and Lyapunov Functions for Langevin Dynamics with Singular Potentials." Communications on Pure and Applied Mathematics, vol. 72, no. 10, WILEY, Oct. 2019, pp. 2231–55. Wos, doi:10.1002/cpa.21862.

Chin, Andrew, et al. "The Signature of Gerrymandering in Rucho v. Common Cause." South Carolina Law Review, vol. 70, 2019.

Herschlag, G., et al. Evaluating Partisan Gerrymandering in Wisconsin. Sept. 2017.

Bakhtin, Y., et al. Smooth invariant densities for random switching on the torus (Submitted). Aug. 2017.

Johndrow, J. E., and J. C. Mattingly. Coupling and Decoupling to bound an approximating Markov Chain (Submitted). July 2017.

Glatt-Holtz, N. E., et al. "Scaling and Saturation in Infinite-Dimensional Control Problems with Applications to Stochastic Partial Differential Equations (Accepted)." Annals of Pde, June 2017.

Hairer, M., and J. Mattingly. The strong Feller property for singular stochastic PDEs (Submitted). 2016.

Tempkin, J. O. B., et al. "Trajectory stratification of stochastic dynamics (Accepted)." Siam Review, Society for Industrial and Applied Mathematics, 2016. PMID: 34650314. PMCID: PMC8514164.

Munch, E., et al. "Probabilistic Fréchet means for time varying persistence diagrams." Electronic Journal of Statistics, vol. 9, Jan. 2015, pp. 1173–204. Scopus, doi:10.1214/15-EJS1030.

Luo, S., and J. C. Mattingly. Scaling limits of a model for selection at two scales. 2015. PMID: 28867875. PMCID: PMC5580332.

Glatt-Holtz, N., et al. On Unique Ergodicity in Nonlinear Stochastic Partial Differential Equations (Submitted). 2015.

Herzog, David P., and Jonathan Christopher Mattingly. Noise-Induced Stabilization of Planar Flows II. Apr. 2014.

Mattingly, Jonathan C., and Etienne Pardoux. “Invariant measure selection by noise. An example.” Discrete and Continuous Dynamical Systems. Series A, vol. 34, 2014, pp. 4223–57. Manual, doi:10.3934/dcds.2014.34.4223.

Lawley, Sean D., et al. “Sensitivity to switching rates in stochastically switched ODEs.” Communications in Mathematical Sciences, vol. 12, 2014, pp. 1343–52. Manual, doi:10.4310/CMS.2014.v12.n7.a9.

Herzog, David P., and Jonathan C. Mattingly. “Noise-Induced Stabilization of Planar Flows I.” Arxiv Preprint Arxiv:1404.0957, 2014.

Bakhtin, Yuri, et al. “Regularity of invariant densities for 1D-systems with random switching.” Arxiv Preprint Arxiv:1406.5425, 2014.

Lawley, Sean D., et al. “Stochastic switching in infinite dimensions with applications to random parabolic PDEs.” Arxiv Preprint Arxiv:1407.2264, 2014.

Herzog, David P., and Jonathan C. Mattingly. “A practical criterion for positivity of transition densities.” Arxiv Preprint Arxiv:1407.3858, 2014.

Huckemann, Stephan, et al. “Sticky central limit theorems at isolated hyperbolic planar singularities.” Arxiv Preprint Arxiv:1410.6879, 2014.

Mattingly, Jonathan C., and Christy Vaughn. “Redistricting and the Will of the People.” Arxiv Preprint Arxiv:1410.8796, 2014.

Hotz, Thomas, et al. “Sticky central limit theorems on open books.” The Annals of Applied Probability, vol. 23, 2013, pp. 2238–58. Manual, doi:10.1214/12-AAP899.

Mattingly, J. C., et al. “Geometric ergodicity of a bead-spring pair with stochastic Stokes forcing.” Stochastic Processes and Their Applications, vol. 122, no. 12, Dec. 2012, pp. 3953–79. Scopus, doi:10.1016/j.spa.2012.07.003.

Luo, Shishi, et al. “The impact of host immune status on the within-host and population dynamics of antigenic immune escape.” J R Soc Interface, vol. 9, no. 75, Oct. 2012, pp. 2603–13. Pubmed, doi:10.1098/rsif.2012.0180. PMID: 22572027. PMCID: PMC3427510.

Athreyaz, A., et al. “Propagating lyapunov functions to prove noise-induced stabilization.” Electronic Journal of Probability, vol. 17, 2012. Scival, doi:10.1214/EJP.v17-2410.

Porporato, A., et al. “Local kinetic interpretation of entropy production through reversed diffusion.” Phys Rev E Stat Nonlin Soft Matter Phys, vol. 84, no. 4 Pt 1, Oct. 2011, p. 041142. Pubmed, doi:10.1103/PhysRevE.84.041142. PMID: 22181122.

Hairer, Martin, and Jonathan C. Mattingly. Yet another look at Harris’ ergodic theorem for Markov chains. Vol. 63, Birkhäuser/Springer Basel AG, Basel, 2011, pp. 109–17. Manual, doi:10.1007/978-3-0348-0021-1_7.

Hairer, M., et al. “Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations.” Probability Theory and Related Fields, vol. 149, no. 1, 2011, pp. 223–59. Scival, doi:10.1007/s00440-009-0250-6.

Anderson, D. F., and J. C. Mattingly. “A weak trapezoidal method for a class of stochastic differential equations.” Communications in Mathematical Sciences, vol. 9, no. 1, 2011, pp. 301–18.

Hairer, M., and J. C. Mattingly. “A theory of hypoellipticity and unique ergodicity for semilinear stochastic PDEs.” Electronic Journal of Probability, vol. 16, 2011, pp. 658–738.

Koelle, K., et al. “A dimensionless number for understanding the evolutionary dynamics of antigenically variable RNA viruses.” Proceedings of the Royal Society B: Biological Sciences, vol. 278, no. 1725, 2011, pp. 3723–30. Scival, doi:10.1098/rspb.2011.0435. PMID: 21543353. PMCID: PMC3203497.

Cooke, Ben, et al. “Geometric Ergodicity of Two–dimensional Hamiltonian systems with a Lennard–Jones–like Repulsive Potential.” Arxiv Preprint Arxiv:1104.3842, 2011.

Mattingly, J. C., et al. “Convergence of numerical time-averaging and stationary measures via Poisson equations.” Siam Journal on Numerical Analysis, vol. 48, no. 2, 2010, pp. 552–77. Scival, doi:10.1137/090770527.

Hairer, M., and J. C. Mattingly. “Slow energy dissipation in anharmonic oscillator chains.” Communications on Pure and Applied Mathematics, vol. 62, no. 8, 2009, pp. 999–1032. Scival, doi:10.1002/cpa.20280.

Martin Hairer, Jonathan C. “Spectral gaps in Wasserstein distances and the 2D stochastic Navier-Stokes equations.” Annals of Probability, no. 6, 2008, pp. 993–1032.

Iyer, Gautam, and Jonathan Mattingly. “A stochastic-Lagrangian particle system for the Navier-Stokes equations.” Nonlinearity, vol. 21, 2008, pp. 2537–53. Manual, doi:10.1088/0951-7715/21/11/004.

Mattingly, J. C., et al. “Simple systems with anomalous dissipation and energy cascade.” Communications in Mathematical Physics, vol. 276, no. 1, Nov. 2007, pp. 189–220. Scopus, doi:10.1007/s00220-007-0333-0.

Anderson, D. F., and J. C. Mattingly. “Propagation of fluctuations in biochemical systems, II: Nonlinear chains.” Iet Systems Biology, vol. 1, no. 6, Nov. 2007, pp. 313–25. Epmc, doi:10.1049/iet-syb:20060063. PMID: 18203578.

Bakhtin, Y., and J. C. Mattingly. “Malliavin calculus for infinite-dimensional systems with additive noise.” Journal of Functional Analysis, vol. 249, no. 2, Aug. 2007, pp. 307–53. Scopus, doi:10.1016/j.jfa.2007.02.011.

Anderson, David F., et al. “Propagation of fluctuations in biochemical systems, I: linear SSC networks.” Bulletin of Mathematical Biology, vol. 69, no. 6, Aug. 2007, pp. 1791–813. Epmc, doi:10.1007/s11538-007-9192-2. PMID: 17457656.

Lamba, H., et al. “An adaptive Euler-Maruyama scheme for SDEs: Convergence and stability.” Ima Journal of Numerical Analysis, vol. 27, no. 3, Jan. 2007, pp. 479–506. Scopus, doi:10.1093/imanum/drl032.

Mattingly, J. C., et al. “Anomalous dissipation in a stochastically forced infinite-dimensional system of coupled oscillators.” Journal of Statistical Physics, vol. 128, no. 5, 2007, pp. 1145–52. Scival, doi:10.1007/s10955-007-9351-8.

Nijhout, H. Frederik, et al. “Erratum to H. Frederik Nijhout, et al. Epigenetics Volume 1, Issue 2; pp. 81-87.” Epigenetics, vol. 1, no. 3, Informa UK Limited, July 2006, pp. 115–115. Crossref, doi:10.4161/epi.1.3.3281.

Hairer, M., and J. C. Mattingly. “Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing.” Annals of Mathematics, vol. 164, no. 3, 2006, pp. 993–1032.

Mattingly, J. C., and É. Pardoux. “Malliavin calculus for the stochastic 2D Navier-Stokes equation.” Communications on Pure and Applied Mathematics, vol. 59, no. 12, 2006, pp. 1742–90. Scival, doi:10.1002/cpa.20136.

Bakhtin, Y., and J. C. Mattingly. “Stationary solutions of stochastic differential equations with memory and stochastic partial differential equations.” Communications in Contemporary Mathematics, vol. 7, no. 5, Oct. 2005, pp. 553–82. Scopus, doi:10.1142/S0219199705001878.

Mattingly, J. C., and T. M. Suidan. “The small scales of the stochastic Navier-Stokes equations under rough forcing.” Journal of Statistical Physics, vol. 118, no. 1–2, Jan. 2005, pp. 343–64. Scopus, doi:10.1007/s10955-004-8787-3.

Hairer, Martin, et al. “Malliavin calculus and ergodic properties of highly degenerate 2D stochastic Navier–Stokes equation.” Arxiv Preprint Math/0409057, 2004.

Hairer, Martin, et al. “Malliavin calculus for highly degenerate 2D stochastic Navier-Stokes equations.” Comptes Rendus Mathématique. Académie Des Sciences. Paris, vol. 339, 2004, pp. 793–96. Manual, doi:10.1016/j.crma.2004.09.002.

Hairer, Martin, and Jonathan C. Mattingly. “Ergodic properties of highly degenerate 2D stochastic Navier-Stokes equations.” Comptes Rendus Mathématique. Académie Des Sciences. Paris, vol. 339, 2004, pp. 879–82. Manual, doi:10.1016/j.crma.2004.09.035.

Mattingly, Jonathan C. On recent progress for the stochastic Navier Stokes equations. Univ. Nantes, Nantes, 2003, p. Exp.No.XI-52.

Mattingly, Jonathan C. “The dissipative scale of the stochastics Navier-Stokes equation: regularization and analyticity.” Journal of Statistical Physics, vol. 108, 2002, pp. 1157–79. Manual, doi:10.1023/A:1019799700126.

Mattingly, J. C., et al. “Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise.” Stochastic Processes and Their Applications, vol. 101, 2002, pp. 185–232. Manual, doi:10.1016/S0304-4149(02)00150-3.

Mattingly, J. C. “Contractivity and ergodicity of the random map $x\mapsto\vert x-θ\vert $.” Rossi\U\I Skaya Akademiya Nauk. Teoriya Veroyatnoste\U\I I Ee Primeneniya, vol. 47, 2002, pp. 388–97. Manual, doi:10.1137/S0040585X97979767.

Mattingly, Jonathan C. “Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics.” Communications in Mathematical Physics, vol. 230, 2002, pp. 421–62. Manual, doi:10.1007/s00220-002-0688-1.

Mattingly, J. C., and A. M. Stuart. “Geometric ergodicity of some hypo-elliptic diffusions for particle motions.” Markov Processes and Related Fields, vol. 8, 2002, pp. 199–214.

Mattingly, Jonathan Christopher. “Contractivity and ergodicity of the random map $x\mapsto|x-\theta|$.” Teoriya Veroyatnostei I Ee Primeneniya, vol. 47, no. 2, Steklov Mathematical Institute, 2002, pp. 388–97. Crossref, doi:10.4213/tvp3671.

E, Weinan, et al. “Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation.” Communications in Mathematical Physics, vol. 224, 2001, pp. 83–106. Manual, doi:10.1007/s002201224083.

E, Weinan, and Jonathan C. Mattingly. “Ergodicity for the Navier-Stokes equation with degenerate random forcing: finite-dimensional approximation.” Communications on Pure and Applied Mathematics, vol. 54, 2001, pp. 1386–402. Manual, doi:10.1002/cpa.10007.

Mattingly, Jonathan C. “Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity.” Communications in Mathematical Physics, vol. 206, 1999, pp. 273–88. Manual, doi:10.1007/s002200050706.

Mattingly, J. C., and Ya G. Sinai. “An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations.” Communications in Contemporary Mathematics, vol. 1, 1999, pp. 497–516. Manual, doi:10.1142/S0219199799000183.

Holmes, Philip J., et al. “Low-dimensional models of coherent structures in turbulence.” Physics Reports. a Review Section of Physics Letters, vol. 287, 1997, pp. 337–84. Manual, doi:10.1016/S0370-1573(97)00017-3.

Johndrow, James E., et al. Optimal approximating Markov chains for Bayesian inference.

Bangia, Sachet, et al. Redistricting: Drawing the Line.

Johndrow, James E., and Jonathan C. Mattingly. Error bounds for Approximations of Markov chains used in Bayesian Sampling.

Wang, Chuang, et al. Scaling Limit: Exact and Tractable Analysis of Online Learning Algorithms with Applications to Regularized Regression and PCA.

Carter, Daniel, et al. A Merge-Split Proposal for Reversible Monte Carlo Markov Chain Sampling of Redistricting Plans.

Herschlag, Gregory, et al. Non-reversible Markov chain Monte Carlo for sampling of districting maps.

Autry, Eric A., et al. Multi-Scale Merge-Split Markov Chain Monte Carlo for Redistricting.

Leimbach, Matti, et al. Noise-induced strong stabilization.

Mattingly, Jonathan C., et al. The Gaussian Structure of the Singular Stochastic Burgers Equation.

Herzog, David P., et al. Gibbsian dynamics and the generalized Langevin equation.

Earle, Gabriel, and Jonathan Mattingly. Convergence of Stratified MCMC Sampling of Non-Reversible Dynamics.

Mattingly, Jonathan C., et al. “Diffusion limits of the random walk Metropolis algorithm in high dimensions.” Annals of Applied Probability, vol. 22, no. 3, pp. 881–930. Arxiv, doi:10.1214/10-AAP754.

Heymann, Matthias, et al. Rare Transition Events in Nonequilibrium Systems with State-Dependent Noise: Application to Stochastic Current Switching in Semiconductor Superlattices.

Theses and Dissertations

Mattingly, Jonathan. The Stochastic Navier-Stokes Equation: Energy Estimates and Phase Space Contraction, under Yakov Sinai.

PROFESSIONAL AWARDS AND SPECIAL RECOGNITION

IE Block Community Lecture. SIAM. 2021

Defenders of Democracy. National Common Cause. 2018

Fellow of the American Mathematical Society . American Mathematical Society. 2015

Simons Visiting Professor . MSRI. 2015

Institute of Mathematical Statistics Fellow. Institute of Mathematical Statistics. 2012

Faculty Early Career Development (CAREER) Program. National Science Foundation. 2005

Presidential Early Career Awards for Scientists and Engineers. National Science Foundation. 2005
Sloan Research Fellowship-Mathematics. Alfred P. Sloan Foundation. 2005
School of Mathematics/ Members. Institute for Advanced Study. 2002

PRESENTATIONS AND APPEARANCES

Sampling to Understand Gerrymandering and Influence Public Policy. MIT. January 1, 2021
Panel on Qunatifying Gerrymandering. Democracy in America. October 1, 2021
Hearing the Will of the People. ISM. August 1, 2021
Non-rversible samplers for Gerrymandering. Netherlands. August 1, 2021
The Gaussian Structure of the Stochastically Forced Burgers Equation. Berlin. May 1, 2021
The Mathematics and Policy of Gerrymandering. IAS. December 1, 2021
Gaussian Structure of Burgers Equation. India (online). January 1, 2021
A new model of randomly forced Fluid equations. Princeton Fluids Seminar. November 1, 2021
A new model of randomly forced Fluid equations. ICEM. October 1, 2021
A new model of randomly forced Fluid equations. IAS. December 1, 2021

Gaussian Structure of Stochastic Burgers. February 1, 2021
New Sampling Methods of Quantifying Gerrymandering . Brown Applied Math Colloquium . October 1, 2020
Interactions between noise and instabilities.. IHP, Paris. July 1, 2018
Quantifying Gerrymandering: A Mathematician Goes to Court. July 1, 2018
Ergodicity of Singular SPDEs. Columbia. May 1, 2018
Approximate/exact controllability and ergodicity for (additive noise) SPDEs/SODEs. CIRM, Marseilles 2018
Discovering the geopolitical structure of the United States through Markov Chain Monte Carlo sampling. The Alan Turing Institute, UK. May 1, 2018
Drawing the line in redistricting (A mathematician's take). Stanford University. March 1, 2018
Ergodic and global solutions for singular SPDEs. Corvallis, Oregon. March 1, 2018
A mathematician Goes to Court. October 1, 2017
Stabilization of Stochastic Dynamics . UCLA. IPAM. January 1, 2017
Stabilization and noise. Berekey Mathematics Department. November 12, 2015
Stochastic PDEs. October 1, 2015
Ergodicity Finite and Infinite dimentional Markov Chains. McGill University. July 1, 2015

Lectures

New Sampling Methods to Quantify Gerymandering. IID. Duke Law and TRIPODS. March 1, 2020
Anatomy of an ergodic theorem. Summer School. June 1, 2018
Dynamics Days 2014. Atlanta GA. January 4, 2014
Stabilization by Noise. November 19, 2013
Uniqueness of the inviscid limit in a simple model damped/driven system.. Probability and Mathematical Physics Seminar. November 5, 2013
Stochastic stabilization of OEDs.. Applied Math Seminar, NYU. September 6, 2013
Stochastic partial differential equations. SPA2013. August 1, 2013
Stabilization by noise. University of Maryland. May 1, 2013
Stablization by Noise. Conférence en l'honneur d'Etienne Pardoux, CIRM, Marseillais France.. February 14, 2013
Perspectives on Ergodicity. Conference on SPDEs, IMA, Minnesota. January 14, 2013

A Numerical Method for the SDEs from Chemical Equations. Probability and Biology section, 2012 Canadian Mathematical society (winter meeting). December 1, 2012

Minerva Lectures: Erodicity of Markov Processes: From Chains to SDEs to SPDEs. Mathematics Department, Columbia University. November 1, 2012

Stochastic Stabilization. Inria - Sophia Antipolis. July 1, 2012

A Menagerie of Stabilization. Joint Probability and Analysis Seminar, Nice, France. July 1, 2012

Building Lyapunov Functions (4 lectures). EPSRC Symposium Workshop – Easter Probability Meeting. March 1, 2012

Noise Induced Stability. MBI. February 1, 2012

A Menagerie of Stochastic Stabilization. CAMP/Probability Seminar, University of Chicago. October 18, 2011

A menagerie of stochastic stabilization. Equadiff 2011, Loughborough University. August 1, 2011

Coarse–graining of many–body systems: analysis, computations and applications. July 1, 2011

Ergodicity of systems with singular interaction terms. Stochastic Dynamics Transition Workshop, SAMSI. November 18, 2010

Oberwolfach Seminar: The Ergodic Theory of Markov Processes. Oberwolfach, Germany. October 1, 2010

Malliavin Calculus to prove ergodic theorems for SPDEs. ICM Satellite Conference on Probability and Stochastic Processes Indian Statistical Institute, Bangalore. August 13, 2010

SPDE scaling limits of an Markov chain Montecarlo algorithm. Stochastic Partial Differential Equations: Approximation, Asymptotics and Computation, Newton Institute. June 28, 2010

The spread of randomness. German-American Frontiers of Science, Potsdam Germany. June 1, 2010

How to prove an ergodic theorem. oberwolfach. May 1, 2010

Coupling at infinity. Seminar on Stochastic Processes. March 30, 2010

Long time stochastic simiulation. Imperial College. March 15, 2010

Spectral Gaps in Wasserstien Distance. Ergodic Theory Seminary, Princeton Mathematics. March 4, 2010

Trouble with a chain of stochastic oscillators. PACM, Princeton University. March 2, 2010

Hypo-ellipticity for SPDEs. SPDE program , Newton Institute. March 1, 2010

Numerics of SDEs. Warwick University, UK. February 24, 2010

Long Time Behavior of Stochastically Forced PDEs.. AMS Joint Meeting, San Francisco. January 14, 2010

Ellipticity and Hypo-ellipticity for SPDEs *or* What is ellipticity in infinite dimensions anyway?. Stochastic Partial Differential Equations, Newton Institute. January 8, 2010

SPDE Limits of the Random Walk Metropolis Algorithm in High Dimensions. SIAM PDE Meeting. December 7, 2009

Stochastic fluctuations in bio chemical networks. MBI: Mathematical Developments Arising from Biology. November 9, 2009

What makes infinite dimensional Markov processes different ?. Stochastic Process and Applications, Berlin. July 1, 2009

Introduction to Ergodicity in Infinite Dimentions. TU Berlin. July 1, 2009

Stochastically forced fluid equations: Transfer between scales and ergodicity.. AMS Sectional Meeting (invited talk). April 4, 2009

Trouble with a chain of stochastic oscillators. Princeton University. PACM. April 3, 2009

What makes the ergodic theory if Markov Chains in infinite dimensions different (and dificult) ?. Princeton Ergodic theory seminar. March 3, 2009

Ergodicity, Energy Transfer, and Stochastic Partial Differential Equations. Columbia University. Columbia University. December 15, 2008

The Spread of Randomness: Ergodicity in Infinite Dimensions. Mathematisches Forschungsinstitut Oberwolfach. December 15, 2008

The spread of randomness through dimensions. IPAM. November 1, 2008

The spread of randomness through dimensions. IPAM- Mathematical Frontiers in Network Multi-Resolution Analysis. November 1, 2008

Troubles with oscillators. Stanford: JBK85, Workshop on Applied Mathematics IN HONOR OF JOSEPH B. KELLER. October 1, 2008

What is different about the ergodic theory of stochastic PDEs (vs ODEs). UC Irvine, PDE and Probabilty Seminar. October 1, 2008

Trouble with a chain of stochastic oscillators. Stochastic Seminar, GaTech. September 1, 2008

Troubles with oscillators. East Midlands Stochastic Analysis Seminars. August 1, 2008

Troubles with chains of anharmonic oscillators. Statisical Mechaniques working group. June 1, 2008

The spread of randomness in infinite dimensions and ergodicity for SPDEs. Stochastic Analysis, Random Fields and Applications, Asscona IT. June 1, 2008

Ergodicity of Degenerately forced SPDEs. Séminaire de Probabilités, Laboratoire de Probabilités et Modèles Aléatoires des Universités Pierre et Marie Curie et Denis Diderot. May 27, 2008

Ergodicity of Degenerately forced SPDEs. ETH, Zurich. May 1, 2008

Named Lectures

Barton Lectures in Computational Mathematics. UNCG. November 1, 2021

IE Block Community Lecture . SIAM Annual Meeting. SIAM. July 1, 2021

Quantifying and Understanding Gerrymandering - How a quest to understand his state's political geography led a mathematician to court. ICERM . October 1, 2020

AMS Regional Meeting Plenary Speaker. Gainesville . AMS. January 1, 2019

Long Time Numerical Simulation of SDEs. Insbruk. SciCADE2019 . January 1, 2019

Quantifying Gerrymandering: A mathematician goes to court. UBC. May 1, 2018

Quantifying Gerrymandering: a mathematician goes to court. Stanford Mathematics Department. March 1, 2018

Stochastic PDEs. July 1, 2016

Event/Org Administration

Co-Organizer . Quantifying Gerrymandering. SAMSI. October 2018

Co-Organizer . Regional Gerrymandering Conference. November 2017

Co-Organizer . Interacting particle systems WITH APPLICATIONS IN BIOLOGY, ECOLOGY, AND STATISTICAL PHYSICS. SEPC 2017. May 2017

Organiser Special Term. MSRI, Berkeley CA. August 2015 - December 2015

Organized invited session at SPA2013. August 2013

Co Organizer (with Amarjit Budhiraja) : Seminar on Stochastic Processes 2013. March 2013

Local Orgnaizer (with Rick Durrett) : Woman in Probability III. October 2012

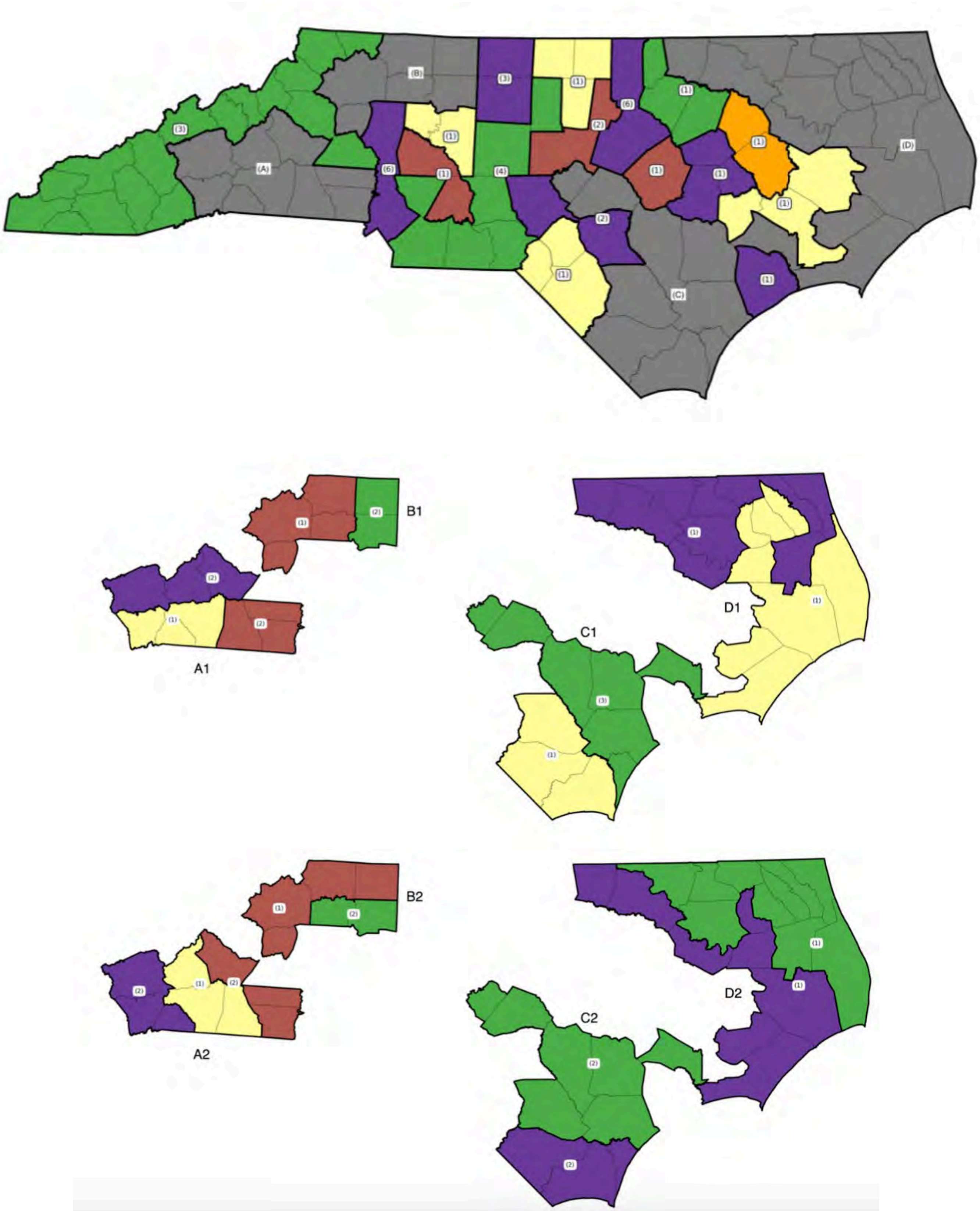
SAMSI Stochastic Dynamics tradition workshop. November 2010

MFO week long school on ergodic theory. October 2010

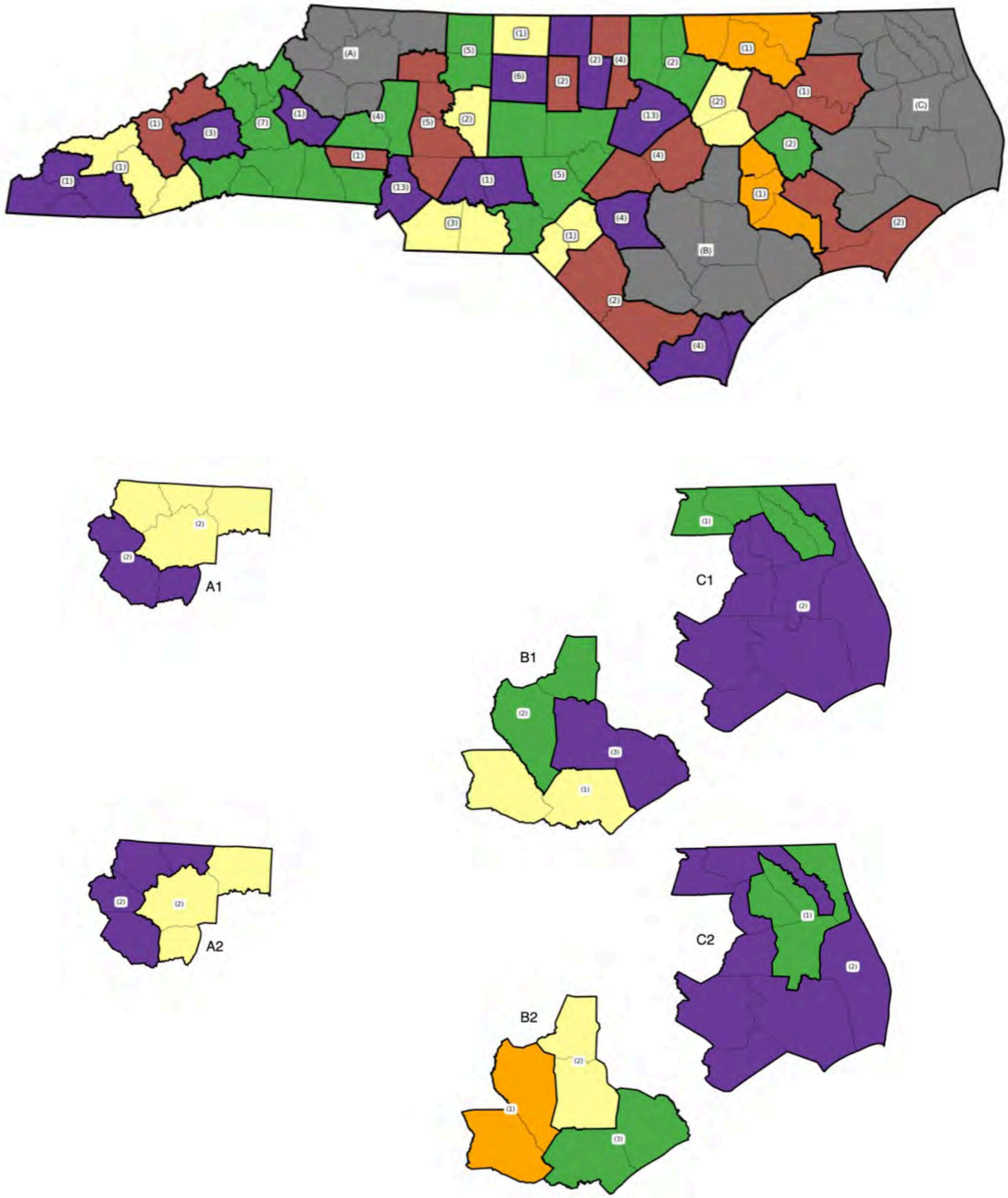
SAMSI Opening Workshop for Stochastic Dynamics. August 2009

local liaison/Organizer SAMSI year on stochastic dynamics. 2009 - 2010

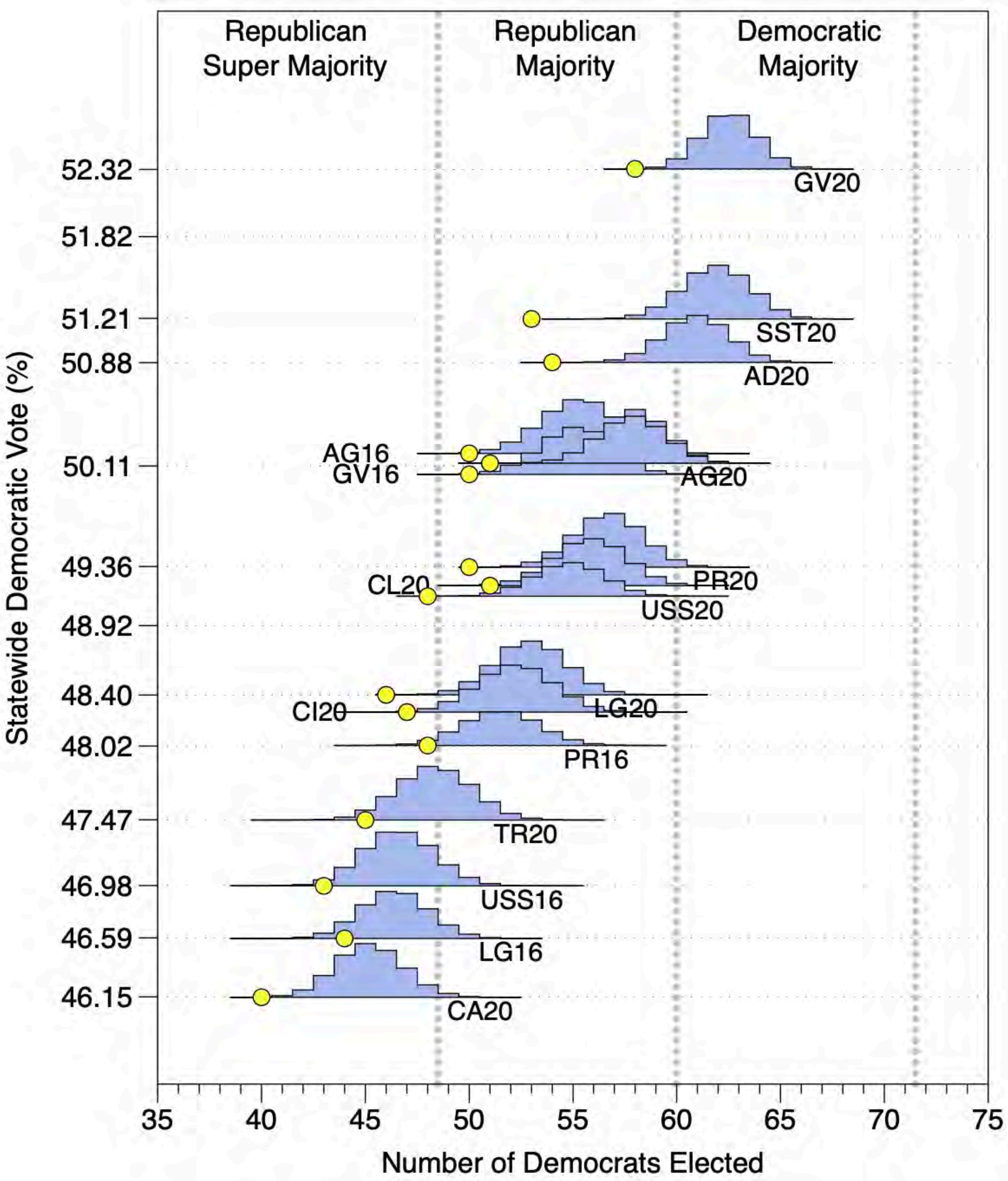
Organiser Special Term. MSRI, Berkeley CA. August 2007 - December 2007



Mattingly Figure 4.3.1: Senate



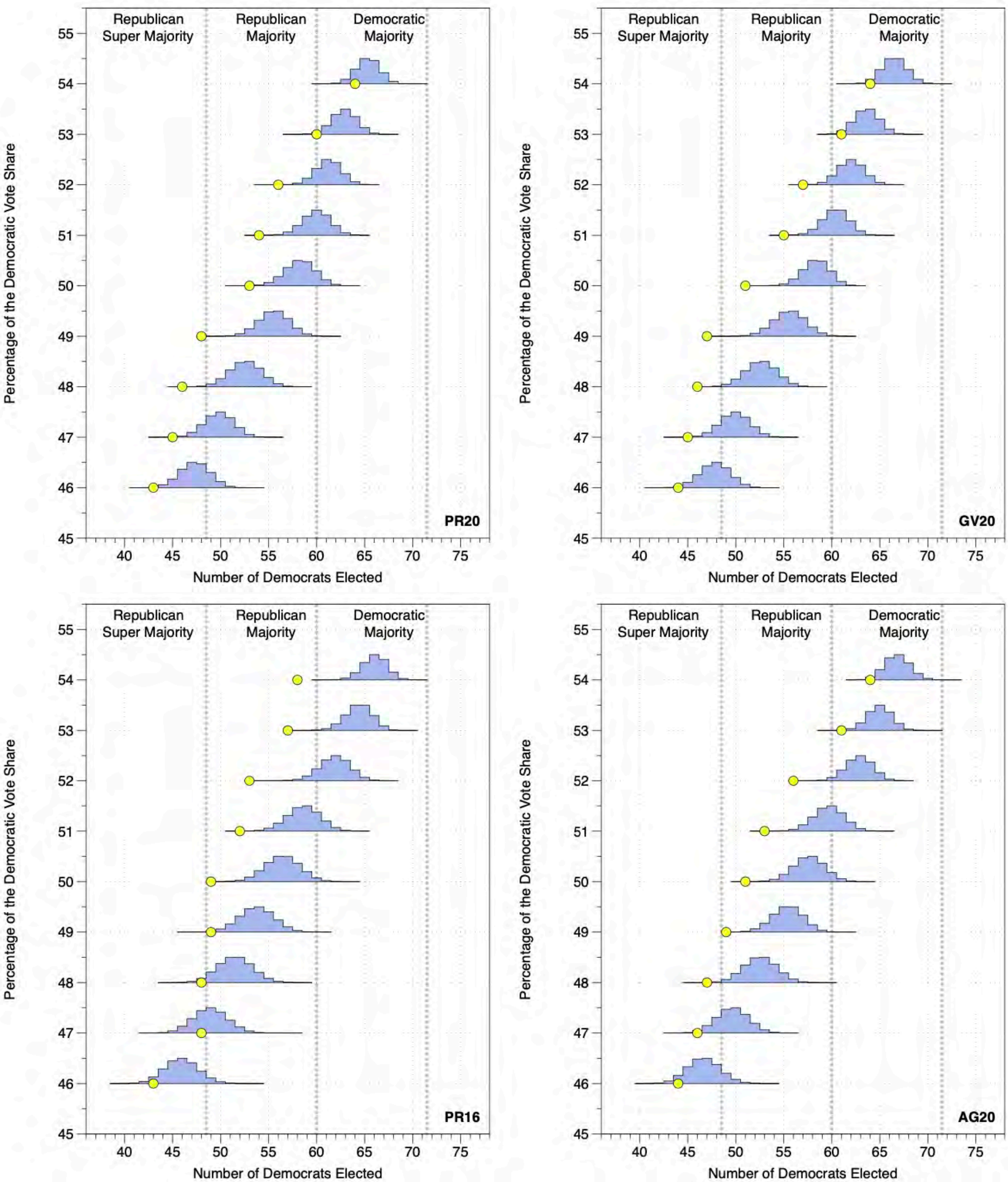
Mattingly Figure 4.3.2: House



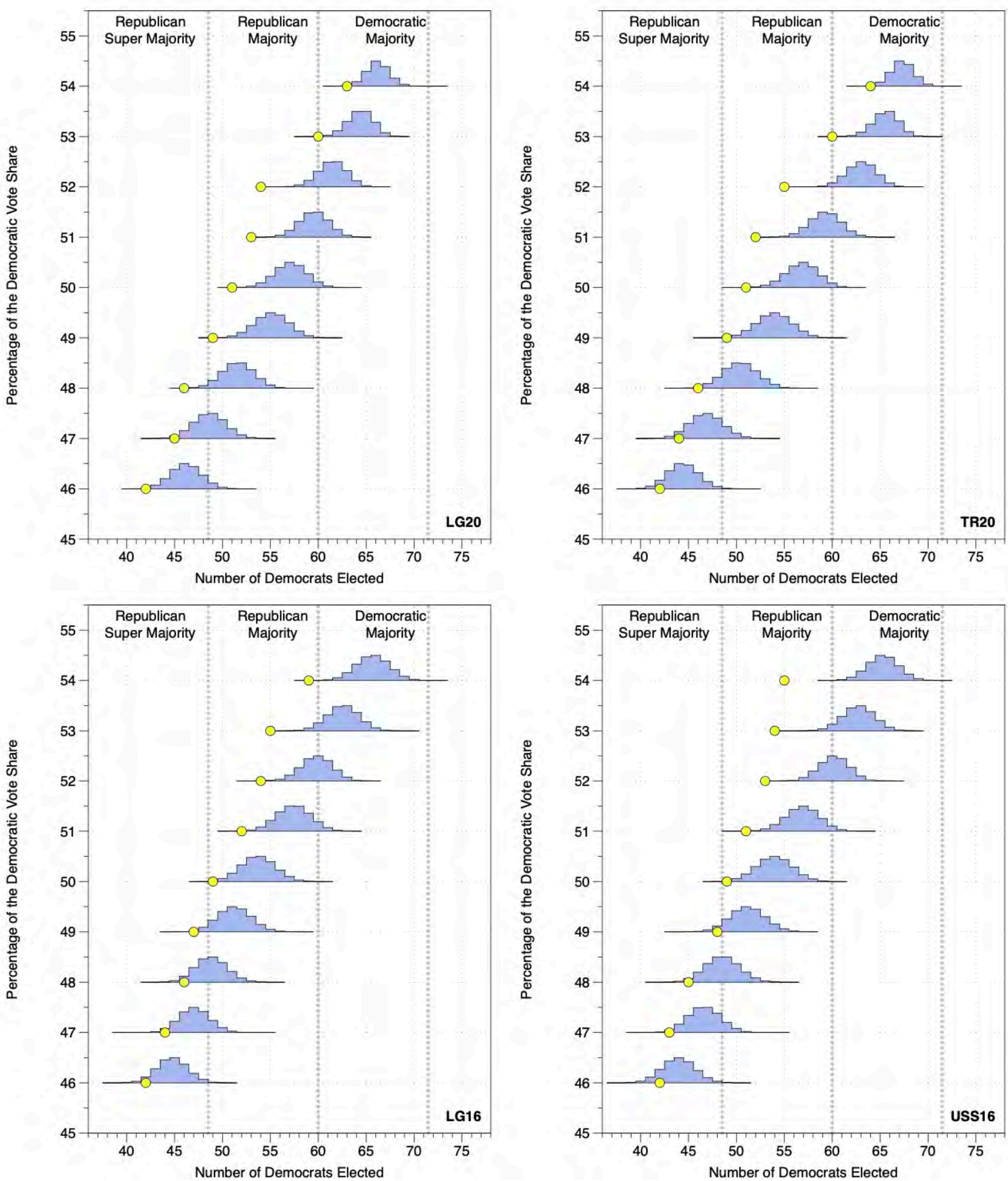
Mattingly Figure 5.1.1: The Collected Seat Histogram for the Primary Ensemble on the NC House. The individual histograms give the frequency of the Democratic seat count for each of the statewide elections considered from the years 2016 and 2020. The histograms are organized vertically based on the statewide partisan vote fraction for each election. The more Republican elections are placed lower on the plot while more Democratic elections are placed higher. Three dotted lines denote the boundary between where the supermajorities and simple majorities are in force. The yellow dot represents the enacted plan.

% Dem	Election	% Outlier	# Outlier	# Samples
52.32%	GV20	0.118%	118	100000
51.21%	SST20	0.000%	0	100000
50.88%	AD20	0.007%	7	100000
50.20%	AG16	0.451%	451	100000
50.13%	AG20	0.005%	5	100000
50.05%	GV16	0.399%	399	100000
49.36%	PR20	0.007%	7	100000
49.22%	CL20	0.759%	759	100000
49.14%	USS20	0.012%	12	100000
48.40%	LG20	0.009%	9	100000
48.27%	CI20	0.461%	461	100000
47.47%	TR20	5.569%	5569	100000
46.98%	USS16	3.066%	3066	100000
46.59%	LG16	11.778%	11778	100000
46.15%	CA20	0.094%	94	100000

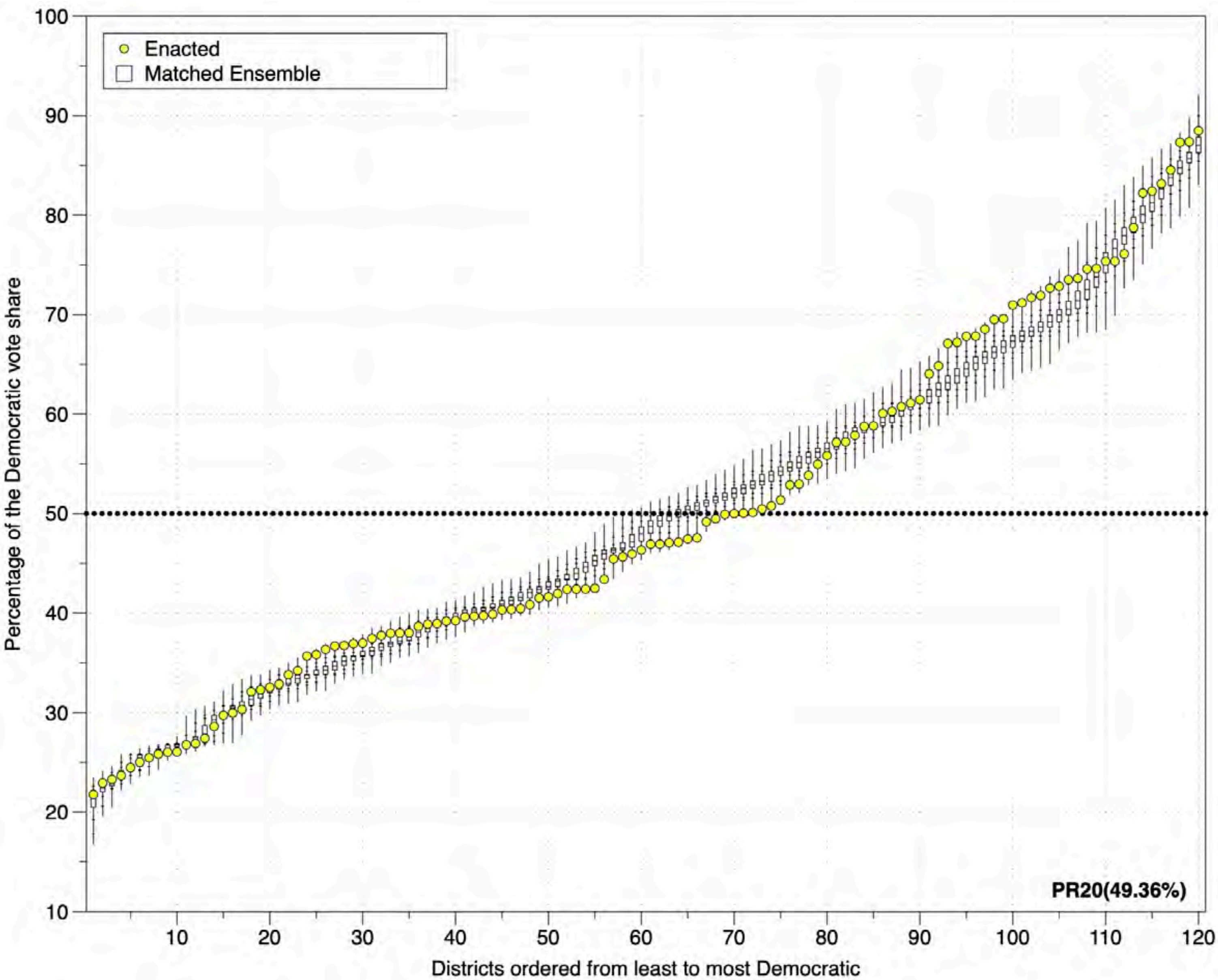
Mattingly Table 1: NC House Collected Seat Histogram Outlier Data. Starting from the left, the first column gives the statewide partisan makeup of the election under consideration whose abbreviation is given in the second column from the left. The right most column gives the total number of plans in the ensemble considered which is 100,000. The second column from the right gives the number of those 100,000 plans which elect the same or less Democrats under the given election. These are the plans which are as much or more of an outlier than the enacted map. The middle column is the percentage of plans which are more or equal of an outlier. (It is calculated by dividing the 2nd column from the right by 100,000 and multiplying by 100 to make a percentage.) The extremely low percentages in the middle column shows that the enacted plan is an extreme outlier across many different electoral settings.



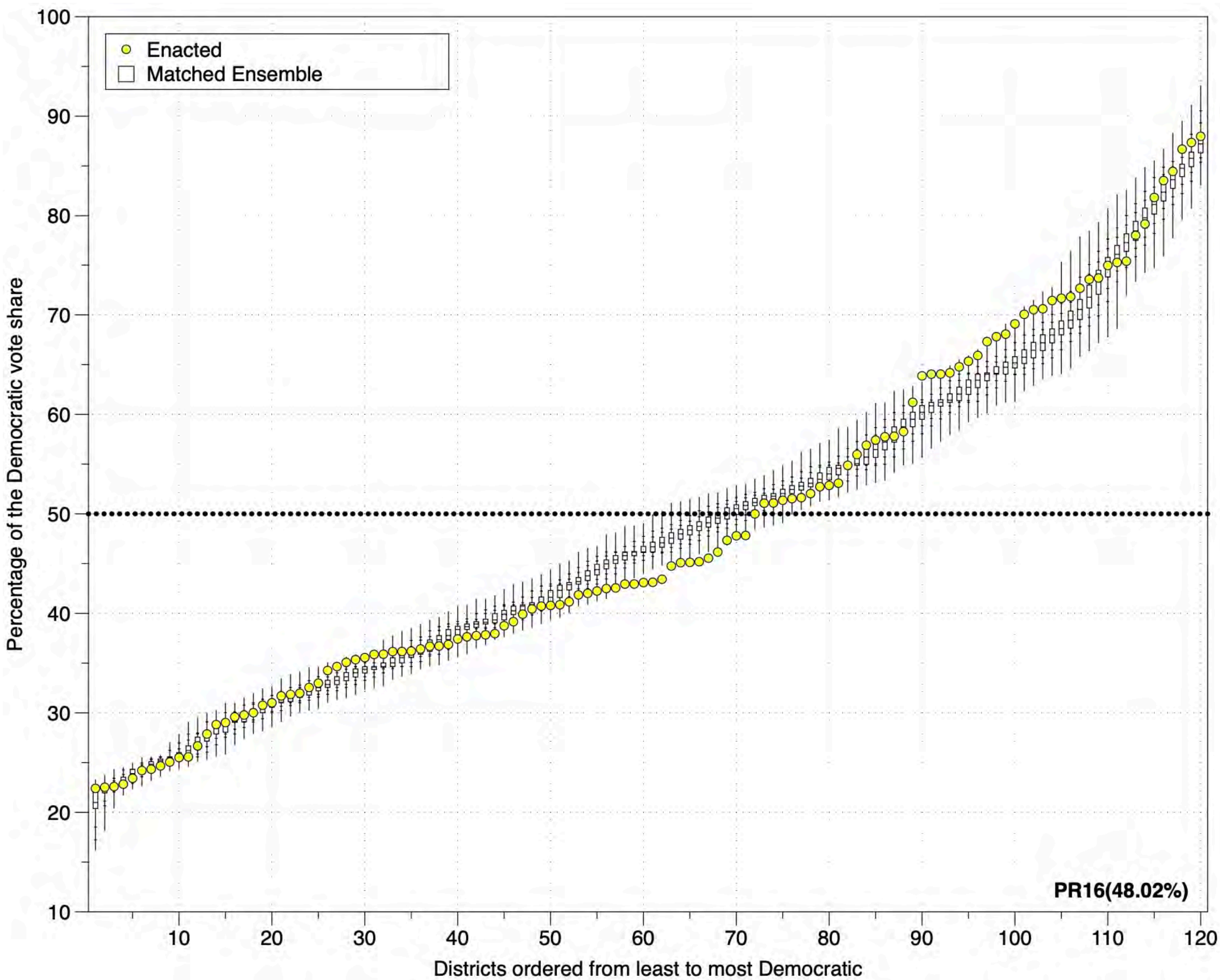
Mattingly Figure 5.1.2: The individual histograms give the frequency of the Democratic seat count in the ensemble for each of the shown statewide elections, with a uniform swing. The histograms are organized vertically based on the statewide partisan vote fraction. The more Republican swings are placed lower on the plot while more Democratic swings are placed higher. Three dotted lines denote the boundary between where the supermajorities and simple majorities are in force. The yellow dot is the enacted plan.



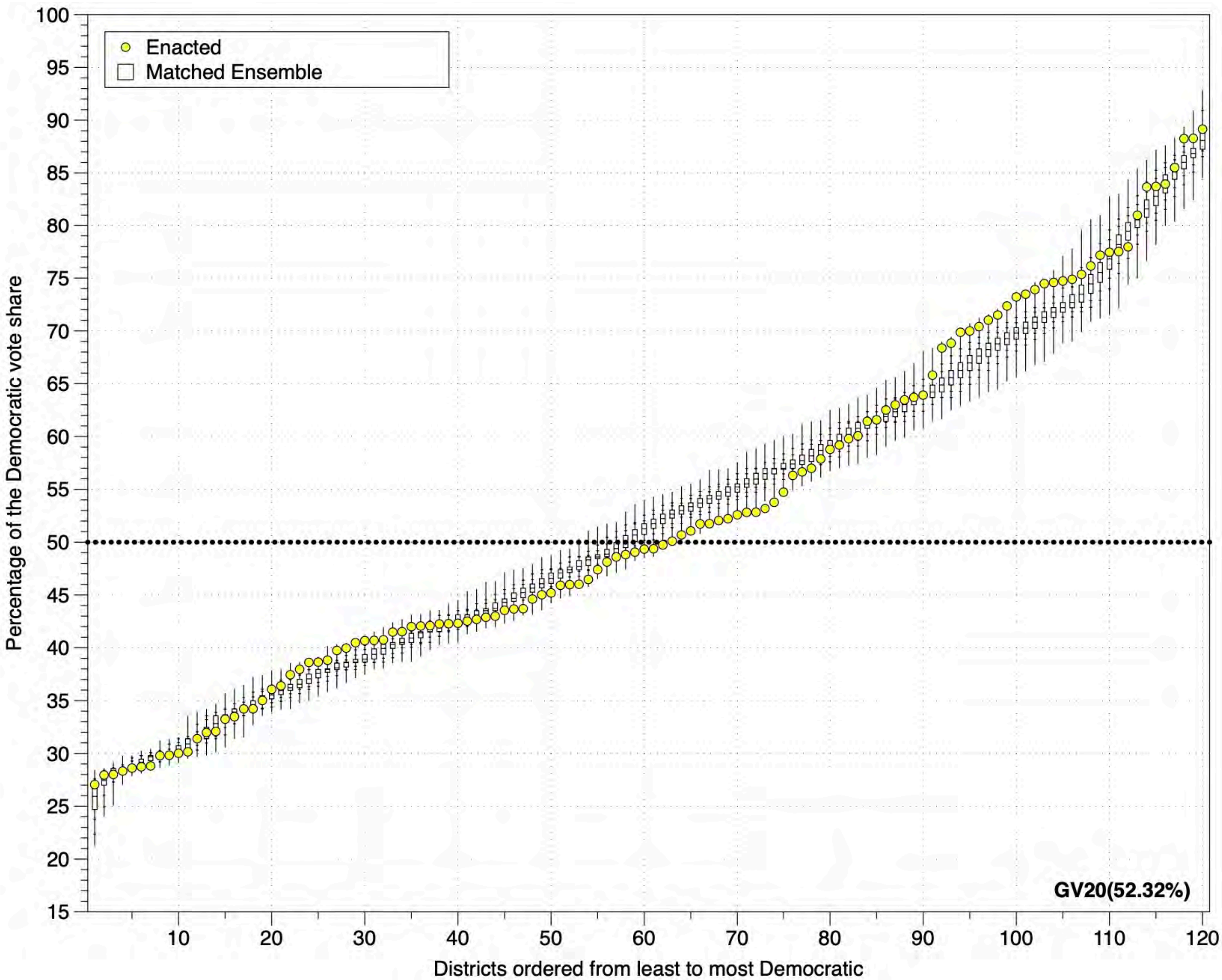
Mattingly Figure 5.1.3: The individual histograms give the frequency of the Democratic seat count in the ensemble for each of the shown statewide elections, with a uniform swing. The histograms are organized vertically based on the statewide partisan vote fraction. The more Republican swings are placed lower on the plot while more Democratic swings are placed higher. Three dotted lines denote the boundary between where the supermajorities and simple majorities are in force. The yellow dot is the enacted plan.



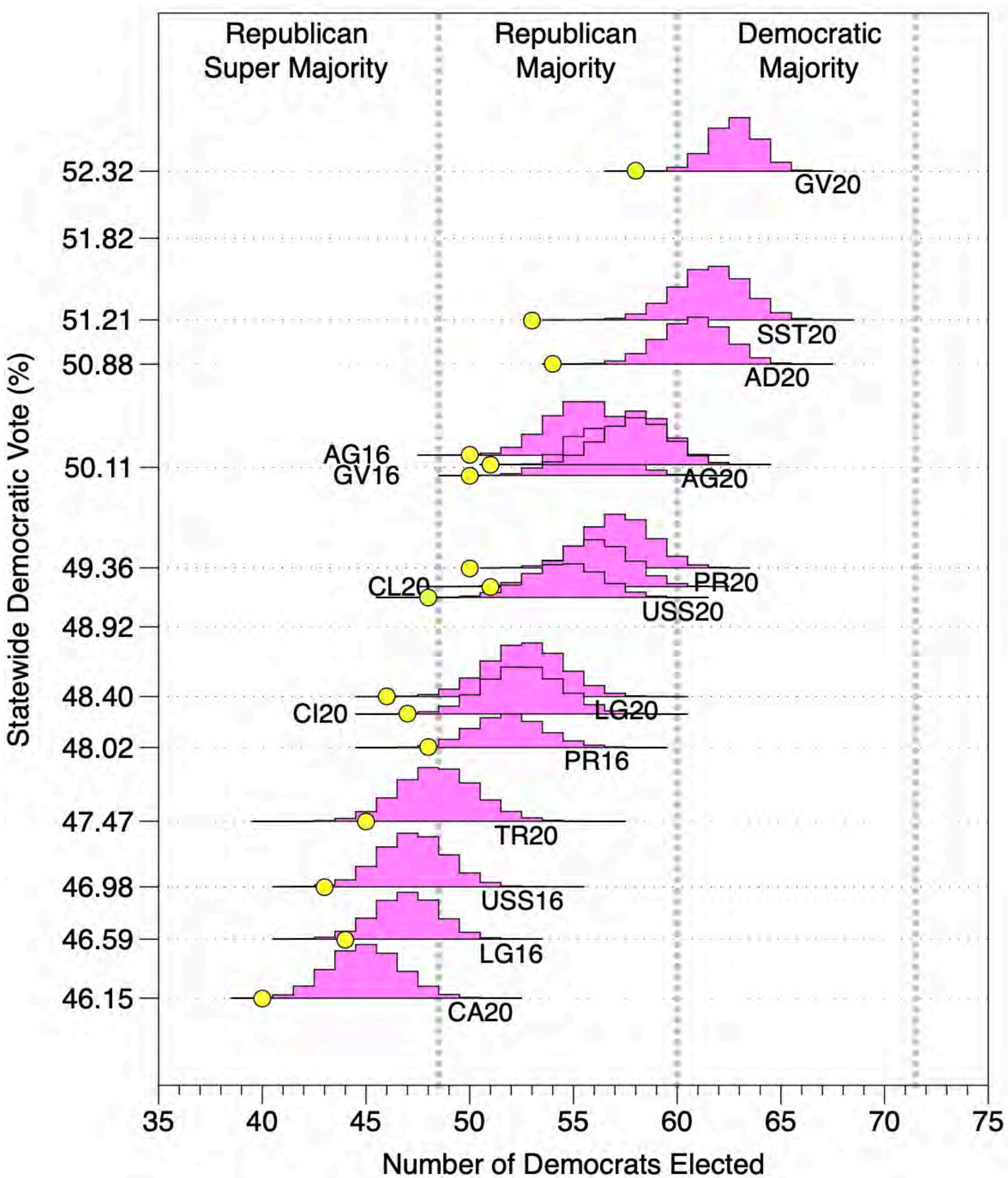
Mattingly Figure 5.1.4: The yellow dots represent the democratic vote fraction of the enacted map under the PR20 vote count when the district are ordered from most Republican on the left to most Democratic in vote share on the right. The box-plots show the range of the same statistic plotted over the primary ensemble. From around the 60th to 80th district the yellow dots all well below the boxplots of the ensemble. This result is that many dots fall well below the dotted 50% line than usually would; and hence more Republicans are elected than typical. To achieve this effect, the fraction of Democrats is increased in the already strongly democratic districts ranging from the 90th to 105th most Democratic districts. This structure does not exist in the non-partisan ensemble and is responsible for the map's extreme outlier behavior.



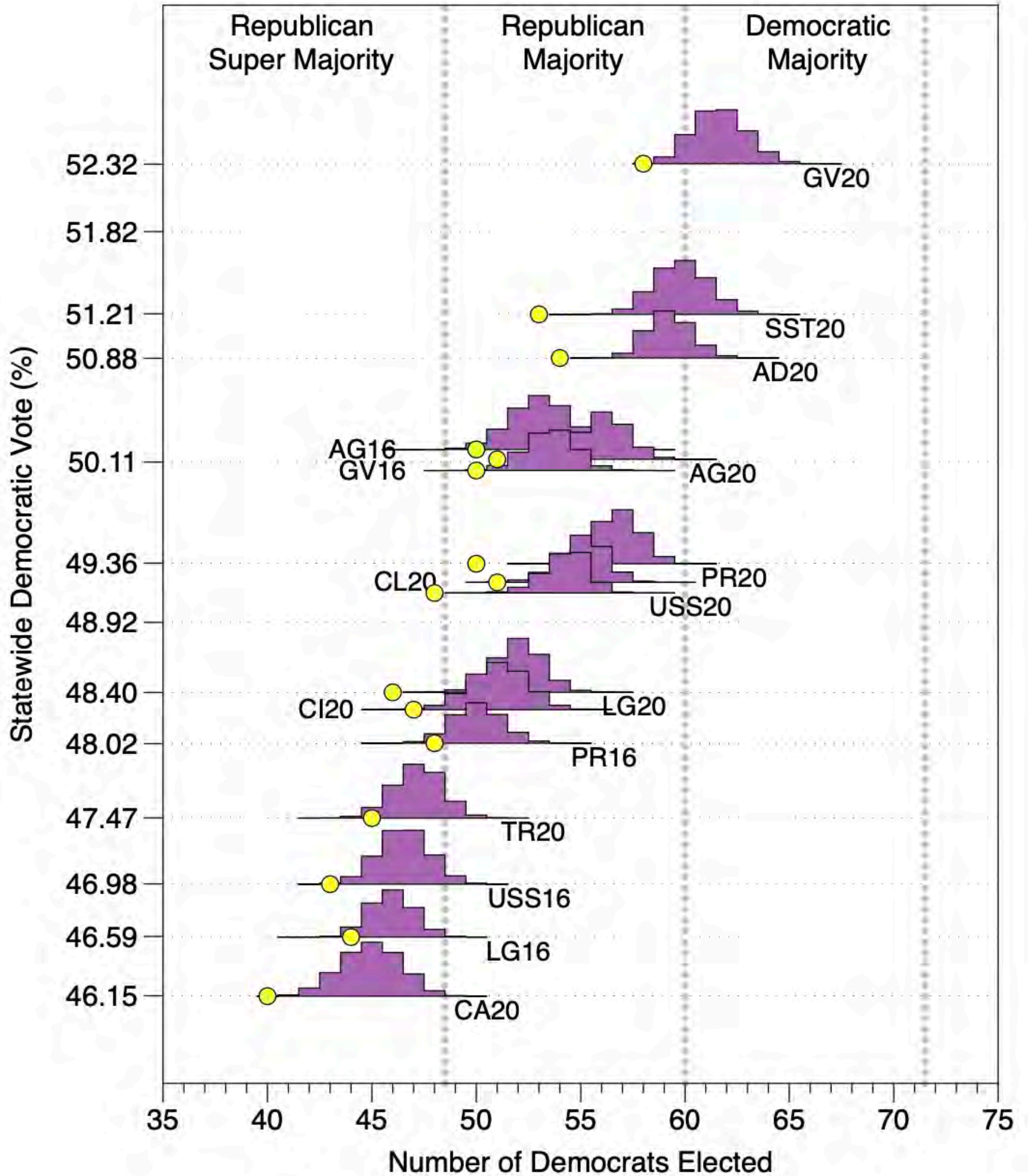
Mattingly Figure 5.1.5: A similar structure to that seen in Figure 5.1.4 is repeated here. The low 50s to the high 70s have had the number of democrats depleted while the districts from the high80s to around 105 have an excess of Democrats.



Mattingly Figure 5.1.6: Mirroring what was seen in Figure 5.1.4 and Figure 5.1.5, we have abnormally few Democrats from around the 60th to the 80th most Republican and abnormally many Democrats packed in the districts in the low 90s to the just below 110.



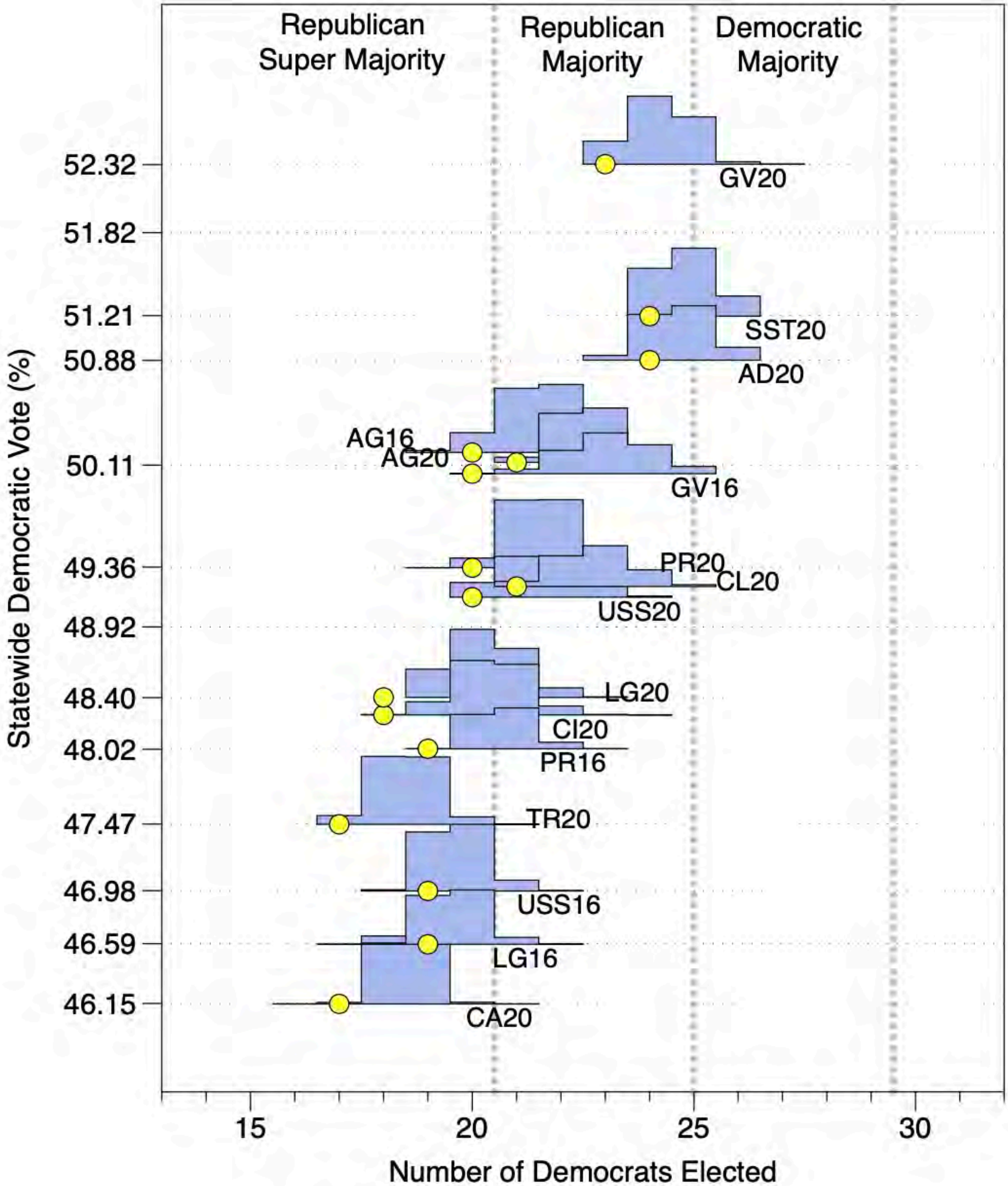
Mattingly Figure 5.1.7: The Collected Seat Histogram for the Primary Ensemble on the NC House with incumbency considerations added. See Figure 5.1.1 for full description.



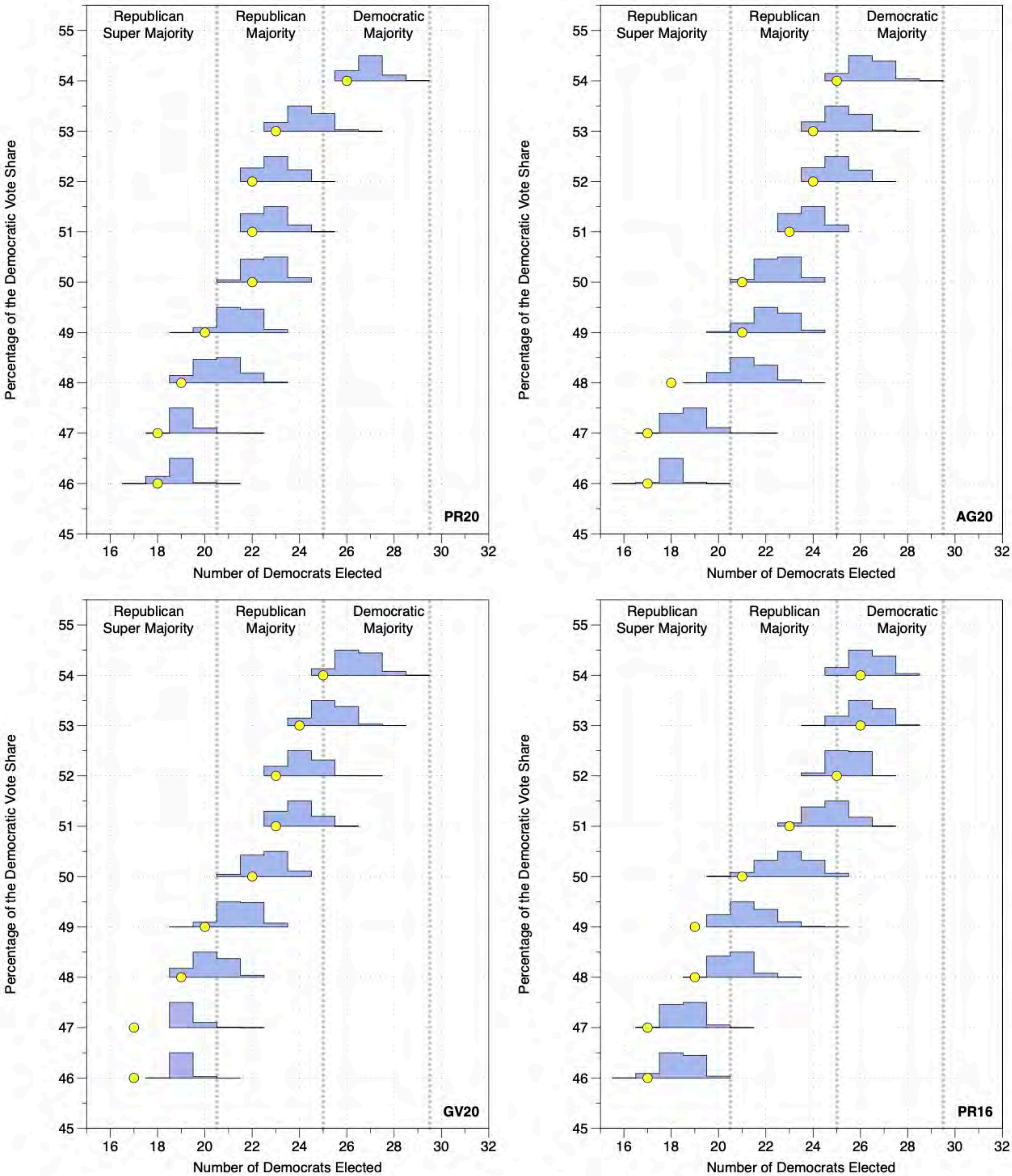
Mattingly Figure 5.1.8: The Collected Seat Histogram for the Secondary Ensemble on the NC House. The Secondary Ensemble for the NC House is centered on distributions which better preserve municipalities than the enacted plan. See Figure 5.1.1 for full description.

% Dem	Election	% Outlier	# Outlier	# Samples
52.32%	GV20	16.343%	16343	100000
51.21%	SST20	35.184%	35184	100000
50.88%	AD20	42.880%	42880	100000
50.20%	AG16	12.129%	12129	100000
50.13%	AG20	4.332%	4332	100000
50.05%	GV16	0.075%	75	100000
49.36%	PR20	6.220%	6220	100000
49.22%	CL20	5.365%	5365	100000
49.14%	USS20	14.052%	14052	100000
48.40%	LG20	0.000%	0	100000
48.27%	CI20	0.322%	322	100000
47.47%	TR20	5.726%	5726	100000
46.98%	USS16	43.176%	43176	100000
46.59%	LG16	44.943%	44943	100000
46.15%	CA20	1.123%	1123	100000

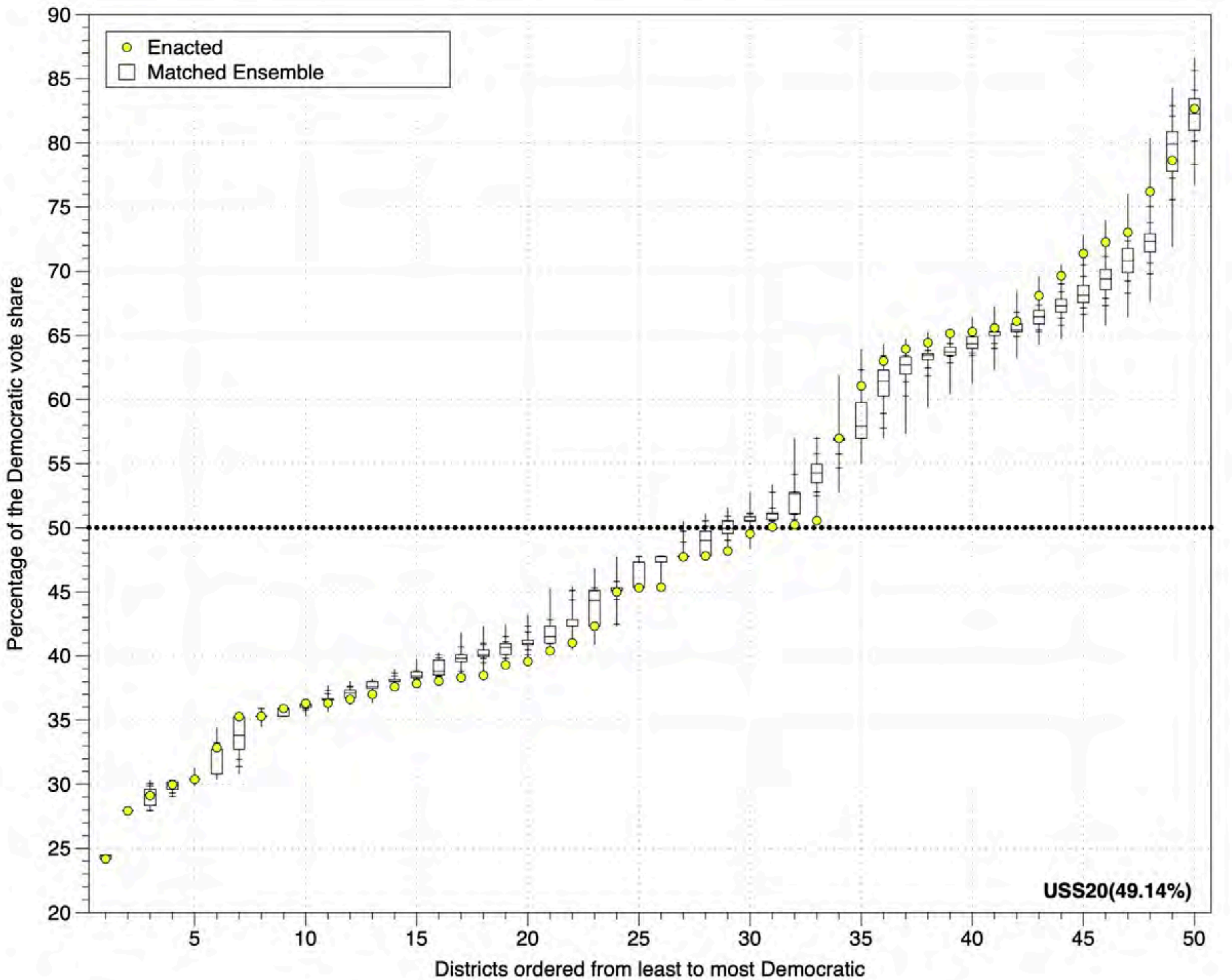
Mattingly Table 2: NC Senate Collected Seat Histogram Outlier Data. Starting from the left, the first column gives the statewide partisan makeup of the election under consideration whose abbreviation is given in the second column from the left. The right most column gives the total number of plans in the ensemble considered which is 100,000. The second column from the right gives the number of those 100,000 plans which elect the same or less Democrats under the given election. These are the plans which are as much or more of an outlier than the enacted map. The middle column is the percentage of plans which are more or equal of an outlier. (It is calculated by dividing the 2nd column from the right by 100,000 and multiplying by 100 to make a percentage.) The number of fairly small to extremely small percentage in the middle column between 50.13% (AG20) and 47.47% (TR20) are another signature of the anomalous behavior seen visually in Figure 5.2.1 over the same range of vote percentages.



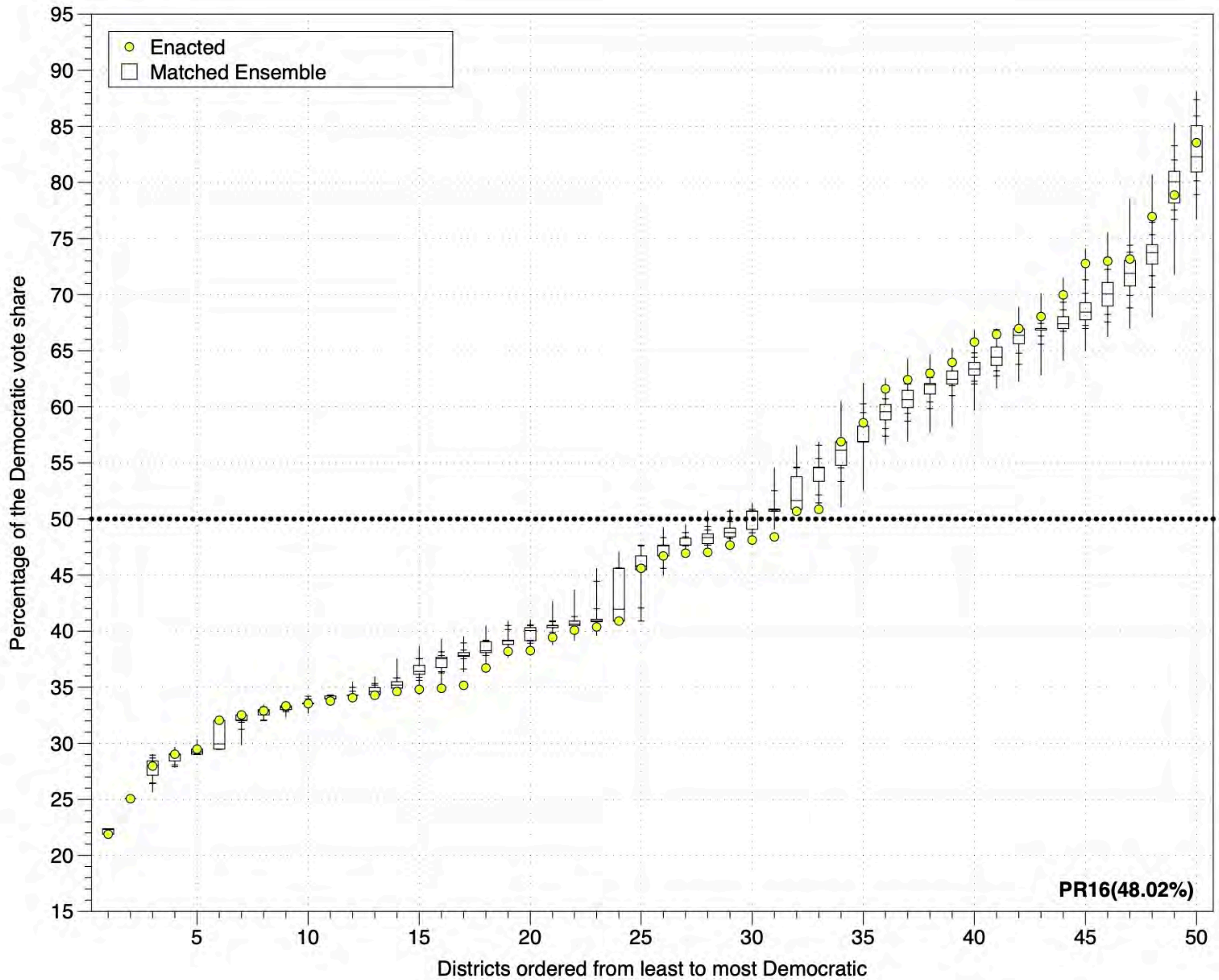
Mattingly Figure 5.2.1: The Collected Seat Histogram for the Primary Ensemble on the NC Senate. The individual histograms give the frequency of the Democratic seat count for each of the statewide elections considered from the years 2016 and 2020. The histograms are organized vertically based on the statewide partisan vote fraction for each election. The more Republican elections are placed lower on the plot while more Democratic elections are placed higher. Three dotted lines denote the boundary between where the supermajorities and simple majorities are in force.



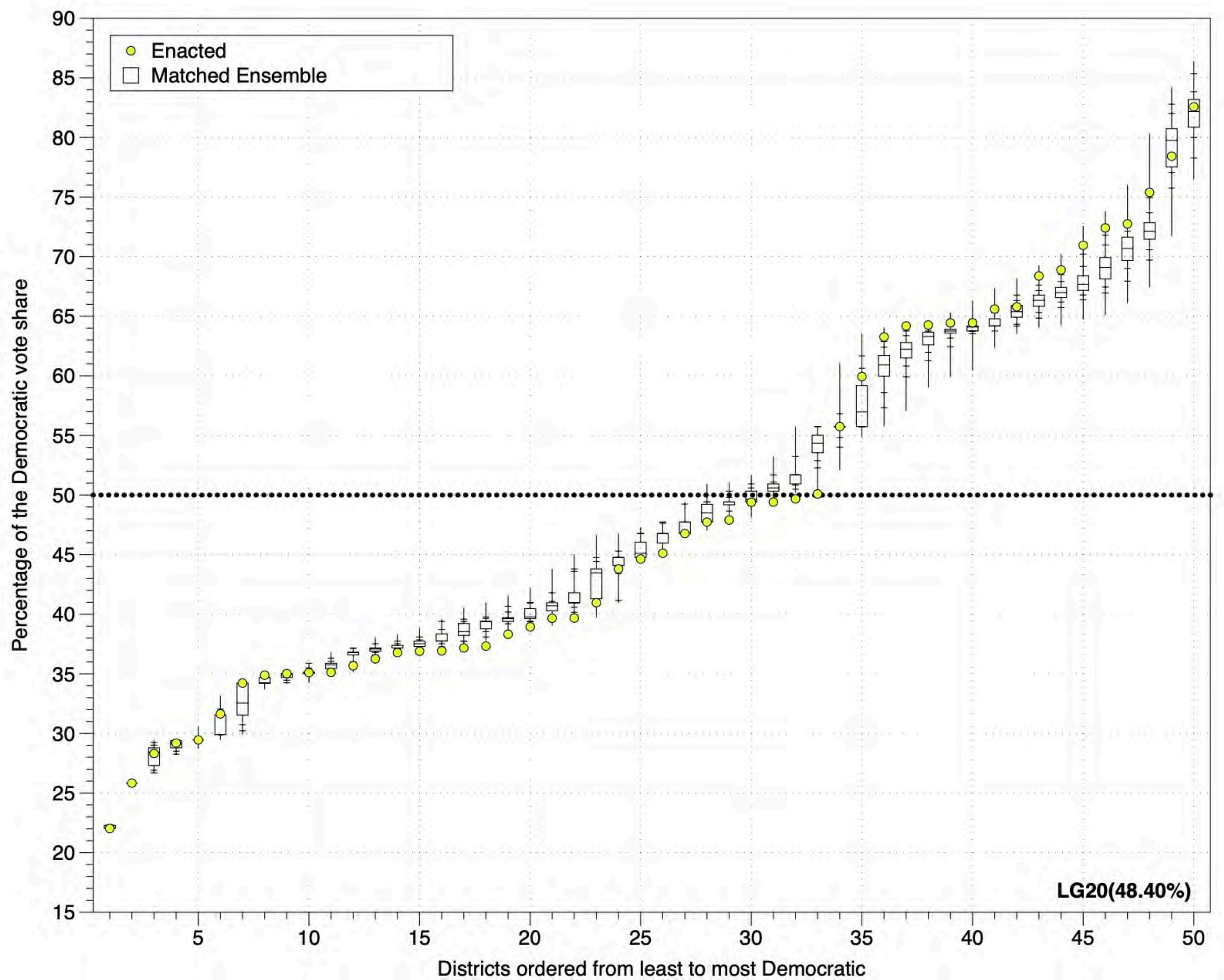
Mattingly Figure 5.2.2: The Collected Seat Histograms for the Primary Ensemble on the NC Senate built from a collection of voting data generated via uniform swing.



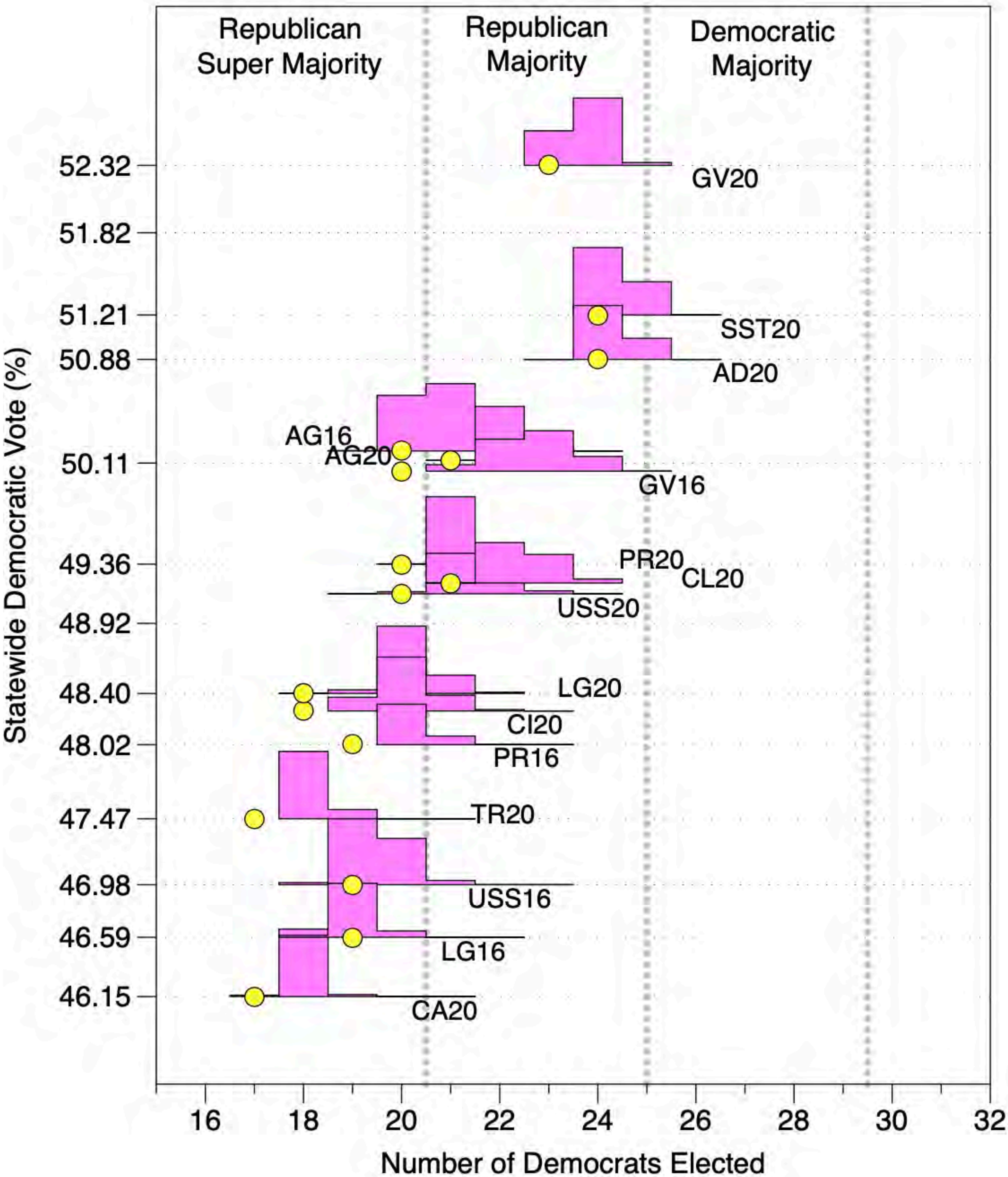
Mattingly Figure 5.2.3: The yellow dots represent the democratic vote fraction of the enacted map under the USS20 vote count when the district are ordered from most Republican on the left to most Democratic in vote share on the right. The box-plots show the range of the same statistic plotted over the primary ensemble. Essentially all of the districts between the 15th most Republican and the 33rd most Republican have abnormally few Democrats. This is compensated by packing abnormally many Democrats the 35th to the 47th most Republican districts. This structure is an extreme outlier and does not occur in the ensemble.



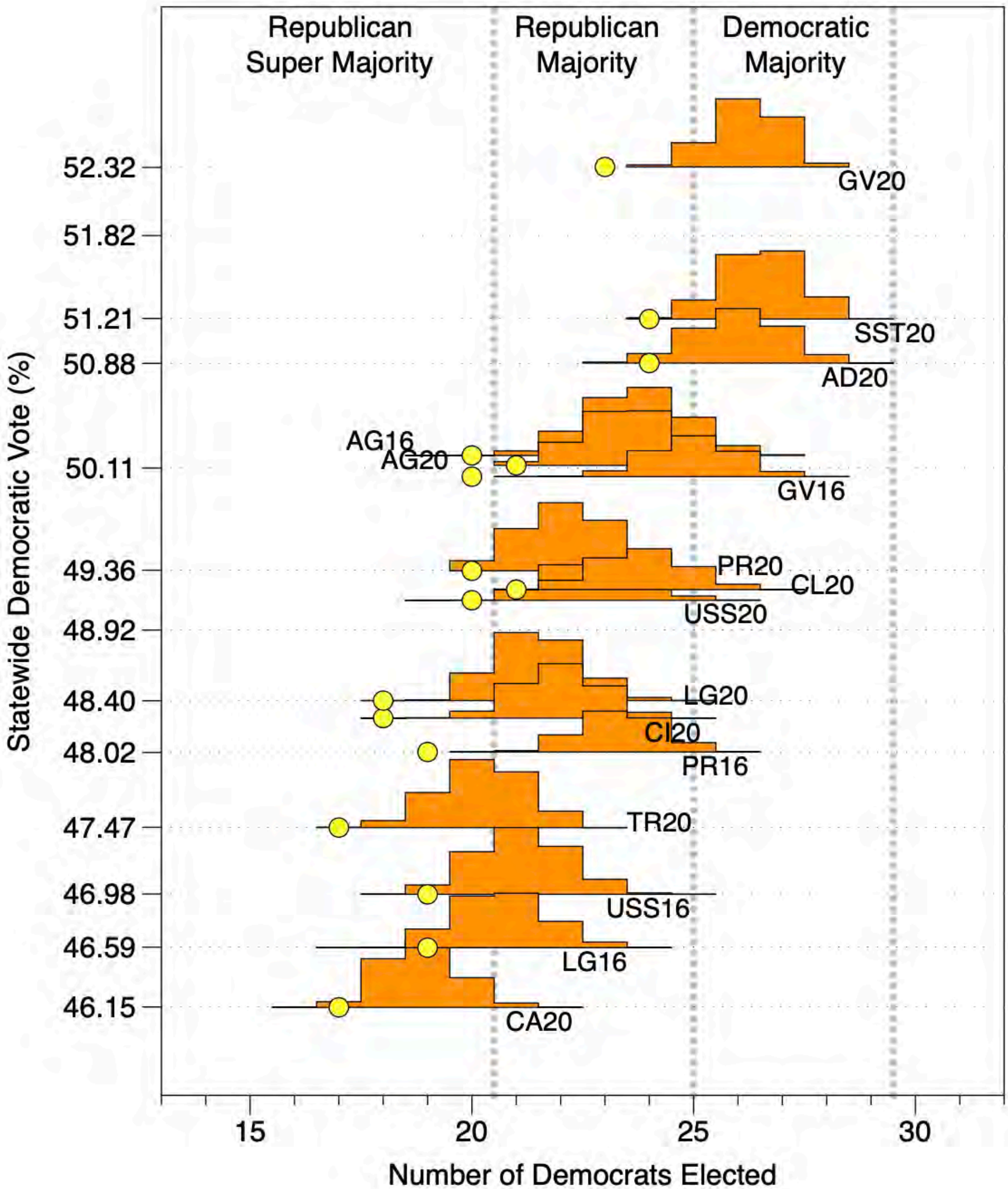
Mattingly Figure 5.2.4: A similar structure to that seen in Figure 5.2.3 is repeated here over a nearly identical range of districts.



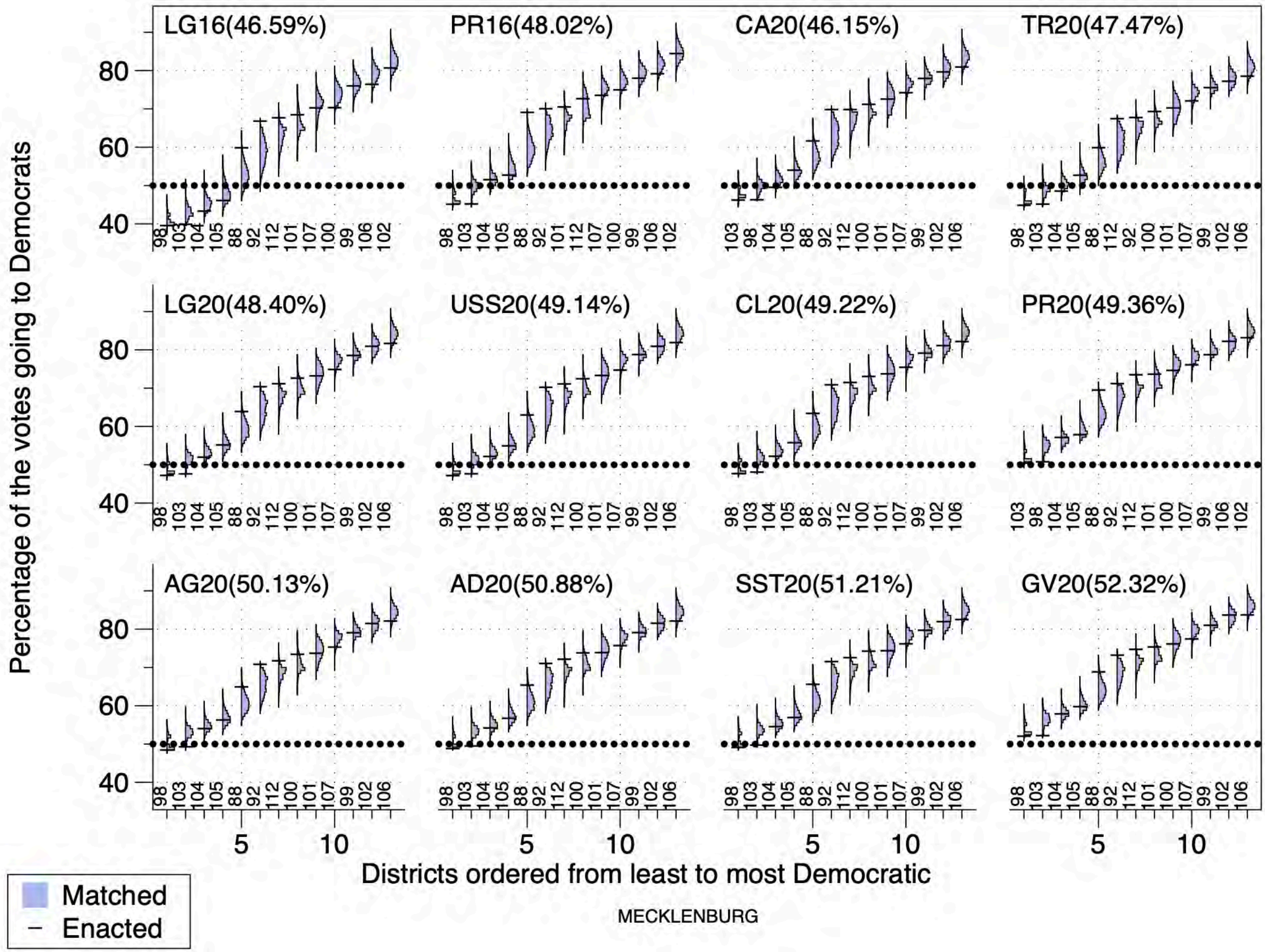
Mattingly Figure 5.2.5: A similar structure to that seen in Figure 5.2.3 is repeated here.



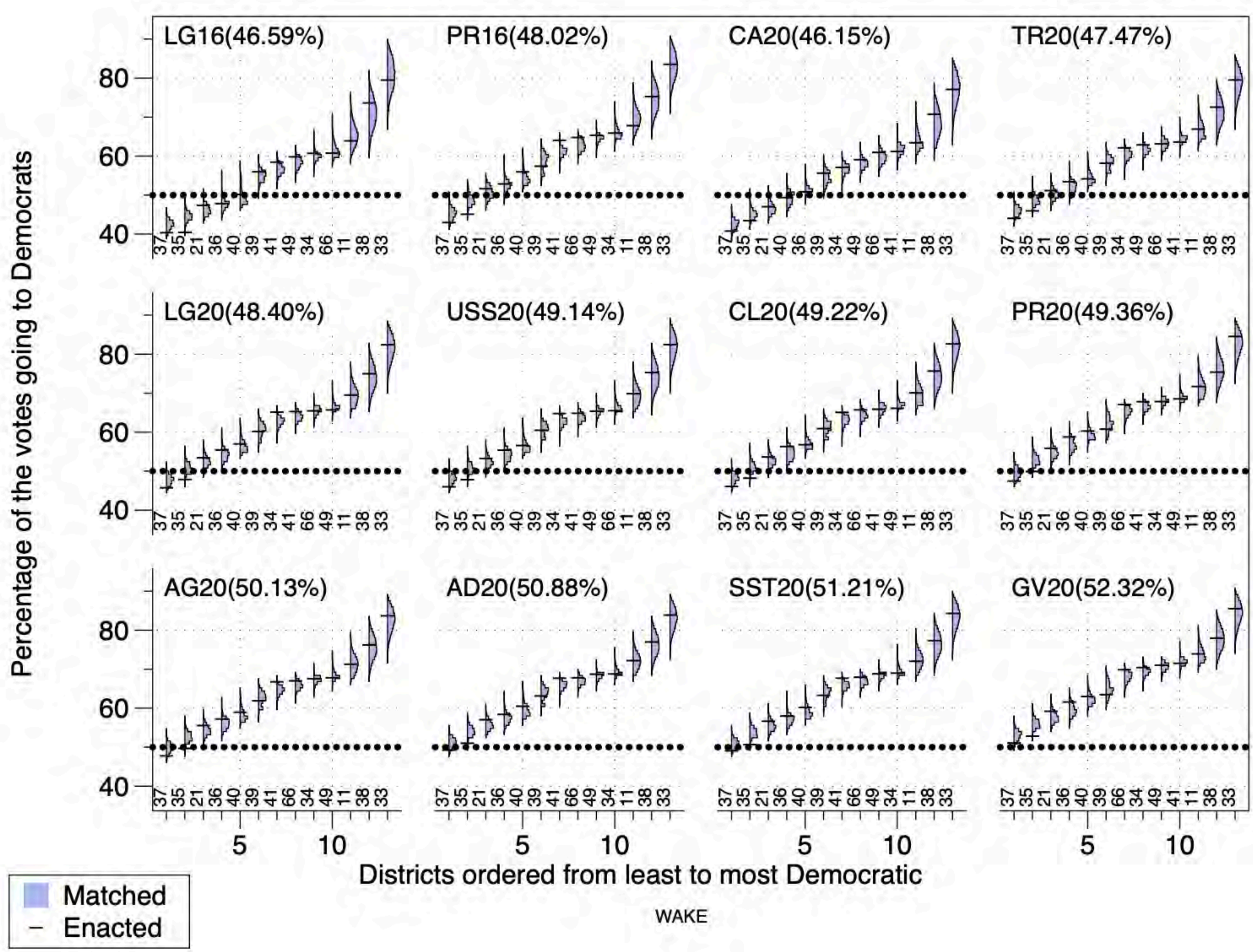
Mattingly Figure 5.2.6: The Collected Seat Histogram for the Primary Ensemble on the NC Senate with incumbency considerations added. See Figure 5.1.1 for full description.



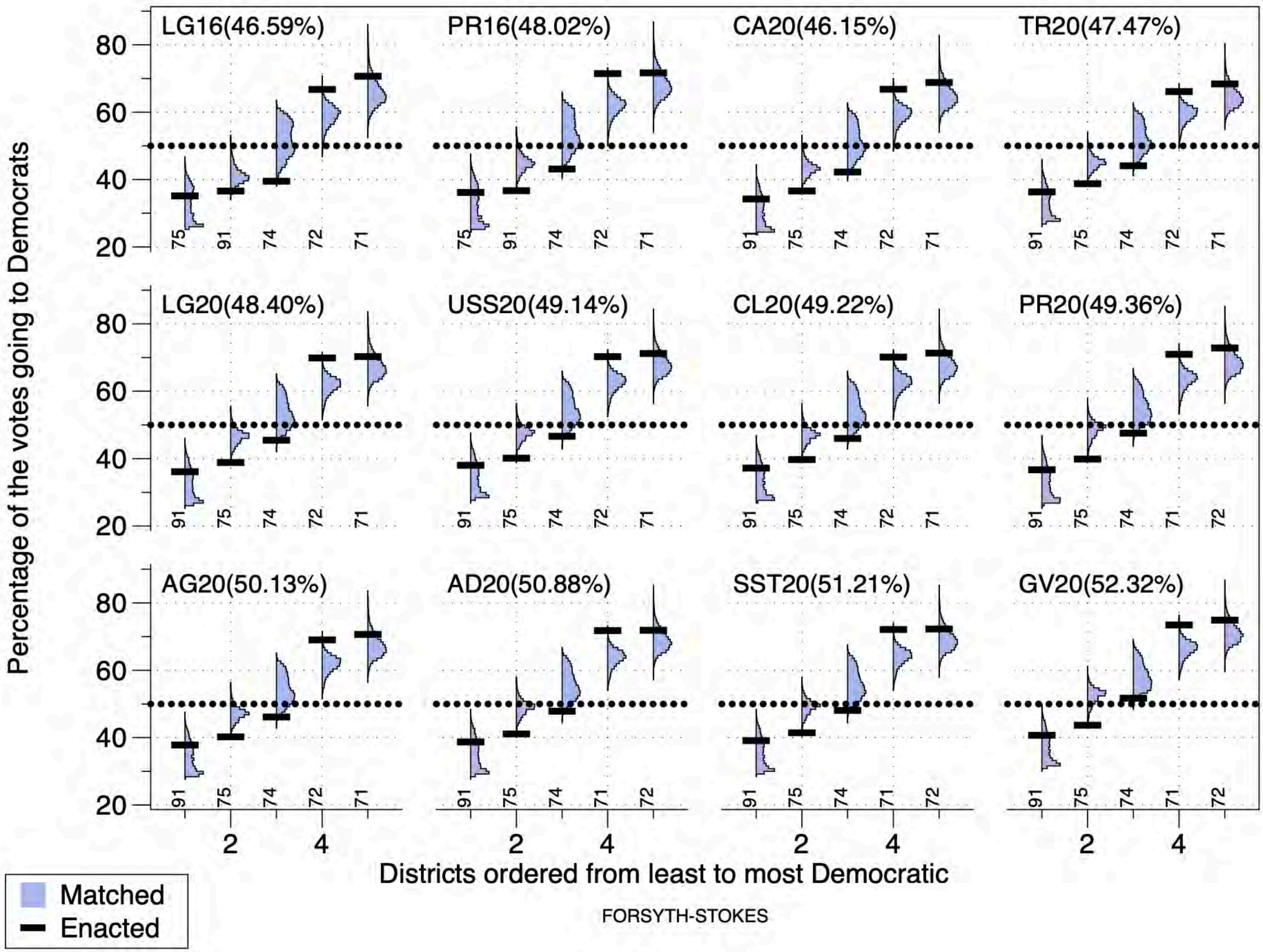
Mattingly Figure 5.2.7: The Collected Seat Histogram for the Secondary Ensemble on the NC Senate. The Secondary Ensemble for the NC Senate is centered on distributions which do not explicitly consider municipality preservation. See Figure 5.1.1 for full description.



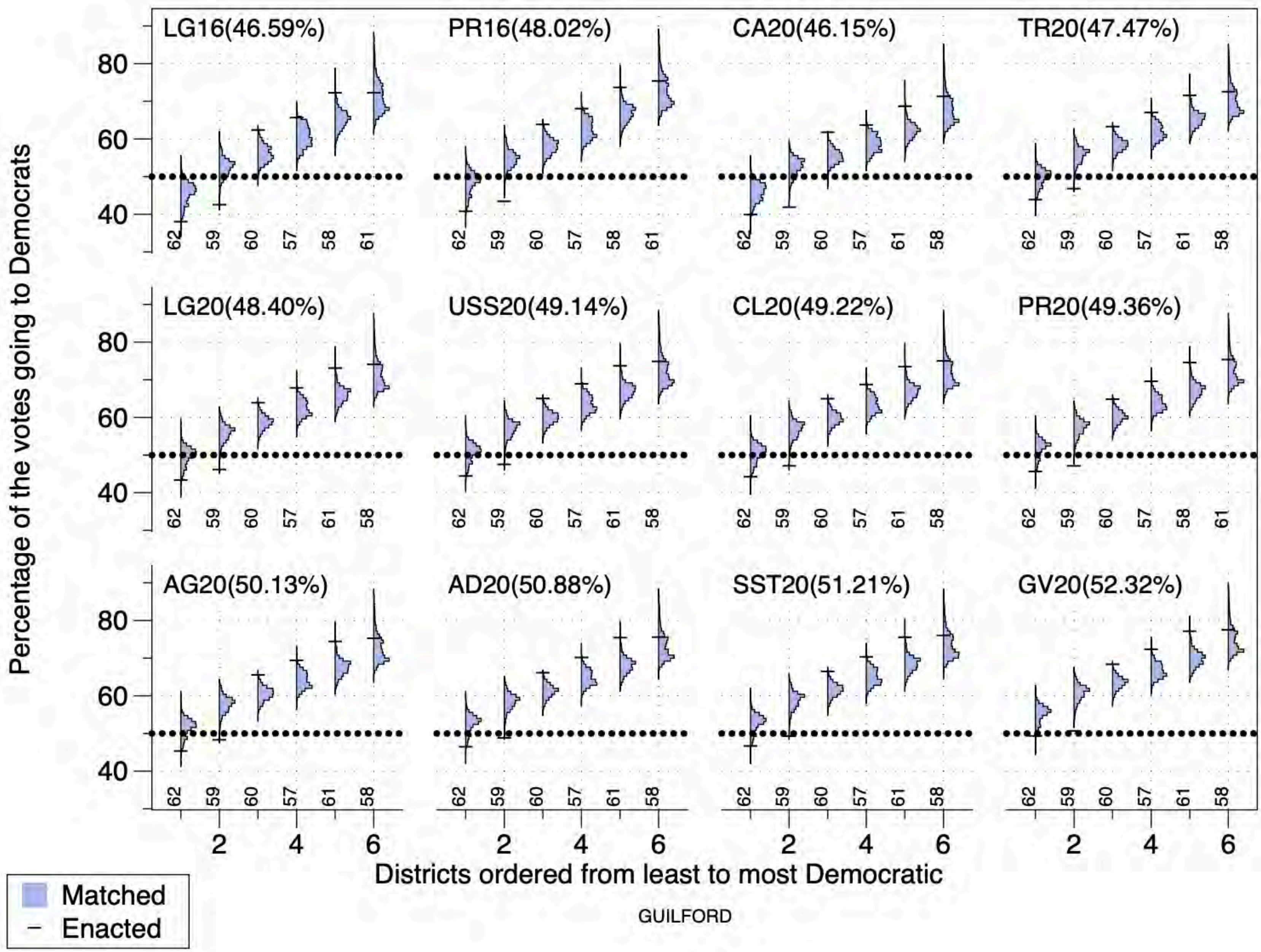
Mattingly Figure 6.1.1: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



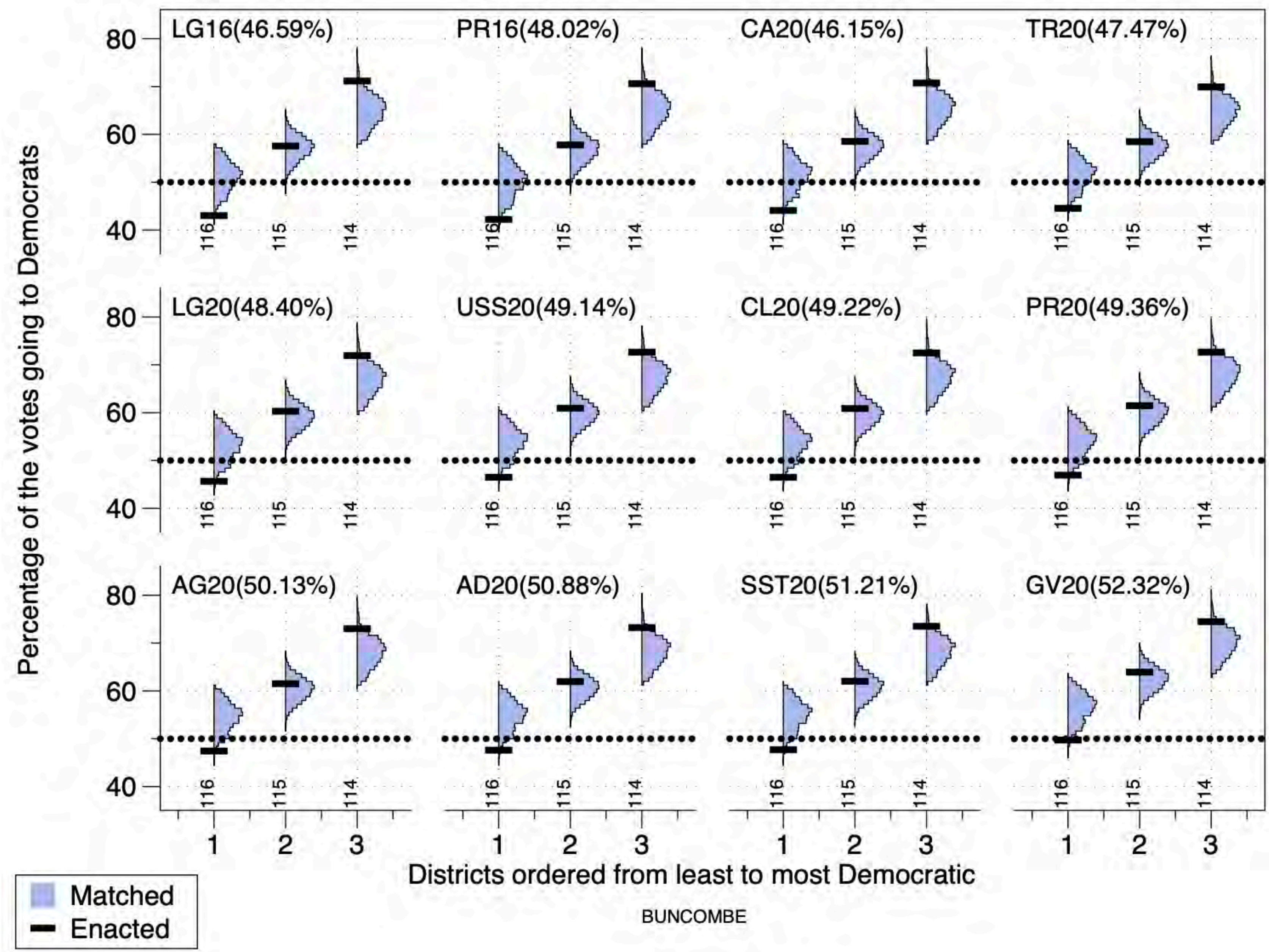
Mattingly Figure 6.1.4: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



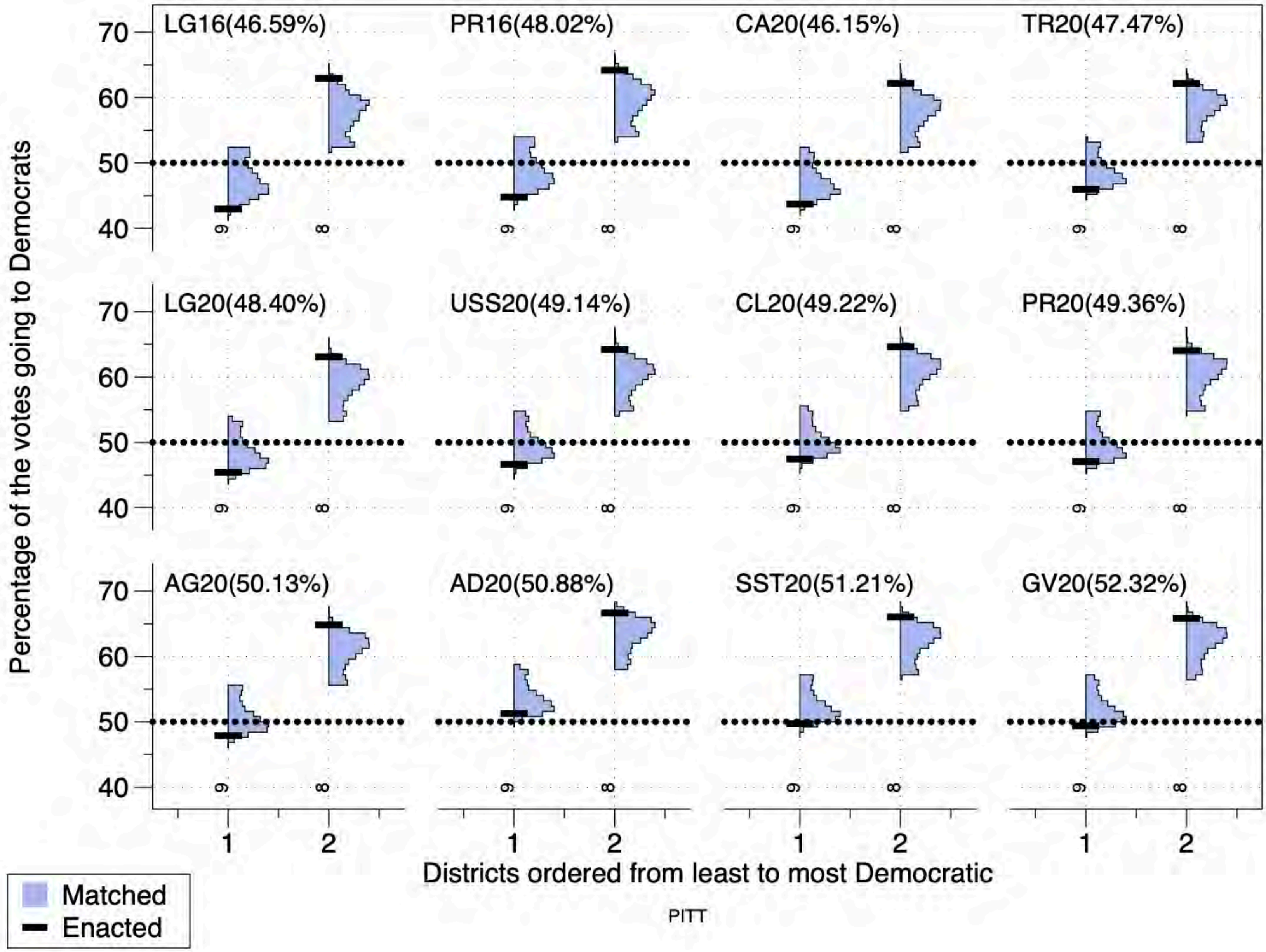
Mattingly Figure 6.1.7: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



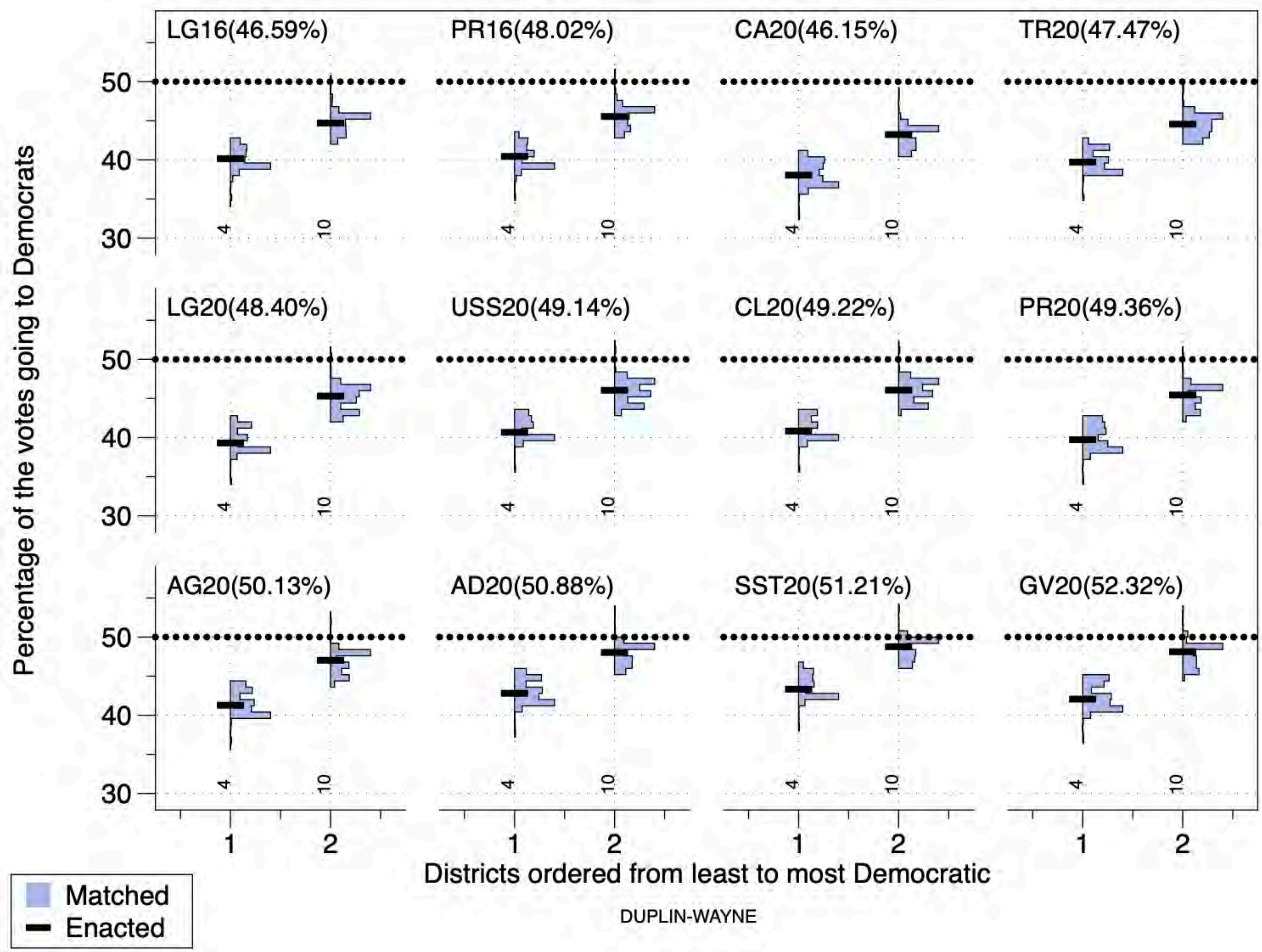
Mattingly Figure 6.1.10: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



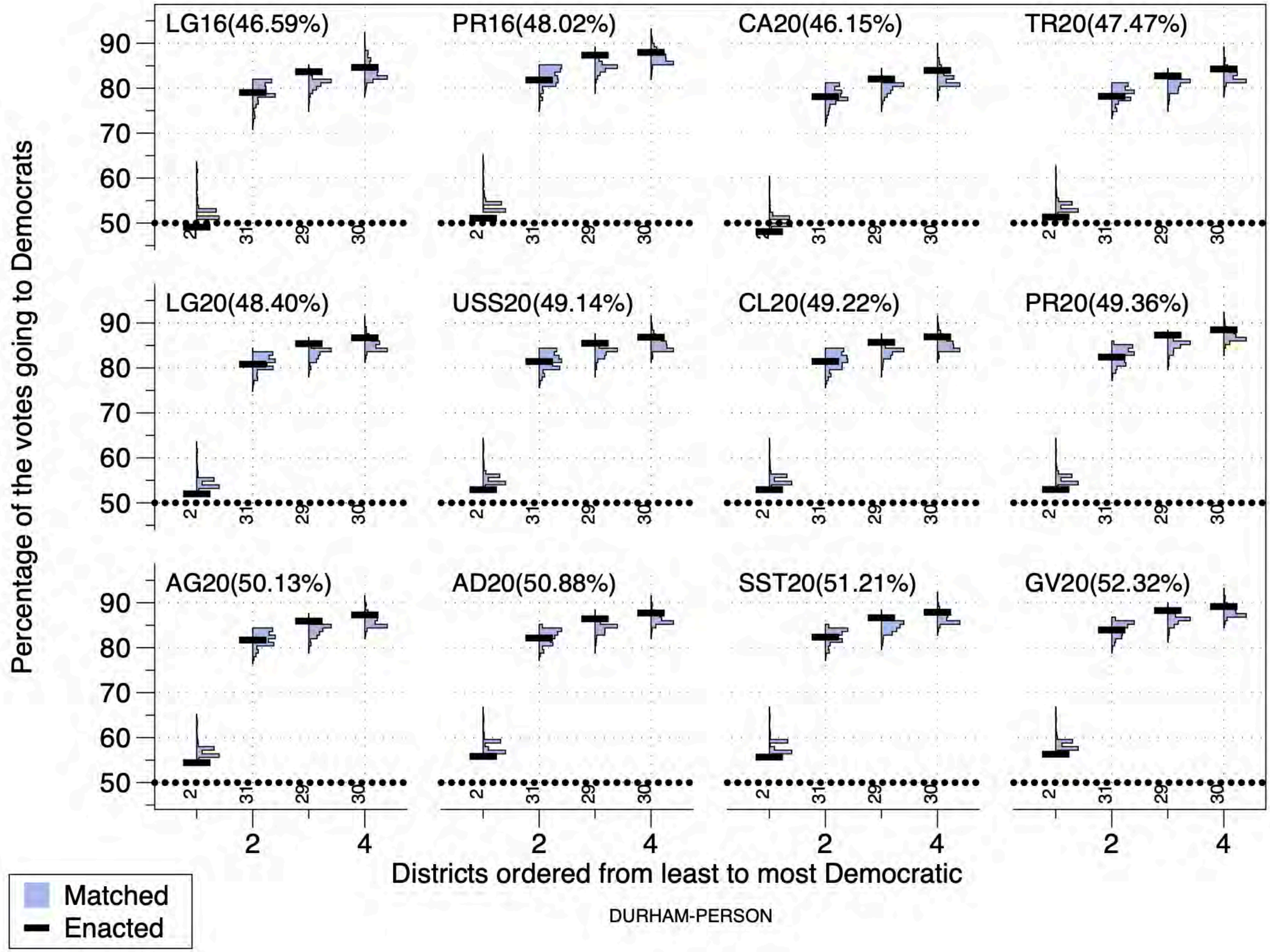
Mattingly Figure 6.1.13: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



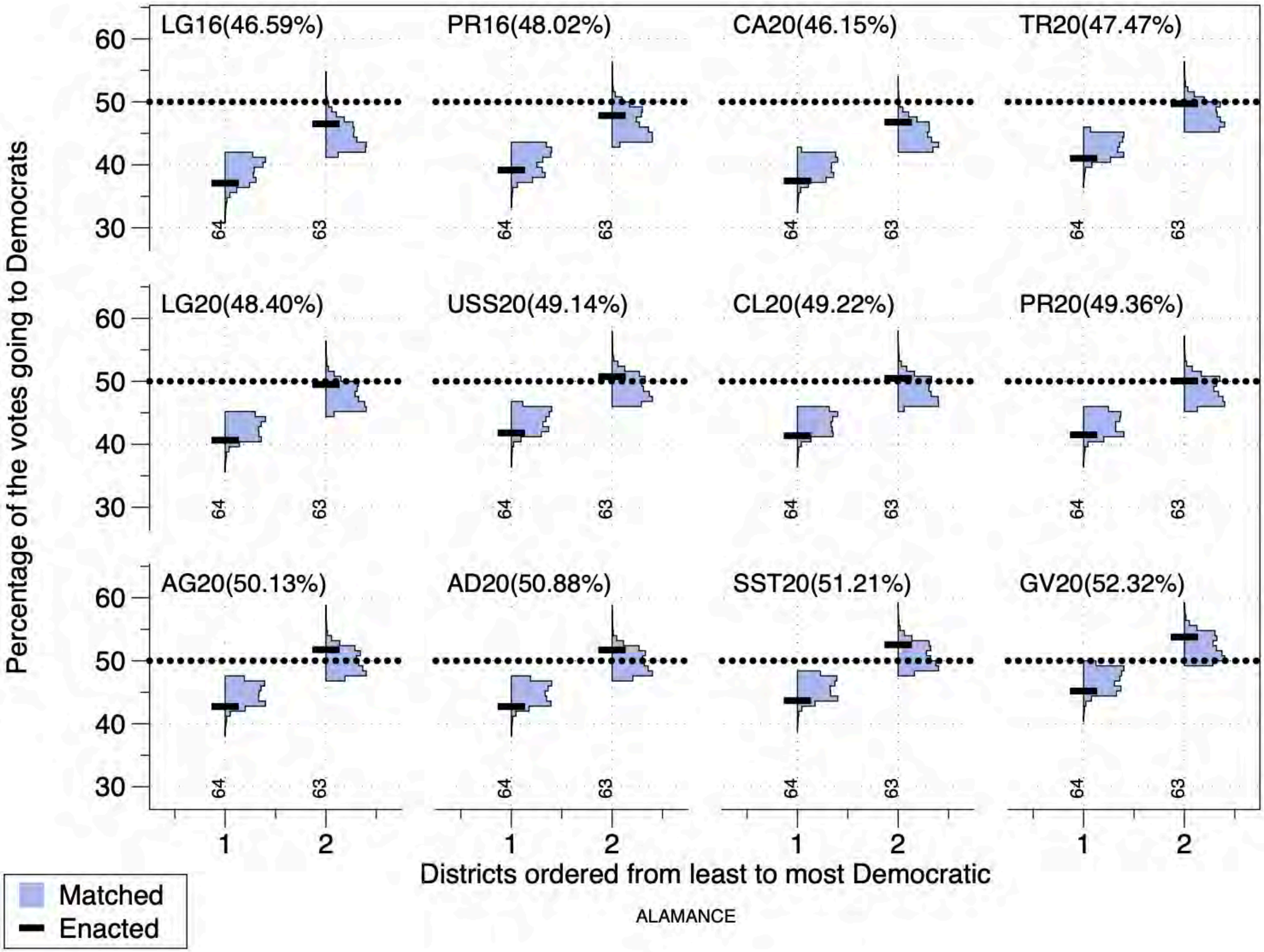
Mattingly Figure 6.1.16: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



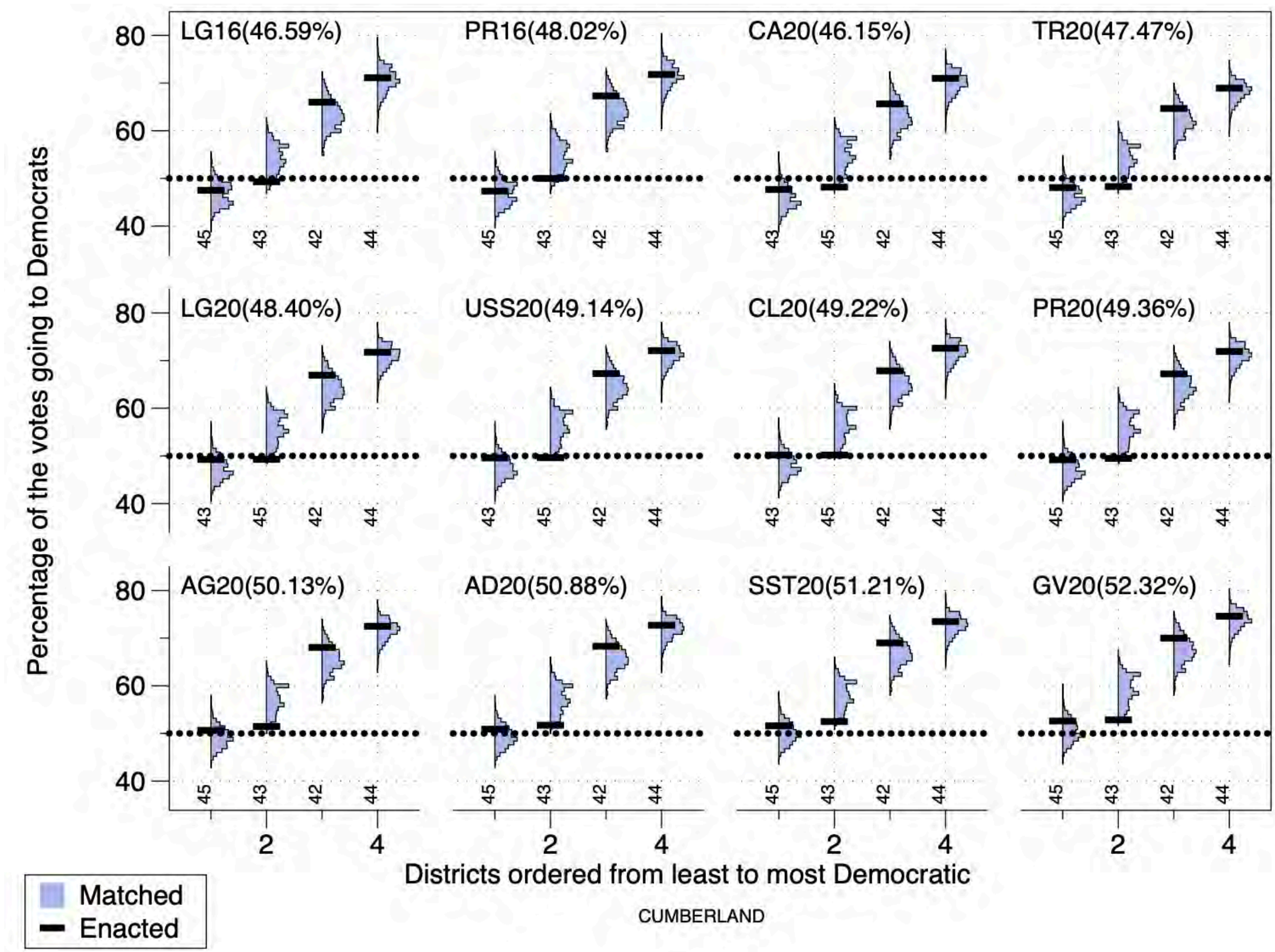
Mattingly Figure 6.1.19: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



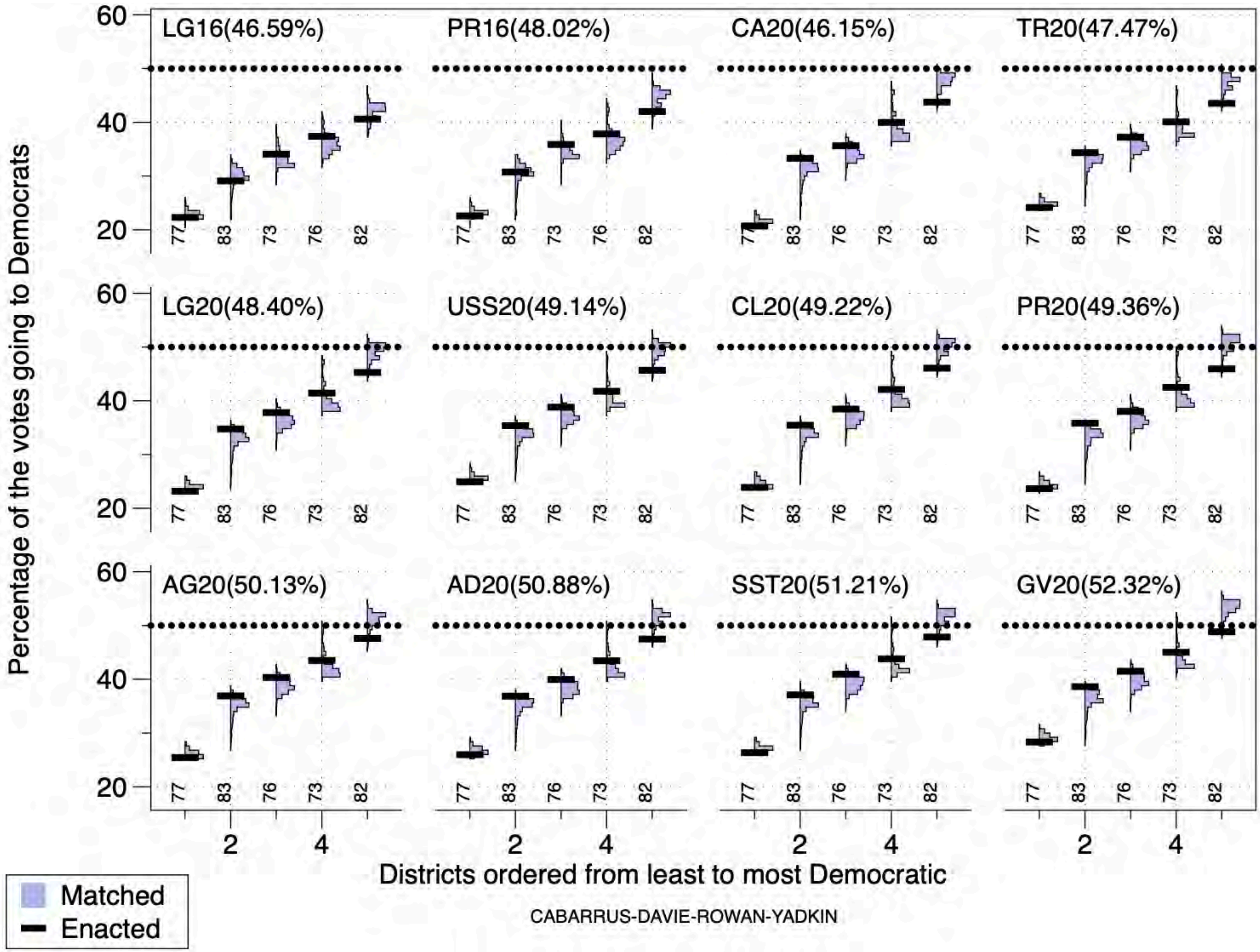
Mattingly Figure 6.1.22: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



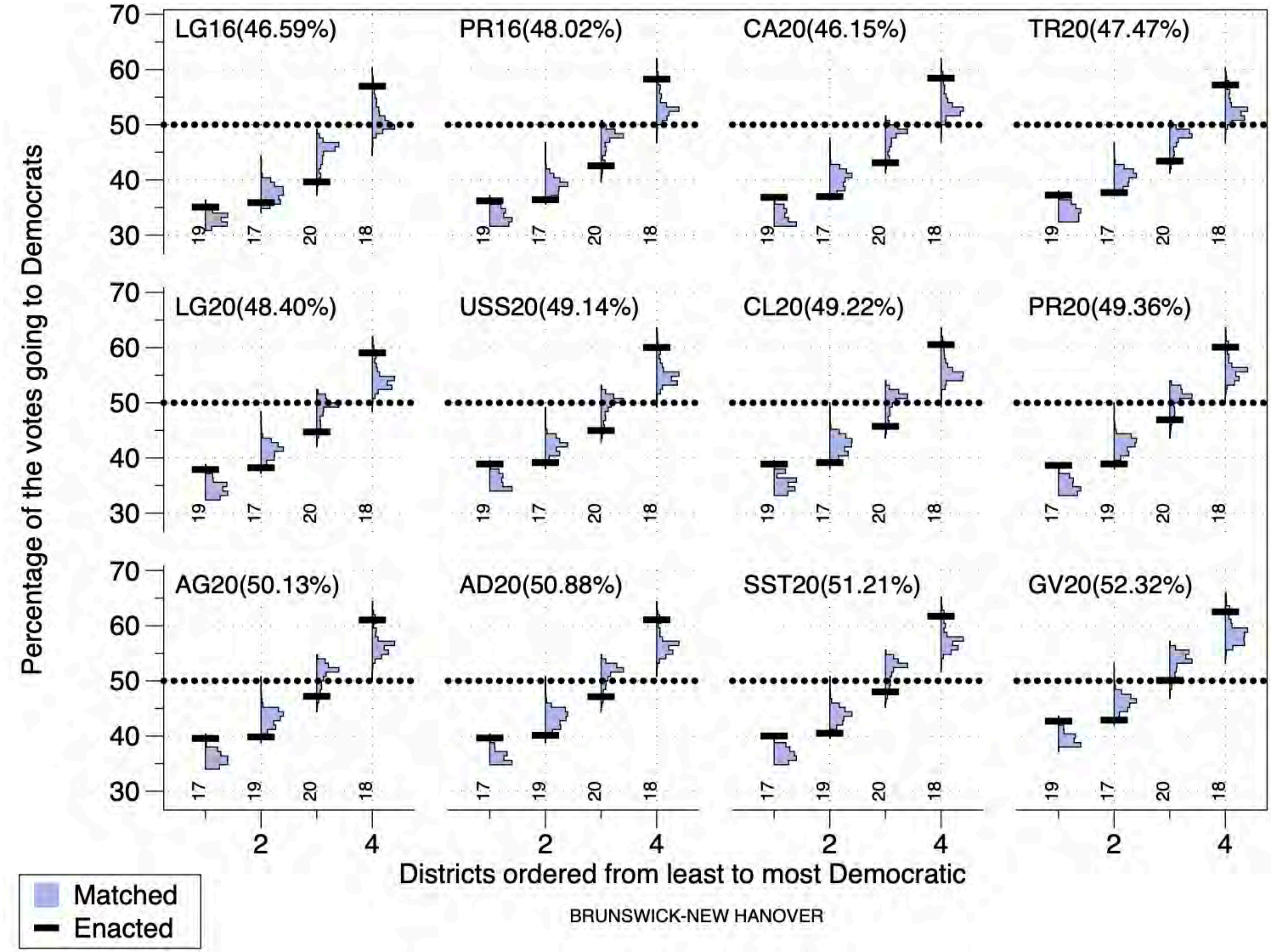
Mattingly Figure 6.1.25: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



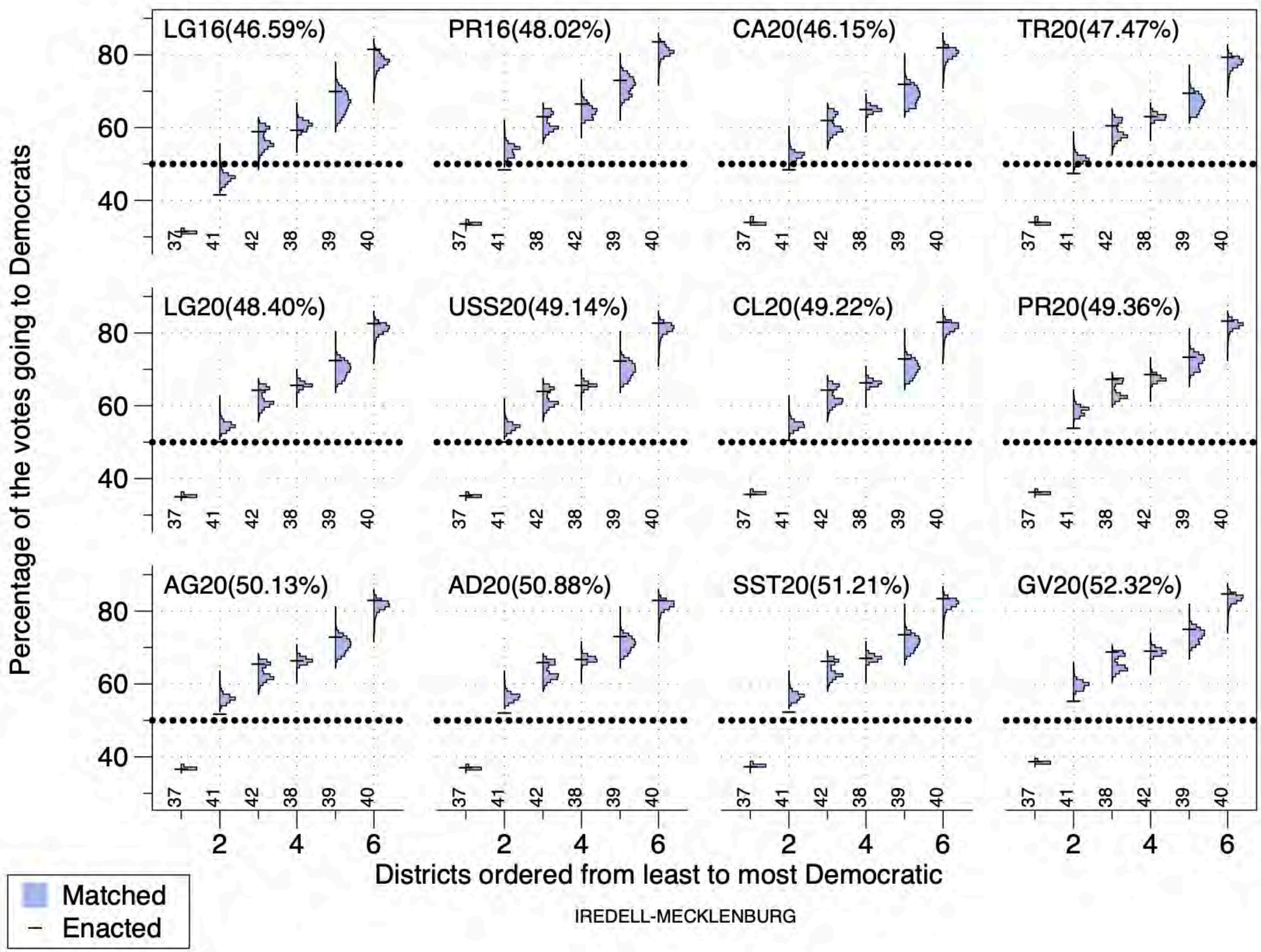
Mattingly Figure 6.1.28: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



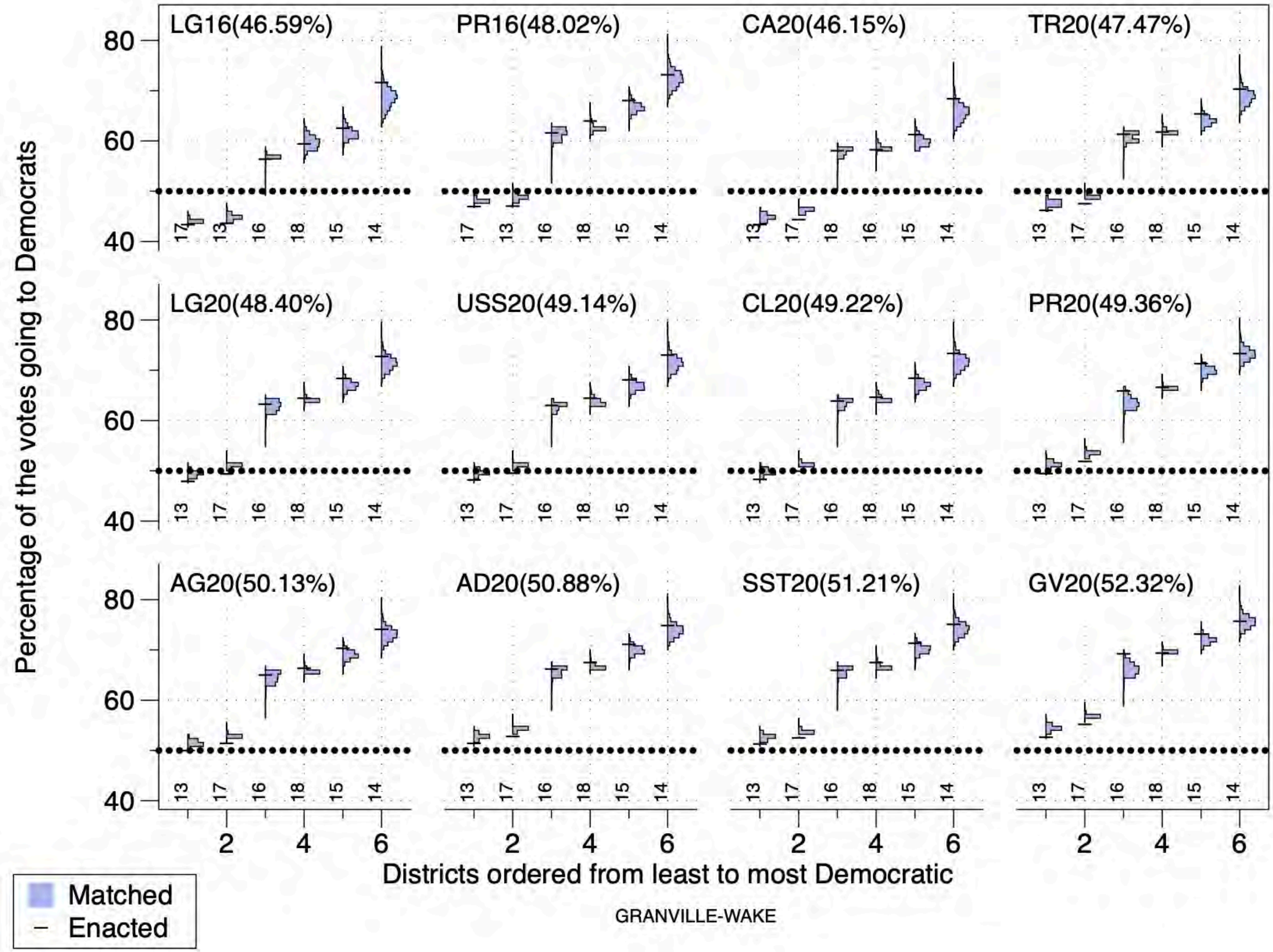
Mattingly Figure 6.1.31: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



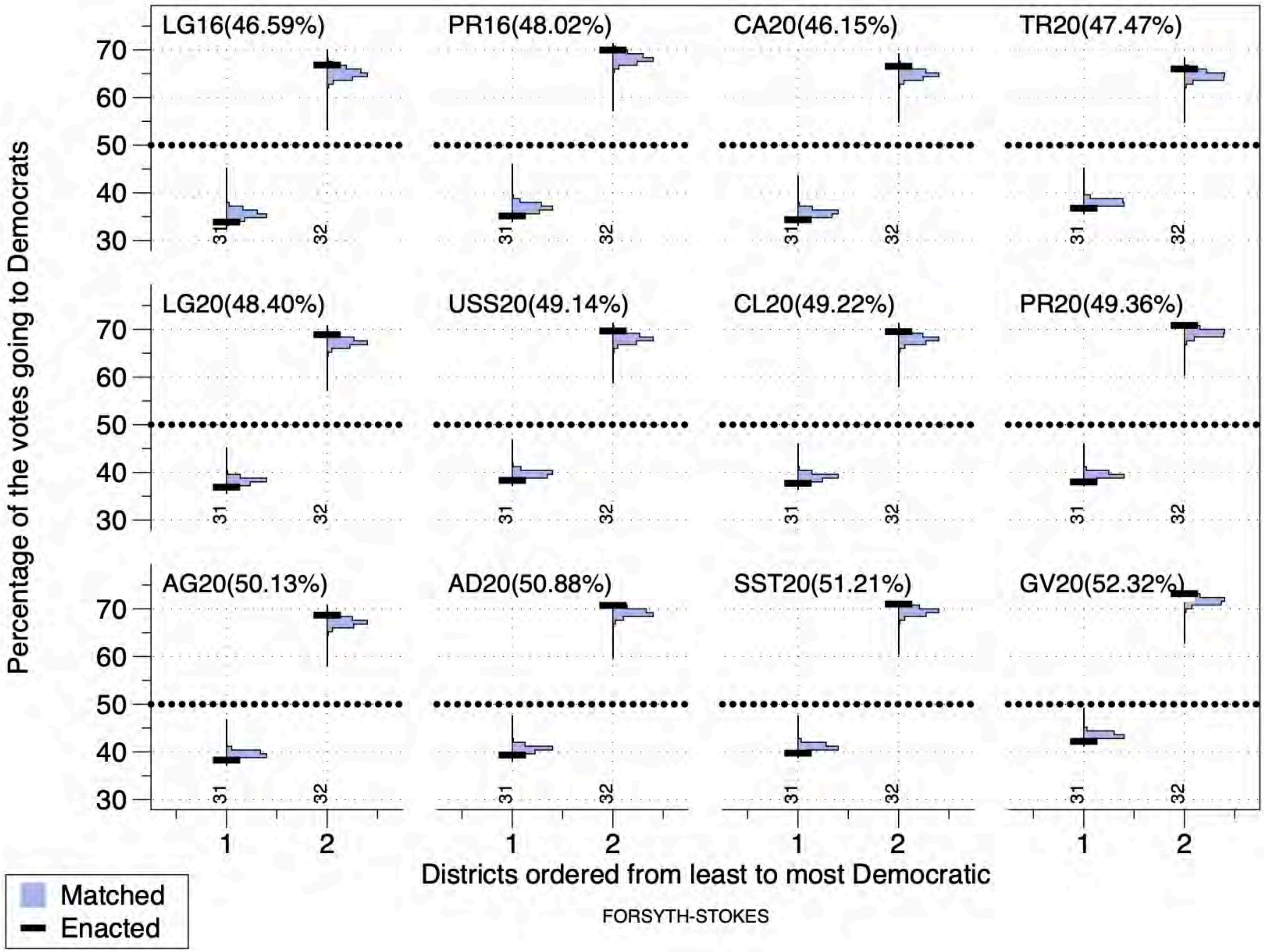
Mattingly Figure 6.1.34: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The “ – ” on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the “ – ”. This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



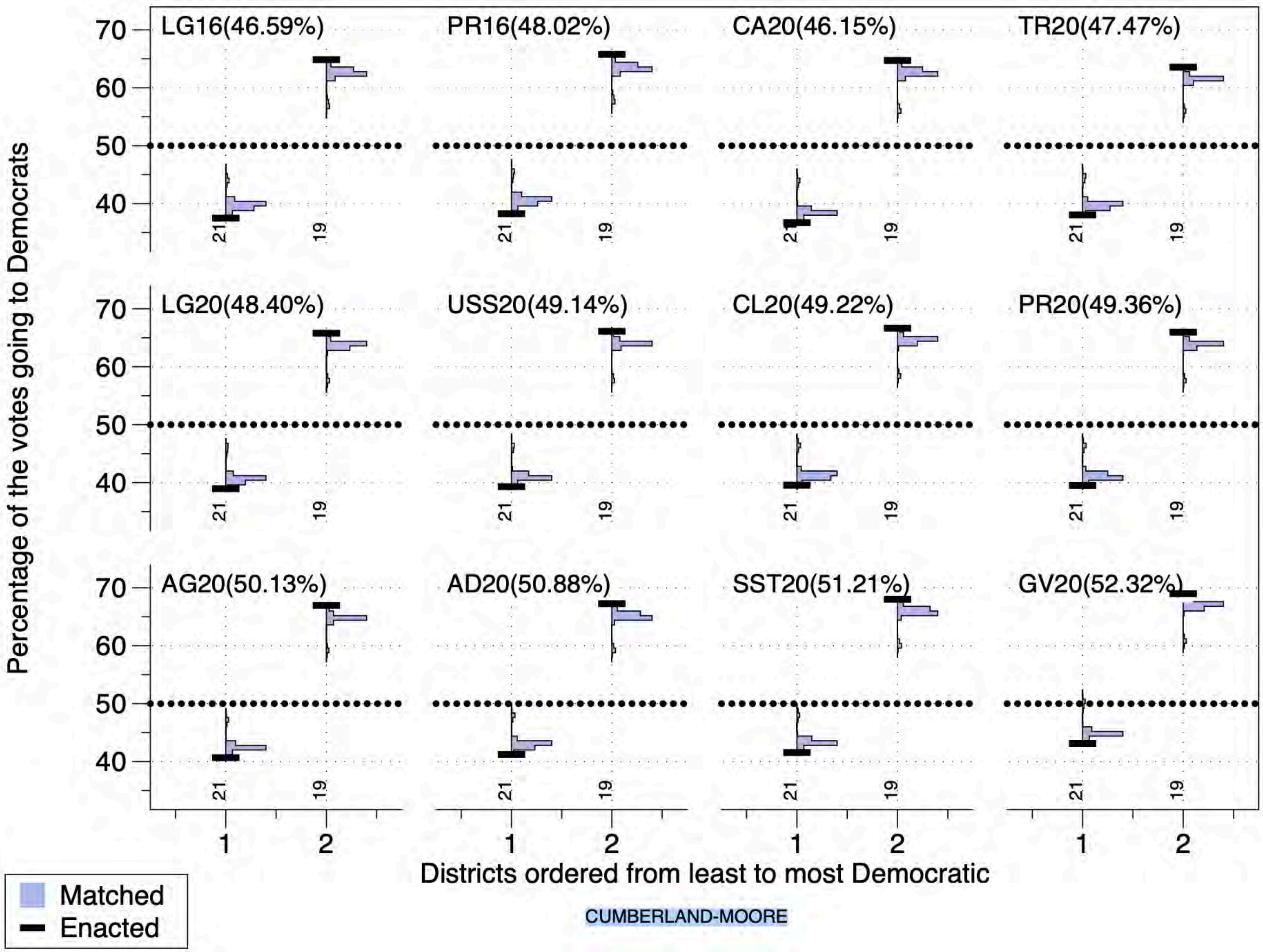
Mattingly Figure 6.2.1: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



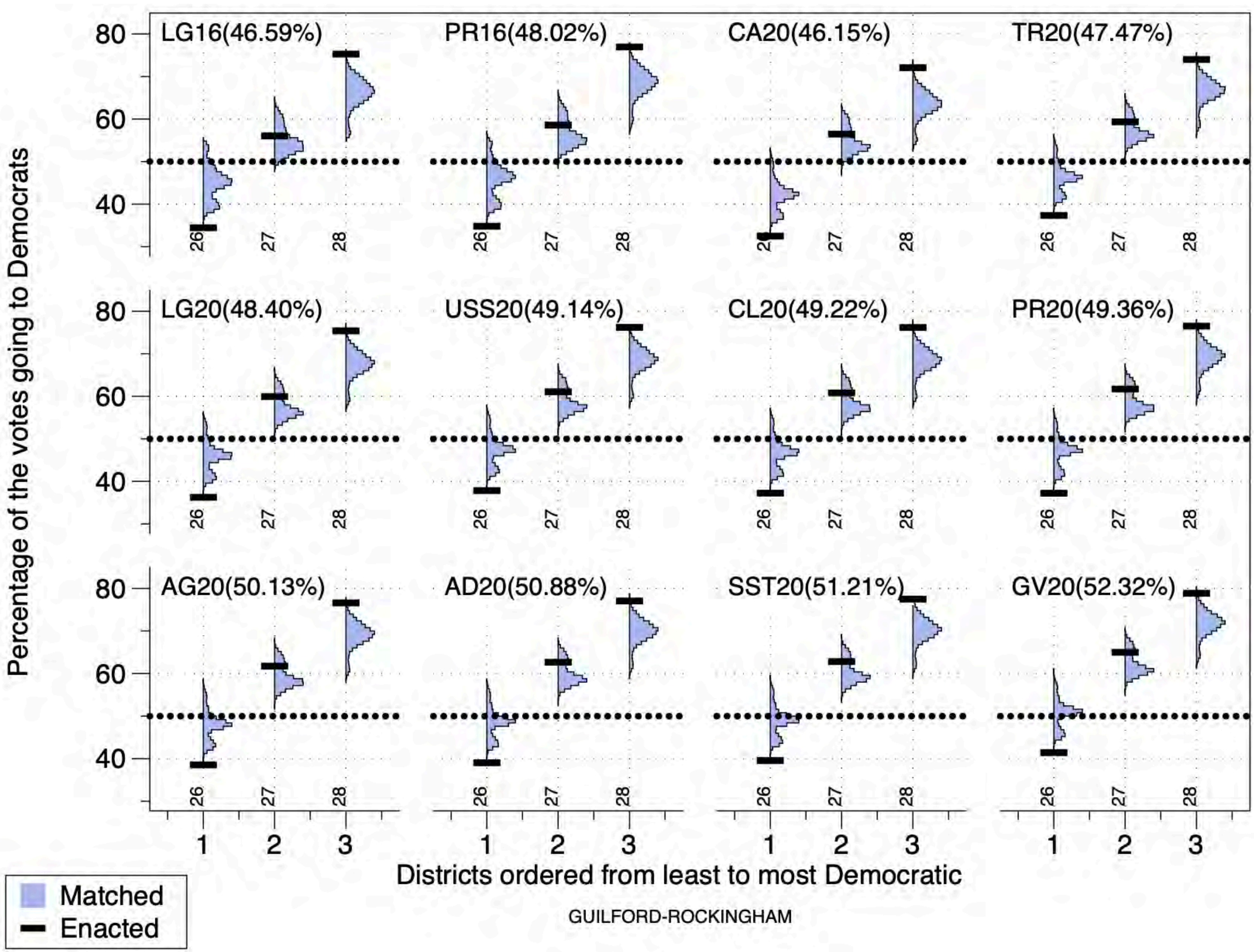
Mattingly Figure 6.2.4: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The “ – ” on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the “ – ”. This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



Mattingly Figure 6.2.7: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



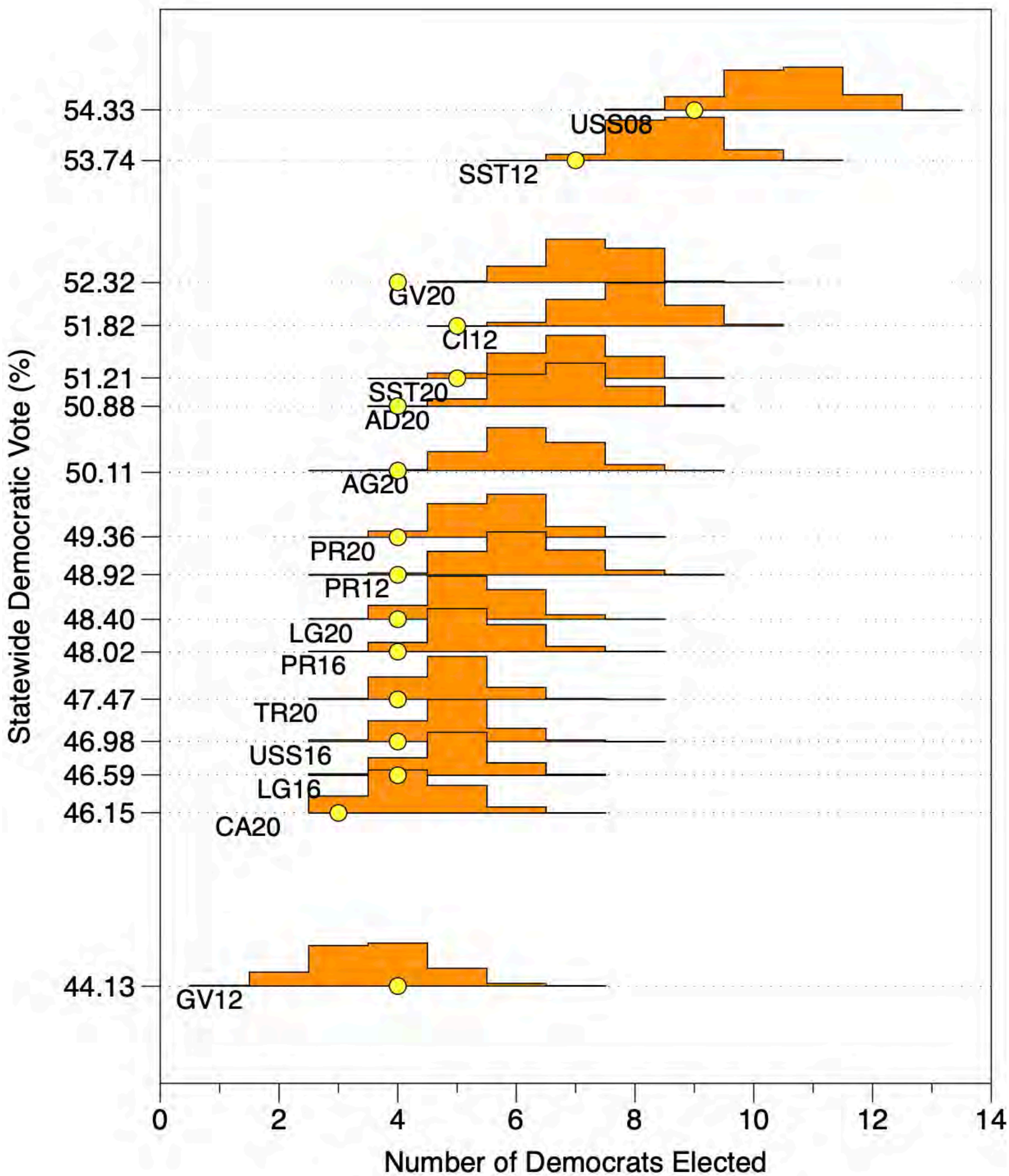
Mattingly Figure 6.2.10: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.



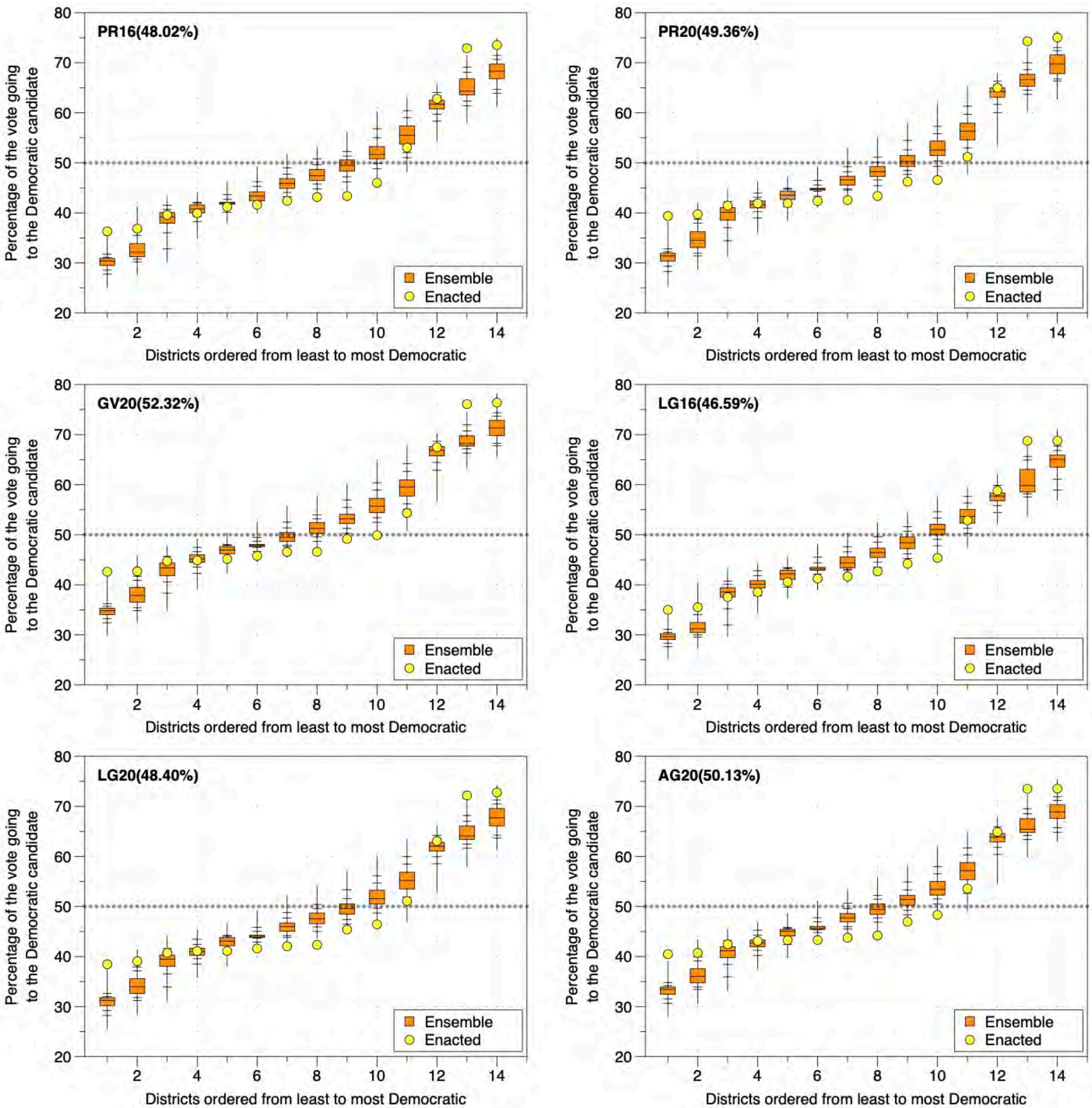
Mattingly Figure 6.2.13: Shown are the distributions of the Democratic vote fraction of the districts in the plan when ordered from most Republican (on the left) to most Democrat (on the right). The " – " on each marginal histogram denotes the vote fraction of the corresponding district in the enacted plan. The numbers along the horizontal axis give the district numbers in the enacted plan corresponding to the " – ". This plot uses the Primary ensemble which was tuned to match the municipal preservation of the enacted plan.

	Enacted Cluster 1	Enacted Cluster 2	Alternative Cluster 1	Alternative Cluster 2
County Clusters	Martin, Warren, Halifax, Hyde, Pamlico, Chowan, Washington, Carteret	Gates Currituck Pasquotank Dare Bertie Camden Perquimans Hertford Tyrrell Northampton	Pasquotank, Dare, Perquimans, Hyde, Pamlico, Chowan, Washington, Carteret	Gates, Currituck, Camden, Bertie, Warren, Halifax, Hertford, Tyrrell, Northampton, Martin
Democratic Vote %(LG16)	46.07%	47.74%	38.51%	55.42%
Democratic Vote % (PR16)	45.60%	46.70%	37.83%	54.59%
Democratic Vote % (CA20)	42.28%	44.47%	36.48%	50.75%
Democratic Vote % (USS20)	45.31%	45.36%	38.45%	52.75%
Democratic Vote % (TR20)	44.12%	44.58%	37.61%	51.59%
Democratic Vote % (GV20)	46.79%	47.56%	40.75%	54.12%
Democratic Vote % (AD20)	47.79%	47.72%	41.02%	54.99%
Democratic Vote % (SST20)	47.56%	47.85%	41.03%	54.89%
Democratic Vote % (AG20)	45.88%	46.11%	39.15%	53.40%
Democratic Vote % (PR20)	44.09%	45.54%	38.30%	51.84%
Democratic Vote % (LG20)	43.80%	45.12%	37.74%	51.69%
Democratic Vote % (CL20)	45.23%	46.42%	39.12%	52.00%

Mattingly Table 3: Voting History for the two different choices of county grouping for the northeast corner in the NC Senate.



Mattingly Figure 9.0.1: Each histogram represents the range and distribution of possible Democratic seats won in the ensemble of plans; the height is the relative probability of observing the result. The yellow dots represent the results from the enacted congressional plan under the various historic votes.



Mattingly Figure 9.0.2: The Ranked Marginal Box-plots for the NC Congressional Plan. The ranked ordered marginals for the enacted map are shown in yellow. 50% of the ensemble is contained within the box. Inside the first pair of tick marks is 80% of the data and inside the second set is 95% of the points.

Election	Plans with the same or more Dem (1-2)	Plans with the same or more Rep (5-11)	Plans with the same or more Dem (12-14)	Total Plans
LG16	18	0	0	79997
PR16	0	0	0	79997
CA20	0	0	0	79997
TR20	0	0	0	79997
LG20	0	0	0	79997
USS20	0	0	0	79997
CL20	0	0	0	79997
PR20	0	0	0	79997
AG20	0	0	0	79997
AD20	0	0	0	79997
SST20	0	0	0	79997
GV20	0	0	0	79997
CI20	0	0	0	79997
USS16	0	0	0	79997
GV16	1	0	0	79997
AG16	15	0	0	79997

Mattingly Table 4: Over the approximately 80,000 plans in our ensemble, we ask how many plans have (1) as high Democratic fraction in the two most Republican districts, (2) as small a fraction of Democrats in the 5th through 11th most Republican districts, and (3) have as high a Democratic fraction in the 12th through 14th most Republican districts. The answer is given in this table along with the total number of plans in our ensemble.

Response to Expert Report by Dr. Barber on the North Carolina State Legislature Redistricting Plans

Jonathan C. Mattingly

December 28, 2021

Contents

1 Introduction

The report by Dr. Michael Barber begins with a discussion of the political geography of the state of North Carolina. He emphasizes the heterogeneity of the state. While he points out the strengths of ensemble methods to separate the effect of natural clustering of votes and other effects due to political geography, Dr. Barber limits its use to analysis of the individual county clusters. Similarly, though he uses a collection of election data at the cluster level, he does not consider a diverse collection of election analyses both at the cluster level and when performing his statewide analysis. Rather, he restricts himself to a single summary statistic, namely, counting the number of Democratic-leaning districts at the individual cluster level based primarily on a composite election obtained through averaging several past statewide elections.

We complete the missing parts of Dr. Barber's analysis using data directly from his report when possible. When needed, we augment this data with an ensemble of maps obtained by running Dr. Barber's code. From this completed analysis, we see that Dr. Barber's ensemble shows both the Enacted NC House and the Enacted NC Senate to be extreme partisan outliers with a clear and systematic tilt in favor of electing Republicans.

When we focus on the structure of the enacted maps in the county clusters under Dr. Barber's analysis, we again see the same structures we observed using the Primary Ensembles from our initial report. These structures showed the enacted map to be an extreme outlier. Due to time constraints, we did not complete cluster level analysis on all clusters using Dr. Barber's simulations; we have, however, performed a cluster level analysis on a diverse collection of clusters in the NC House. Our cluster level analysis considers not only seat counts, but also the margins of victory within those seats. By examining the margins, we identify extreme partisan behavior at the cluster level using the very sampling code that Dr. Barber created.

We conclude that Dr. Barber's ensembles provide another independent verification that the enacted plans for the NC House and NC Senate are extreme gerrymanders.

2 Comment on Political Geography of State

In Section 3 of Dr. Barber's report, he discusses the political geography of the state. He made a number of statewide evaluations of the partisan structure using a single average of 11 statewide elections from 2014-2020. As his analysis in later sections makes clear, the political climate varies significantly from year to year and election to election. The average of these elections creates a new set of voting data, possibly quite district from those averaged to create it. I see no reason to elevate the behavior and properties of a map under the one particular political environment signified by this vote over other elections. It is important that the map used to translate our election votes into elected officials act in a non-biased way across a number of elections which represent different political climates seen in North Carolina, not just one.

In the rest of the report, Dr. Barber does switch to considering a number of distinct elections. However, he does not return to any aggregate statewide discussion using these individual elections and the diversity of election environments they represent. He does firmly endorse the use of a computer drawn ensemble of maps to create a base line against which the enacted map can be compared. He correctly represents that this method has the advantage of taking into account all of the political geography of the state, such as the concentrating of particular voters in some regions of the state or the preservation of counties and the like. Hence, when a map is an outlier compared to a computer drawn ensemble, these natural clustering or political geography considerations cannot be the explanation.

Dr. Barber never conducts any statewide analysis under his ensemble using different election results. However, all of the components necessary to perform such analysis are present in his report. Utilizing Dr. Barber's cluster-by-cluster ensembles, we complete the absent statewide analysis to examine the number of Democratic leaning seats under various elections. This analysis demonstrates that the enacted map *is* an extreme outlier when compared to Dr. Barber's ensemble.

3 Nonpartisan Ensemble Generated by Dr. Barber

In analyzing the North Carolina State House and Senate maps, Dr. Michael Barber generates an ensemble of non-partisan redistricting maps via the Sequential Monte Carlo (SMC) procedure in the *redist* R-package developed and maintained by a research group at Harvard University. When used to sample from a known distribution in a moderate sized problem, this method has been shown to faithfully sample the target distribution. This was validated on moderate sized examples using an enumeration algorithm developed by the same group that developed the *redist* R-package at Harvard. The method we used has similarly been validated using this and other methods. Dr. Barber used the ensemble method only at the cluster level and does not use it to perform a statewide analysis based on a statewide ensemble. Rather he just summarizes the cluster by cluster results in a few tables (Table 2 and Table 32) instead of performing any analysis which would show the cumulative effect at the statewide level. The coin flipping analogy we offer below shows why this is so inadequate. In utilizing Dr. Barber's ensemble, we demonstrate that he would have concluded the enacted map was an extreme outlier at the statewide level. This is not an endorsement of any of the particular algorithm choices he has made, but rather to demonstrate that this conclusion is available from his findings.

By taking the percentages in the cluster-by-cluster tables in Dr. Barber's report, we were able to perform the statewide analysis he neglected using his data. We were also able to perform this for the collection of different statewide elections Dr. Barber used in his analysis. This allowed us to see the behavior of the maps under different types of elections. Both of these considerations are important and we briefly discuss them individually before turning to the statewide analysis using Dr. Barber's data.

- **Importance of statewide analysis:** Dr. Barber analyzes each cluster one-by-one and concludes that the majority of them are not extreme outliers so under his election composite the map is not an outlier. However, in almost every case, he finds that the more Republican of the non-outlying options is selected. Consider the following analogy. Someone flips a coin that they claim is fair but is in fact biased to produce heads more often. They flip the coin and produce 40 heads and zero tails. On each flip, the chance of getting a head from a fair coin is 50%. Hence the outcome on each flip is not that surprising. Dr. Barber's analysis is analogous to looking at each flip alone and then claiming that the coin is fair because the outcome was a head and the chance of a fair coin producing a head was reasonable. However, taking a more global view one can an easily see that the chance of getting 40 heads in a row is astronomically small. And thus, one can conclude the coin is biased. This would even be true if there were only 35 heads and 5 tails.

  Analogously, each cluster taken individually might not be an extreme outlier, but it is extremely unlikely that all of these clusters woud exist together in a statewide map drawn without partisan intent.

  We will also see that some of the local clusters are extreme outliers in their own right using Dr. Barber's data and extending his analysis to look at the margins of victory (or the extent of the partisan lean) rather than only focusing on the number of seats won by either party (or the direction of the partisan lean). This extended analysis agrees with the finding in our initial report.

- **Often extreme behavior is apparent in only some elections:** If one wanted to rig a card game by colluding with some of the other players, the group would only need to act when none of the group was going to win. The group need only act when cards were aligned against them. Hence, the behavior of a gerrymandered map might appear typical in settings where the gerrymandering party is content with the outcome that one would typically expect without gerrymandering. Furthermore, it is possible that whatever system the card players are using is not sufficient to counteract some hands. In other words, even a card player that is cheating might not be able to win when their opponent draws a royal flush. Hence, it is not to be expected that in all cases a gerrymandered map is effective in supporting the gerrymandering party.

  In particular, one can not simply declare that a map is not gerrymandered because it is fair in some fraction (even a relatively large fraction) of the election environments. If it is clearly gerrymandered in some reasonable and pertinent election environments, then the map should be seen as gerrymandered. To do otherwise would be to argue that a casino would be happy with card players who only cheated 30% of the time and in particular did not cheat when they were already winning or had an unsalvageable hand.

In addition to generating a statewide analysis using the actual data from Dr. Barber's report, we also employ ensembles generated from the *redist* code base, set up according to Dr. Barber's analysis scripts.[1] We then show that well-established methods of probing for gerrymandering reveal that many of the individual clusters are indeed extreme gerrymanders. In doing so, we consider the partisan seat counts of each party and also extend the analysis to consider *how* the seats are won. The latter is important as it shows the degree that a given district is politically safe as well as determines how future political swings, unseen at present, might affect political outcomes. For example, atypically polarized districts can lead to maps which do not respond to the shifts in the electorate's preferences, and effectively lock in a particular outcome. Additionally, when a map has an extremely partisan structure, this can speak to the intent of the map makers even if the structure would be unlikely to affect some collection of elections such as wave elections in favor of the gerrymandering party.

[1] Dr. Barber did include a R Data file which might have included the maps he generated in his run. However, since our version of R was slightly different than his, it would not load. Hence we were forced to re-run his code.

4 Statewide Analysis of Dr. Barber's Ensemble of NC House Plans

Within each cluster, Dr. Barber presents the fraction of plans in his ensembles that would lead to a certain number of Democratic districts under each set of historic and averaged vote counts. These tables can be used to construct the probability of drawing a non-partisan plan at the statewide level that would yield a certain number of Democratic leaning districts under various elections.

Beginning with his averaged statewide vote counts, we construct the statewide probabilities of electing various numbers of representatives and present them in Figure 1 in terms of the number of Democrats elected. Only 0.177% of all of the plans in Dr. Barber's ensemble elect the same or more Republicans than the enacted plan.

Note that our count of Democrats elected includes the Democrats elected in single-district clusters, which are omitted from Dr. Barber's Table 2. So our Figure 1 reports that the enacted plan elects 49 Democrats under Dr. Barber's composite of elections, which is the four Democrats elected in single-district clusters that Dr. Barber reports in his Table 1 plus the 45 Democrats elected in multi-district clusters that Dr. Barber reports in his Table 2.

We repeat the above analysis with the 2016 and 2020 election data used by Dr. Barber. The only supplemental data we introduce is the number of single district Democratic clusters in each election which we have taken from our previous analysis. We summarize the 10 elections in Figure 2 and Table 1.

As in our previous analysis, we find that the outlier status of the ensemble has a significant impact on the amount of power the Republicans can amass in the House. For example, under the votes of the 2020 Lt. Governor race, 2016 Presidential race, and 2020 US Senate race, the ensemble breaks a Republican supermajority in 99.3937%, 98.976, and 99.992% of the plans in Dr. Barber's ensemble, respectively. However, the enacted plan would elect a Republican supermajority under each of these votes. Similarly, under the 2020 Governor race, the Republican majority would have been broken in 96.42% of the plans in Dr Barber's ensemble, yet they would have maintained the majority using the enacted map under these votes.

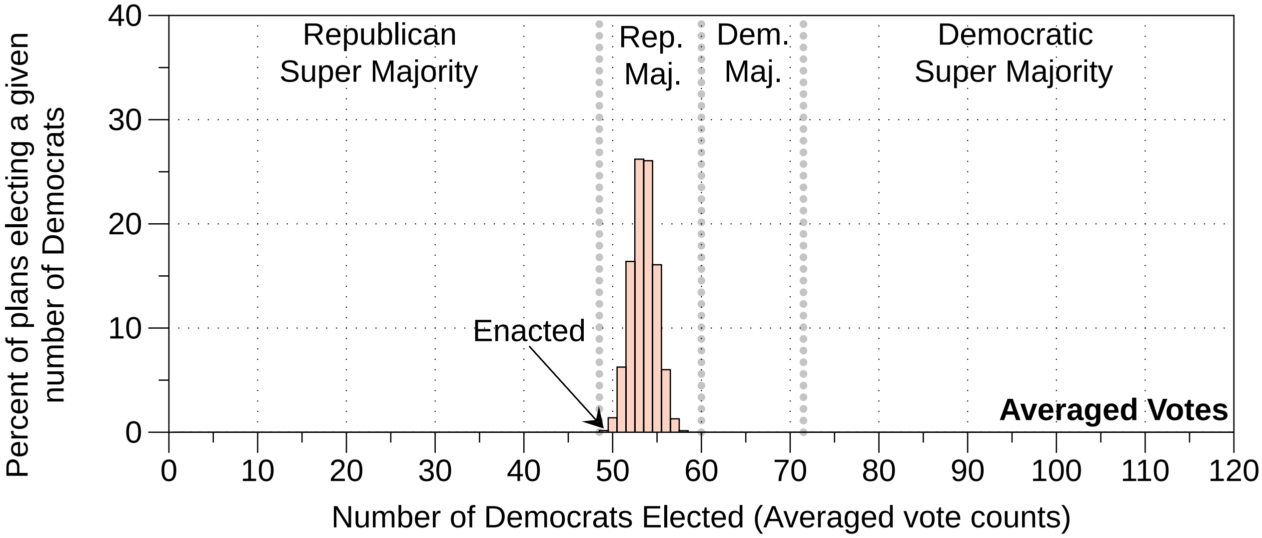


Figure 1: We compare Dr. Barber's statewide ensemble with the enacted plan under the Averaged election results used in his report. We find that only 0.177% of all of the plans in his ensemble would elect the same or more Republicans.

Election	Statewide Dem. Vote	% of Dr. Barber's Plans electing the same or more Republicans than the enacted plan
Barber's Average Vote	-	0.177%
2020 Governor	52.32%	0.204%
2016 Attorney General	50.20%	1.34%
2020 Attorney General	50.13%	0.00684%
2016 Governor	50.047%	0.215%
2020 President	49.36%	0.000146%
2020 Senate	49.14%	0.00804%
2020 Lt. Governor	48.40%	0.000377%
2016 President	48.024%	1.02%
2016 Senate	46.98%	0.223%
2016 Lt. Governor	46.59%	0.518%

Table 1: When considered at the statewide level, the ensembles produced by Dr. Barber are all extreme outliers. The chance that a plan drawn from the ensemble would elect the same or more Republicans as the enacted plan is, at most, 1.34%; in all but three of the elections it is less than 0.25%. We have ordered the elections with the election with the largest Democratic statewide vote fraction at the top and the election with largest Republican statewide vote fraction at the bottom. It is worth noting that many of the most extreme outliers happen for those between 50% and 48%. Looking at Figure 2, we see that this is the range where the Republicans would typically lose the super majority according to Dr. Barber's analysis. Though "Barber's Average Vote" which he used as a partisan index might or might not represent an actual plausible voting pattern, we have included it for comparison.

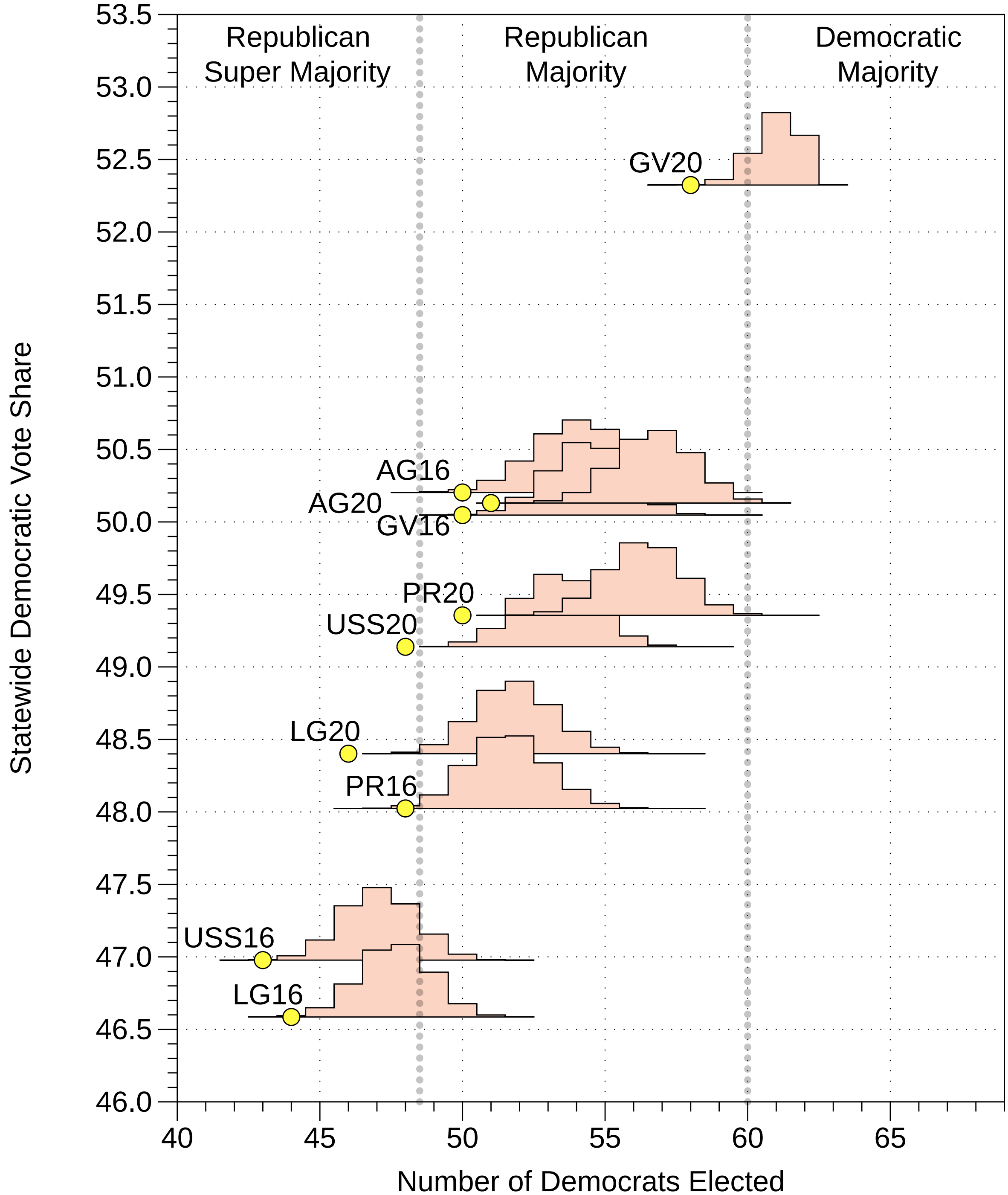


Figure 2: We compare Dr. Barber's statewide ensemble with the enacted plan under the ten 2016 and 2020 elections used in his report. Yellow dots show the result of the enacted plan. The enacted plan is an extreme outlier when considering the same data under a statewide lens. We summarize the numerical extent of the outliers in Table 1. The elections are abbreviated with the last two digits signifying the year, and the first letters representing Lt. Governor (LG), Governor (GV), President (PR), and US Senate (USS).

5 Statewide Analysis of Dr. Barber's Ensemble of NC Senate Plans

Repeating the above analysis for Dr. Barber's ensemble of Senate plans, we begin with the averaged statewide vote counts. We construct the statewide probabilities of electing various numbers of Senators and present them in Figure 3. Once again, our count of Democrats elected includes the Democrats elected in single-district Senate clusters, which are omitted from Dr. Barbers Table 32. So our Figure 3 reports that the enacted plan elects 20 Democrats under Dr. Barbers composite of elections, which is the four Democrats elected in single-district clusters that Dr. Barber reports in his Table 31 plus the 16 Democrats elected in multi-district clusters that Dr. Barber reports in his Table 32. Only 0.00385% of all of the plans in Dr. Barber's ensemble elect the same or more Republicans. Furthermore, this is the percentage of plans that lead to a Republican supermajority under these votes (which the enacted plan would produce as well). In other words, while the enacted plan always produces a Republican supermajority under Dr. Barber's analysis, only .00385% of the non-partisan plans that Dr. Barber simulates would produce a Republican supermajority.

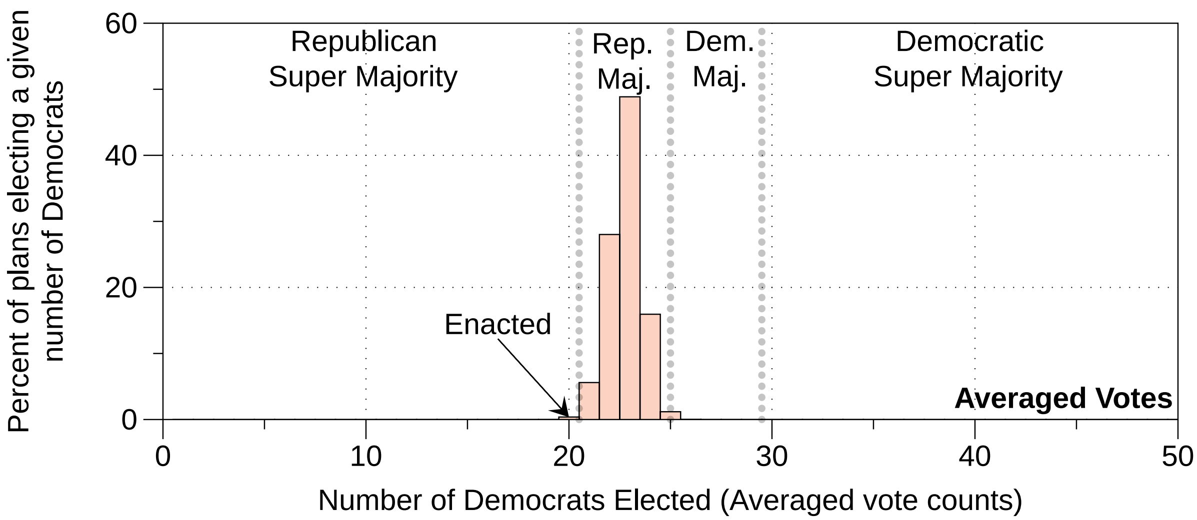


Figure 3: We compare Dr. Barber's statewide ensemble with the enacted plan under the Averaged election results used in his report. We find that only 0.00385% of all of the plans in his ensemble would elect the same or more Republicans than the enacted plan.

We repeat the above analysis with the 2016 and 2020 election data used by Dr. Barber. The only supplemental data we introduce is the number of single district Democratic clusters in each election which we have taken from our previous analysis. We summarize the 10 elections in Figure 4 and Table 2.

Again, we find that the outlier status of the ensemble has a significant impact on the amount of power the Republicans can amass in the Senate. Under the votes of the 2016 Governor race and 2016 Attorney General races, the Republicans lose their supermajority in 99.9544% and 98.9501% of the plans in Dr. Barber's ensemble, respectively. However, the enacted plan would elect a Republican supermajority under each of these voting patterns.

Election	Statewide Dem. Vote	% of Dr. Barber's Plans electing the same or more Republicans than the enacted plan
Averaged	-	0.00385%
2020 Governor	52.32%	1.92%
2016 Attorney General	50.20%	1.05%
2016 Governor	50.047%	0.047%
2020 Attorney General	50.13%	3.74%
2020 President	49.36%	9.92%
2020 Senate	49.14%	5.76%
2020 Lt. Governor	48.40%	0.250%
2016 President	48.024%	0.16%
2016 Senate	46.98%	1.22%
2016 Lt. Governor	46.59%	10.9%

Table 2: When considered at the statewide level, many of the ensembles produced by Dr. Barber are extreme outliers. In six of the ten elections, there is less than a 2% chance that a plan drawn from the ensemble would elect the same or more Republicans as the enacted plan; in three of the ten elections, there is less than a 0.251% chance that a plan drawn from the ensemble would elect the same or more Republicans than the enacted plan. As we have remarked in both our original report and in the analysis below, this *does not* mean that the enacted plan is not an extreme partisan gerrymander under the other four elections; it only indicates that the plan is not as extreme of an outlier in these elections under the particular lens of seat counts.

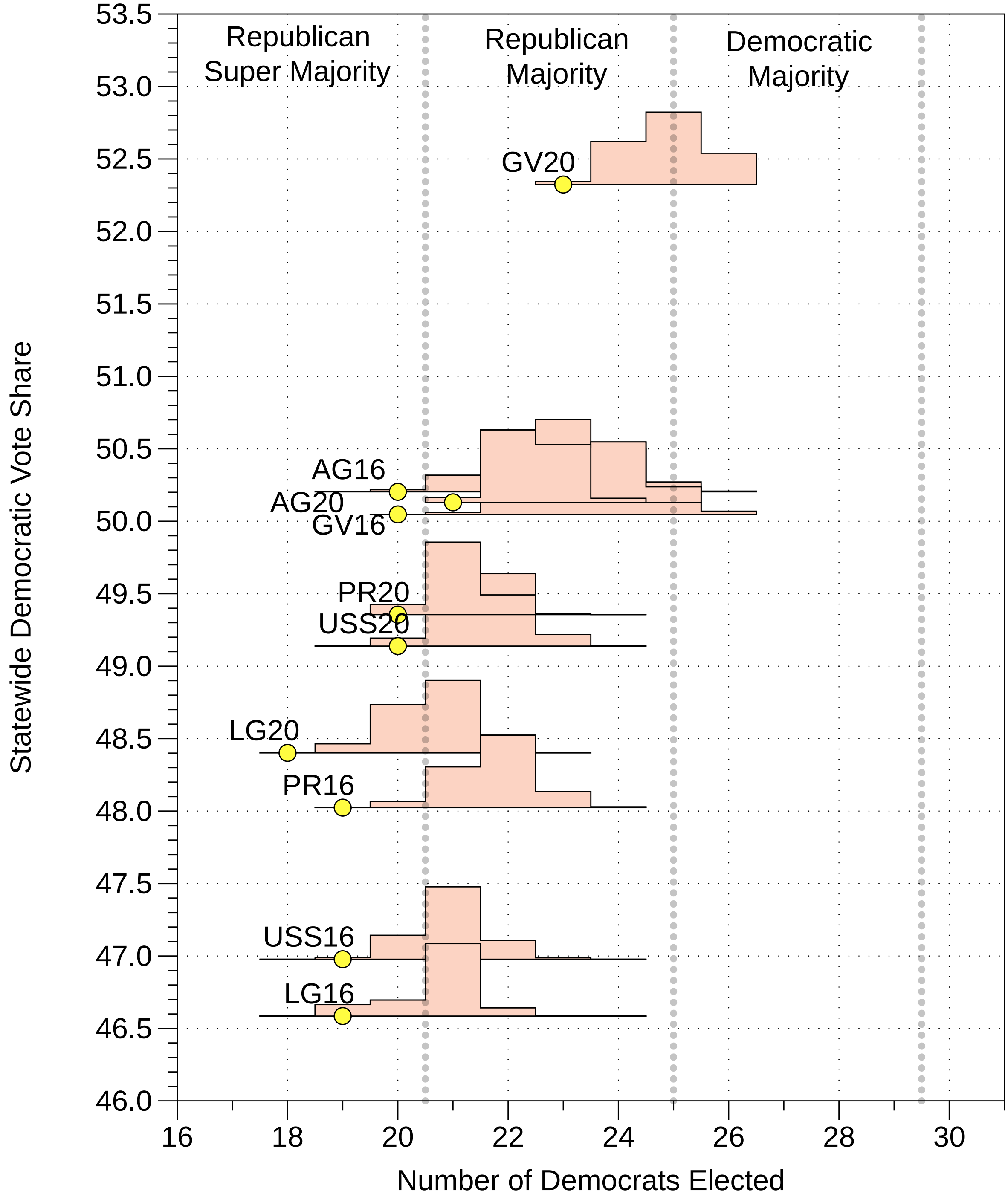


Figure 4: We compare Dr. Barber's statewide ensemble with the enacted plan under the ten 2016 and 2020 elections used in his report. Yellow dots show the result of the enacted plan. The enacted plan is an extreme outlier when considering the same data under a statewide lens. We summarize the numerical extent of the outliers in Table 1. The elections are abbreviated with the last two digits signifying the year, and the first letters representing Lt. Governor (LG), Governor (GV), President (PR), and US Senate (USS).

6 Cluster by Cluster Analysis

We now turn to examining certain clusters presented in Dr. Barber's work. We do not exhaustively examine all of the clusters. Rather, we select certain clusters to demonstrate how the lens that Dr. Barber chooses to use (namely only looking at the number of Democratic districts) yields an incomplete picture of the partisan make up of the districts even with respect to the individual districts.

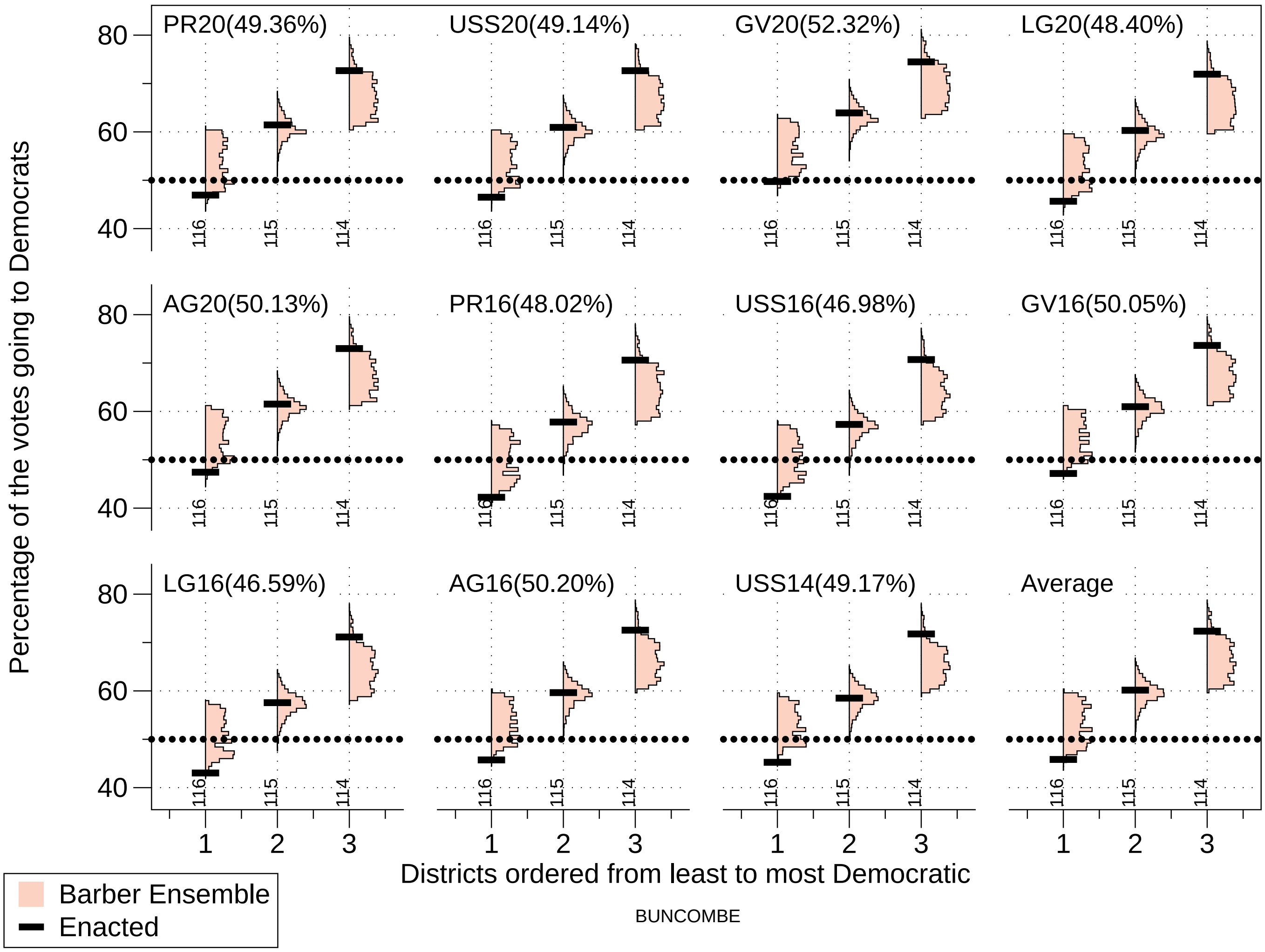
For a more complete picture, one would need to look at the actual partisan make-up of each district within a cluster. In fact, Dr. Barber reported on these values for the enacted plan, but did not compare these values to those found in his ensemble. One way of comparing these numbers is to examine the rank ordered marginal distributions of the vote fraction in each district. To do this, we order the districts from least to most Democratic (what Dr. Barber calls the Partisan Lean of Districts), and then look at the distribution of the most Republican, second most Republican, etc..., all the way until we reach the most Democratic district.

This type of analysis reveals not only how many Democratic leaning districts are within Dr. Barber's ensemble, but also *how much* they lean Democratic (or Republican). As we have demonstrated in our report, this is also relevant at a statewide level.

Note that all of our previous statewide analysis of seat counts simply relied on the numbers presented in Dr. Barber's report, i.e., the exact same ensemble that he relies on. The analysis below uses an ensemble of plans derived from running Dr. Barbers code (we were unable to extract his ensembles he used from the data he provided).[2] However, re-running his same code with his exact same input parameters should produce a comparable ensemble to the one he generated from the report, assuming that his code performs in the way he represents.

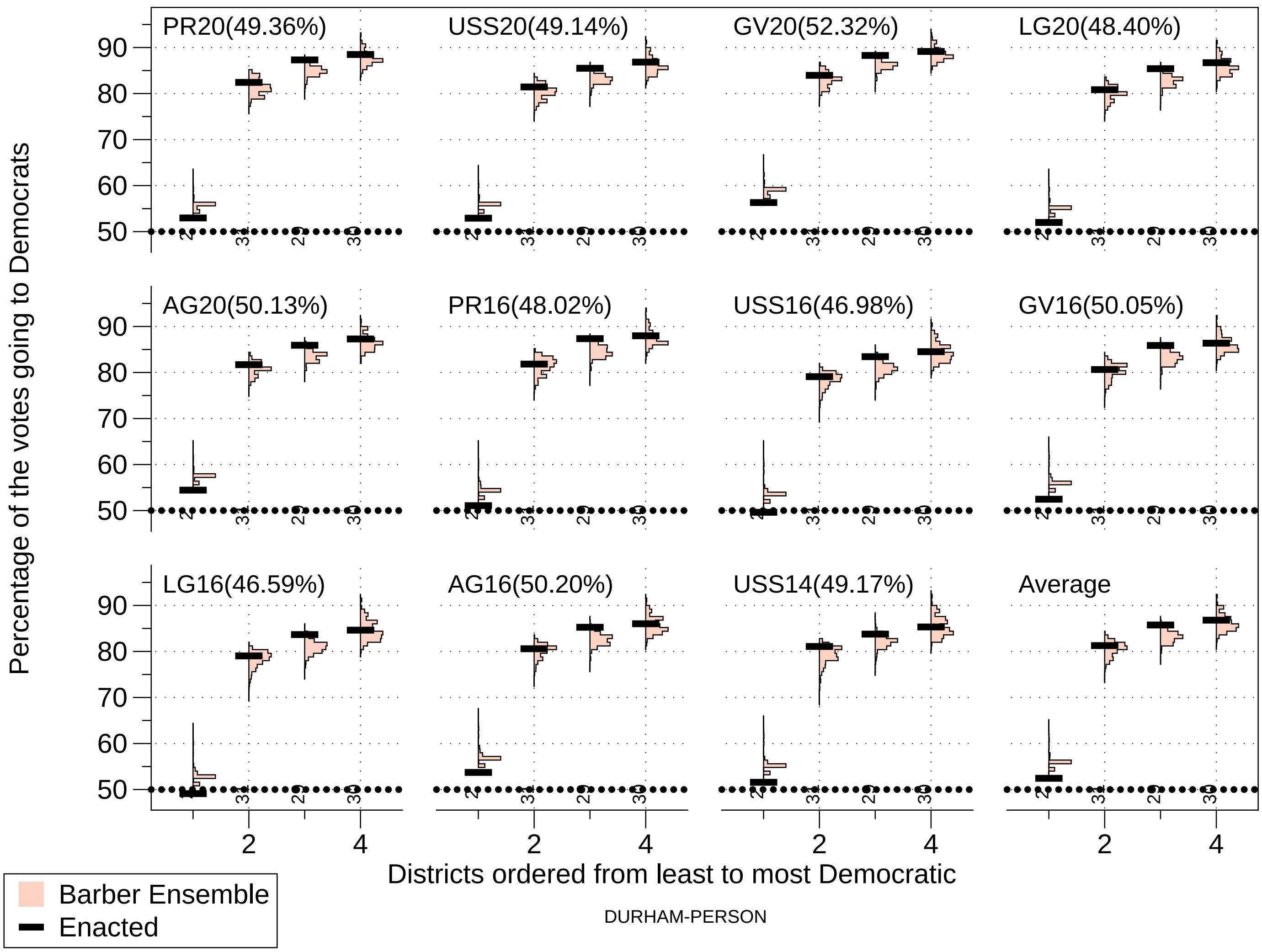
The main conclusion is that when comparing the cluster-by-cluster results from Dr. Barber's ensemble to those in our report, we find the qualitative structure to be the same. We again conclude that the enacted map is an extreme outlier when using Dr. Barber's ensemble with this additional analysis. We include a number of county clusters from the NC House. We make a number of comments in the caption of each figure. We refer the reader to our initial report to the court for a description of these Ranked-Ordered-Marginal-Histograms.

[2]We obtained the ensemble data from runs of Dr. Barber's code from Wes Pegden (CMU) who ran the code on his R installation as we did not have a computing environment able to run the code conveniently during the window when the rebuttal reports were due.



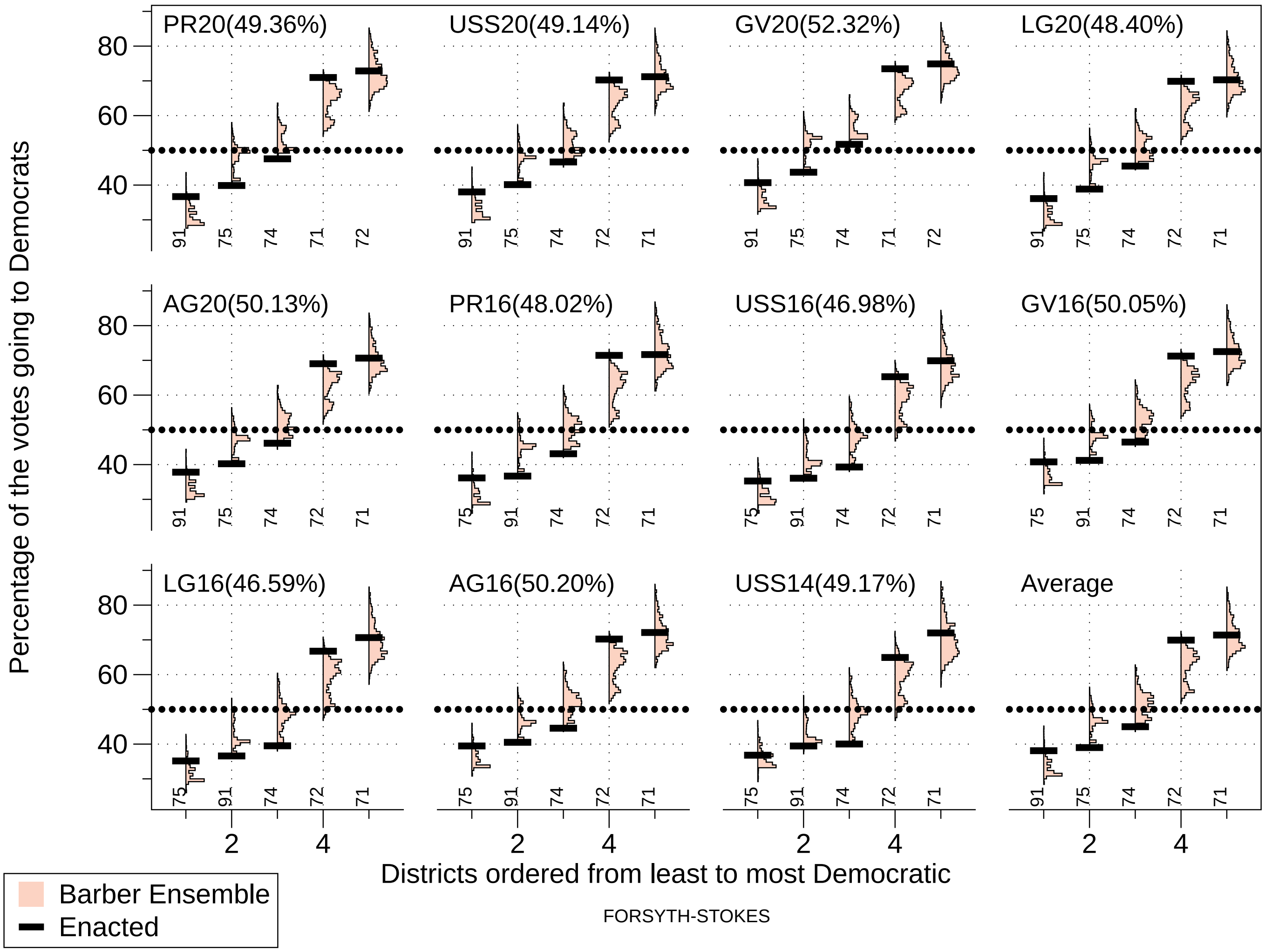
Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
Average	107	0.277	2409	6.23	38664	1	3
PR20	756	1.96	3095	8.0	38664	1	3
USS20	409	1.06	2529	6.54	38664	1	3
GV20	662	1.71	3200	8.28	38664	1	3
LG20	424	1.1	2624	6.79	38664	1	3
AG20	534	1.38	2655	6.87	38664	1	3
PR16	321	0.83	2701	6.99	38664	1	3
USS16	17	0.044	2062	5.33	38664	1	3
GV16	18	0.0466	2067	5.35	38664	1	3
LG16	18	0.0466	1998	5.17	38664	1	3
AG16	17	0.044	1992	5.15	38664	1	3
USS14	3	0.00776	1807	4.67	38664	1	3

Figure 5: In Buncombe County, the Enacted maps is an extreme outlier under Dr. Barber's ensemble. We see the same structure as we saw when compared with the probability ensemble our initial report. The most Republican district in the enacted plan has exceptionally few Democrats while the most Democratic district has exceptionally many Democrats. The result is that the Democrats never win three seats in the enacted plan under any of the elections considered, including Dr. Barber's composite "Averaged Election", even though they would typically do so under a number of elections under Dr. Barber's ensemble.



Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
Average	0	0.0	1396	3.69	37800	1	3 4
PR20	0	0.0	790	2.09	37800	1	3 4
USS20	0	0.0	1326	3.51	37800	1	3 4
GV20	0	0.0	1123	2.97	37800	1	3 4
LG20	0	0.0	1199	3.17	37800	1	3 4
AG20	0	0.0	1205	3.19	37800	1	3 4
PR16	0	0.0	1184	3.13	37800	1	3 4
USS16	0	0.0	2932	7.76	37800	1	3 4
GV16	0	0.0	1382	3.66	37800	1	3 4
LG16	0	0.0	2675	7.08	37800	1	3 4
AG16	0	0.0	1931	5.11	37800	1	3 4
USS14	0	0.0	10357	27.4	37800	1	3 4

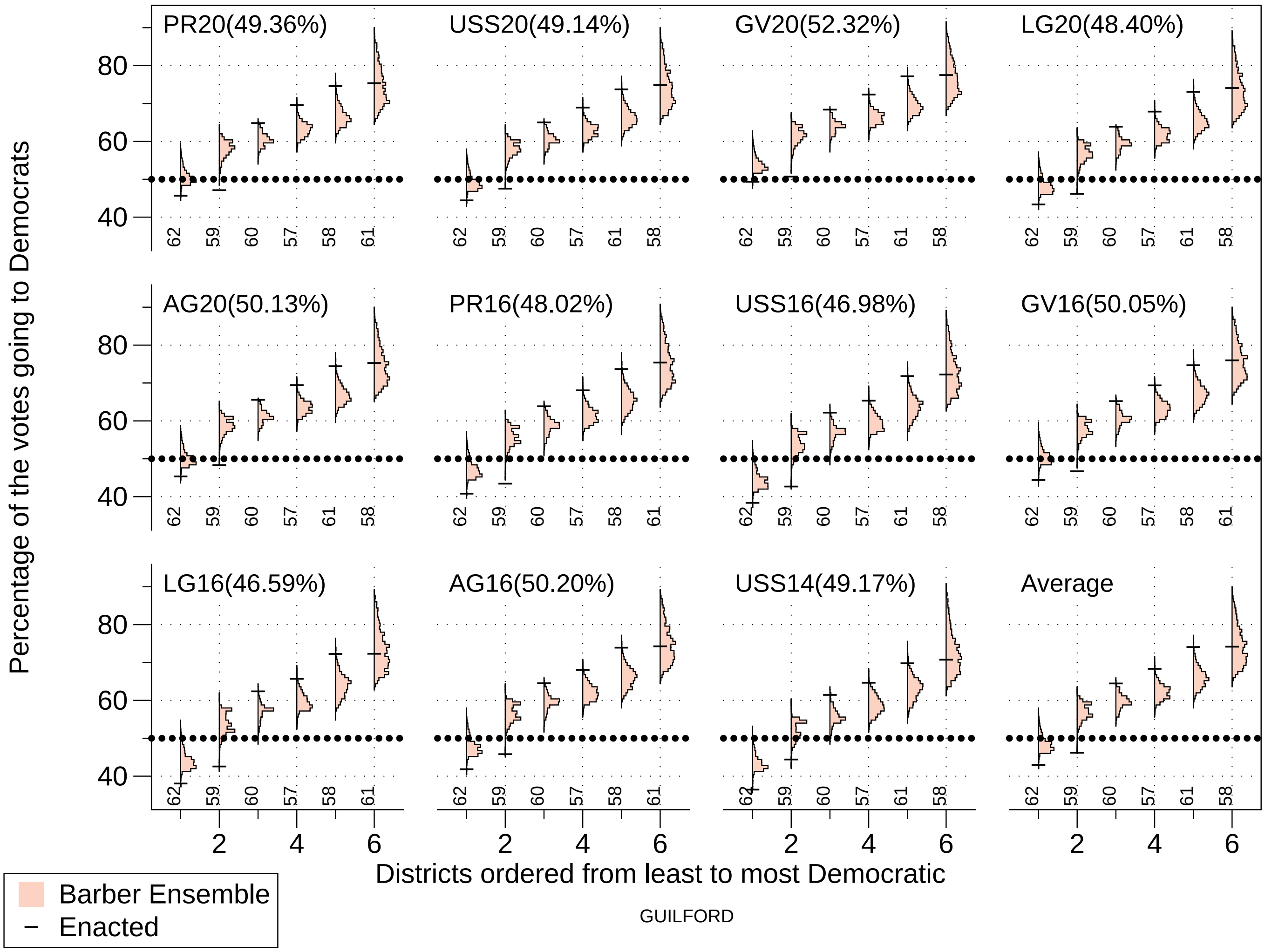
Figure 6: In the Durham-Person cluster, we the same outlier structure in the enacted map when compared to Dr. Barber's ensemble as when compared to the primary ensemble in our orignal report. We see that the most Republican district has been depleted of Democrates. This makes the district much more competitive than it typically would be under a non-partisan redistricting plan.



FORSYTH-STOKES

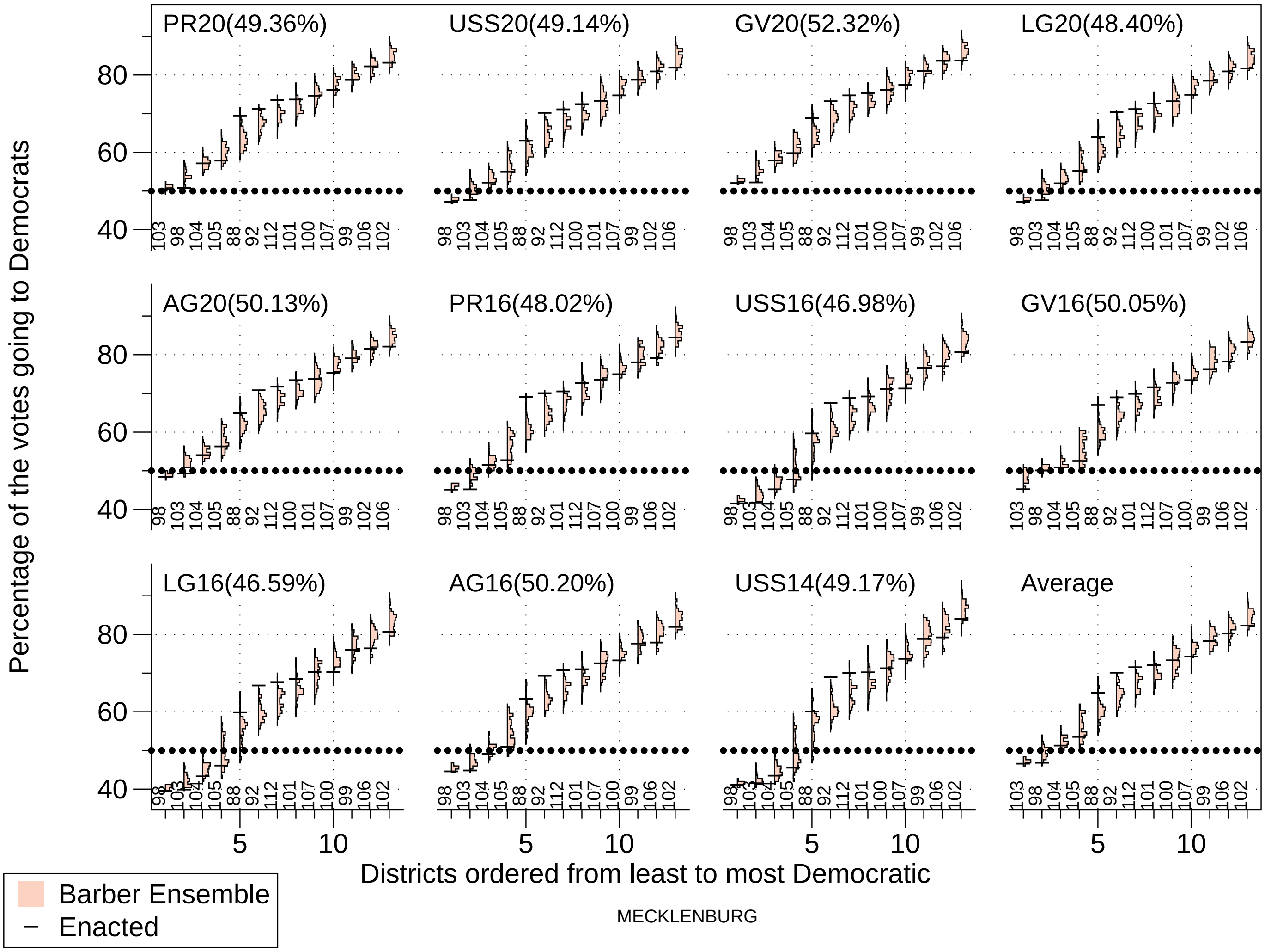
Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
Average	17	0.456	317	8.51	3726	1 2 3	4 5
PR20	4	0.107	349	9.37	3726	1 2 3	4 5
USS20	60	1.61	429	11.5	3726	1 2 3	4 5
GV20	2	0.0537	357	9.58	3726	1 2 3	4 5
LG20	21	0.564	376	10.1	3726	1 2 3	4 5
AG20	47	1.26	395	10.6	3726	1 2 3	4 5
PR16	7	0.188	284	7.62	3726	1 2 3	4 5
USS16	44	1.18	280	7.51	3726	1 2 3	4 5
GV16	11	0.295	292	7.84	3726	1 2 3	4 5
LG16	30	0.805	269	7.22	3726	1 2 3	4 5
AG16	25	0.671	263	7.06	3726	1 2 3	4 5
USS14	13	0.349	351	9.42	3726	1 2 3	4 5

Figure 7: In the Forsyth-Stokes cluster, We again see the same structure in Dr. Barber's ensemble as in the primary ensemble from our initial report. We see abnormally few Democrats in the second and third most Republican districts while we see abnormally many Democrats in the most Republican district and in the two most Democratic districts. The effect is to regularly flip the 3rd most Republican district to the republicans under the enacted map even under elections where many to almost all of the plans in Dr. Barber's ensemble would have awarded the seat to the Democrats.



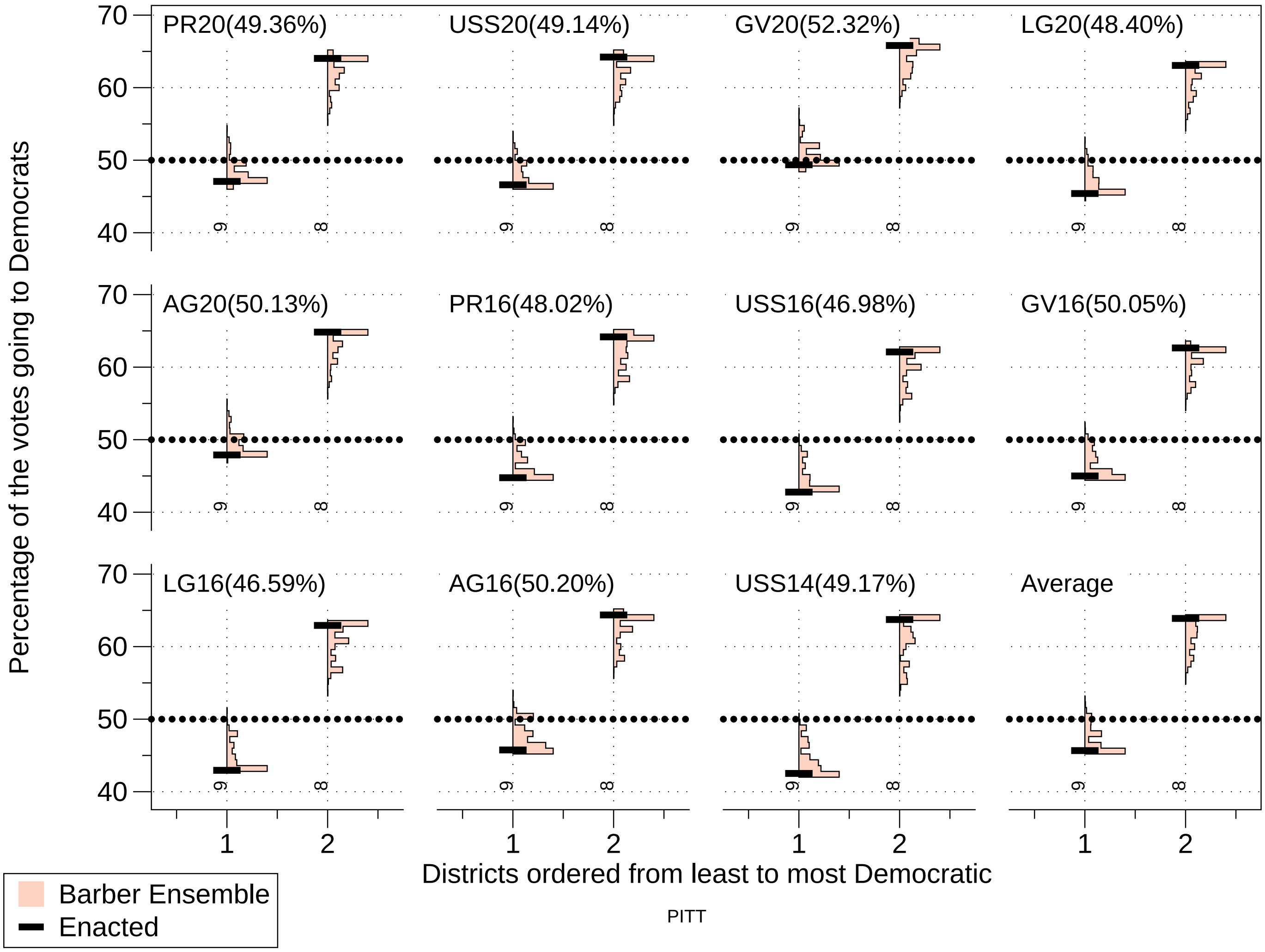
Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
Average	0	0.0	0	0.0	15489	1 2	3 4 5 6
PR20	0	0.0	0	0.0	15489	1 2	3 4 5 6
USS20	0	0.0	0	0.0	15489	1 2	3 4 5 6
GV20	0	0.0	0	0.0	15489	1 2	3 4 5 6
LG20	0	0.0	0	0.0	15489	1 2	3 4 5 6
AG20	0	0.0	0	0.0	15489	1 2	3 4 5 6
PR16	0	0.0	0	0.0	15489	1 2	3 4 5 6
USS16	0	0.0	0	0.0	15489	1 2	3 4 5 6
GV16	0	0.0	0	0.0	15489	1 2	3 4 5 6
LG16	0	0.0	0	0.0	15489	1 2	3 4 5 6
AG16	0	0.0	0	0.0	15489	1 2	3 4 5 6
USS14	0	0.0	0	0.0	15489	1 2	3 4 5 6

Figure 8: Dr. Barber did identify Guilford county as a Republican Gerrymander in the enacted map. The structure which produces this result is clear when compared with this plot of Dr. Barber's ensemble. We see that the two most Republican districts have abnormally few Democrats and the next three Republican districts have abnormally many Democrats. The effect is that the second most Republican seat reliably goes to the Republican party even though in some elections almost all of the maps in Dr. Barber's ensemble would award the seat to the Democrats. This was the same structure seen in the plots of our primary ensemble from our initial report.



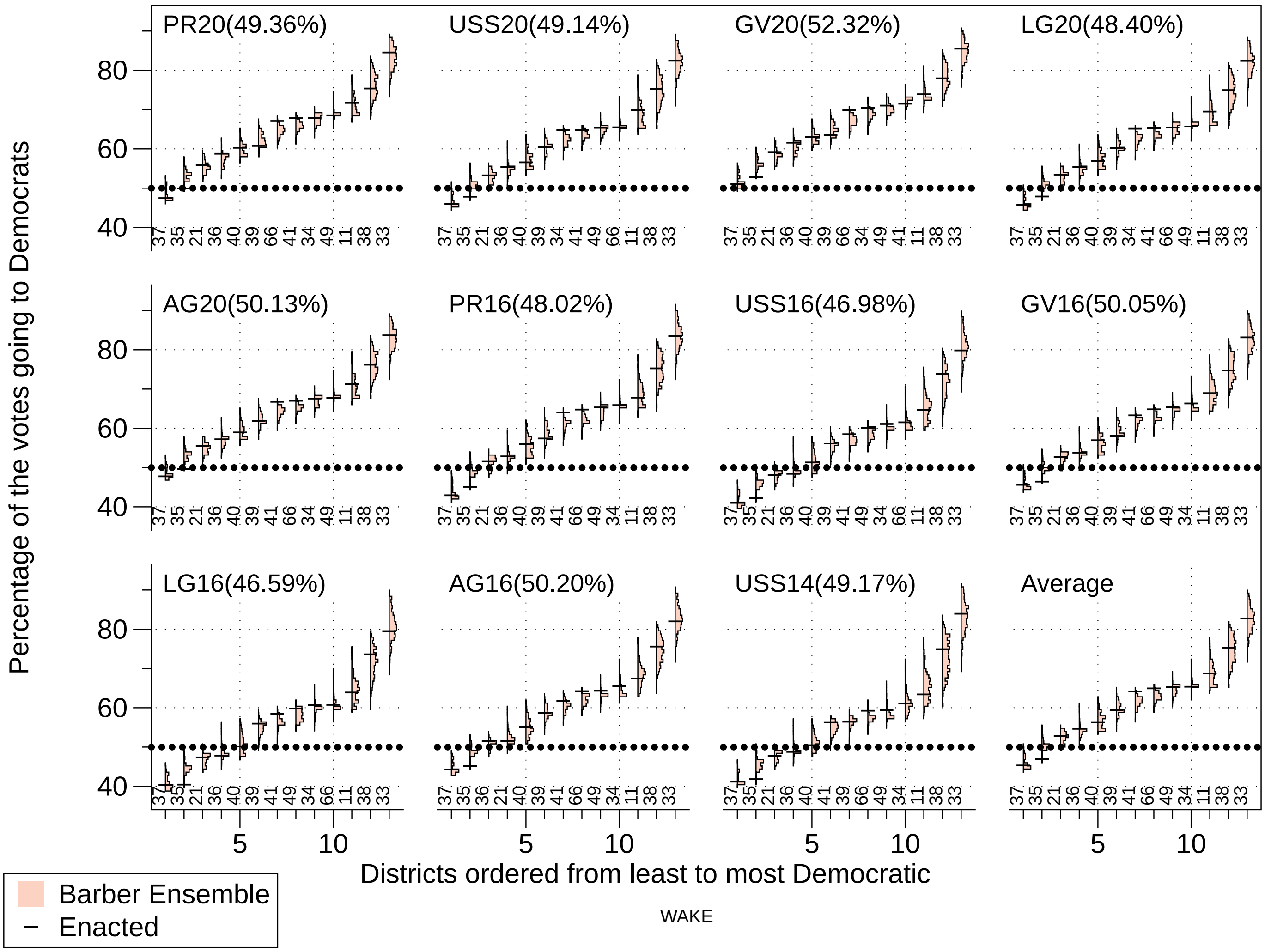
Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
Average	139	4.4	14	0.443	3161	1 2 3 4	5 6 7 8
PR20	105	3.32	18	0.569	3161	1 2 3 4	5 6 7 8
USS20	145	4.59	29	0.917	3161	1 2 3 4	5 6 7 8
GV20	114	3.61	17	0.538	3161	1 2 3 4	5 6 7 8
LG20	117	3.7	17	0.538	3161	1 2 3 4	5 6 7 8
AG20	119	3.76	17	0.538	3161	1 2 3 4	5 6 7 8
PR16	23	0.728	18	0.569	3161	1 2 3 4	5 6 7 8
USS16	74	2.34	15	0.475	3161	1 2 3 4	5 6 7 8
GV16	56	1.77	23	0.728	3161	1 2 3 4	5 6 7 8
LG16	68	2.15	18	0.569	3161	1 2 3 4	5 6 7 8
AG16	52	1.65	15	0.475	3161	1 2 3 4	5 6 7 8
USS14	153	4.84	16	0.506	3161	1 2 3 4	5 6 7 8

Figure 9: In Mecklenburg county, we again have that the four most Republican districts have abnormally few Democrats in them while the next four most Republican districts have abnormally many Democrats. This is the same structure as we saw under our primary ensemble in our initial report. The effect is that in a number of elections the Republican party wins one to two more seats than the typical plan from Dr. Barber's ensemble would award.



Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
Average	314	6.05	1929	37.2	5189	1	2
PR20	1539	29.7	1974	38.0	5189	1	2
USS20	1525	29.4	1929	37.2	5189	1	2
GV20	1556	30.0	1974	38.0	5189	1	2
LG20	1537	29.6	1974	38.0	5189	1	2
AG20	1537	29.6	1974	38.0	5189	1	2
PR16	483	9.31	1929	37.2	5189	1	2
USS16	0	0.0	1660	32.0	5189	1	2
GV16	483	9.31	1929	37.2	5189	1	2
LG16	0	0.0	1660	32.0	5189	1	2
AG16	169	3.26	1660	32.0	5189	1	2
USS14	0	0.0	1660	32.0	5189	1	2

Figure 10: In Pitt county we see that same structure we found in our Primary ensemble repeated in Dr. Barber's ensemble. In particular, we see the districts pulled to the extremes of what is seen in Dr. Barber's ensemble. The depletion of Democrats from the more Republican district protects it from electing a Democrat in the enacted plan even though it would elect a Democrat in many of the plans in Dr. Barber's ensemble in a few of the elections we considered.



Election	No. plans w/ ≤ Dems (First Cluster)	% of plans w/ ≤ Dems (First Cluster)	No. plans w/ ≥ Dems (Second Cluster)	% of plans w/ ≥ Dems (Second Cluster)	Total Plans	First Cluster	Second Cluster
Average	159	1.11	2649	18.5	14305	1 2	3 4 5 6 7 8
PR20	140	0.979	1872	13.1	14305	1 2	3 4 5 6 7 8
USS20	209	1.46	2961	20.7	14305	1 2	3 4 5 6 7 8
GV20	145	1.01	1772	12.4	14305	1 2	3 4 5 6 7 8
LG20	159	1.11	2240	15.7	14305	1 2	3 4 5 6 7 8
AG20	165	1.15	2260	15.8	14305	1 2	3 4 5 6 7 8
PR16	137	0.958	2264	15.8	14305	1 2	3 4 5 6 7 8
USS16	196	1.37	3774	26.4	14305	1 2	3 4 5 6 7 8
GV16	220	1.54	3504	24.5	14305	1 2	3 4 5 6 7 8
LG16	196	1.37	2707	18.9	14305	1 2	3 4 5 6 7 8
AG16	205	1.43	3076	21.5	14305	1 2	3 4 5 6 7 8
USS14	287	2.01	3632	25.4	14305	1 2	3 4 5 6 7 8

Figure 11: In Wake county, we see that the number of Democrats in the first two districts is exceptionally low. Looking across the different Ranked Ordered Marginal Histograms, we see that this increases the electoral environments (as captured in different elections) in which the Republican party wins one of these two districts. In particular, Dr. Barber's ensemble would lead to the Democrats typically winning one of these two districts in cases where the enacted plan does not.

7 Comments on Sampling Methods

We now give some additional details to clarify some of the terms we used and the procedures we followed in sampling of the legislative maps in our original report in light of the discussion in Dr. Barber's report.

We recall that in the Legislative case we used parallel tempering to interpolate between a base measure equal to the uniform measure on spanning forests given the county and population constraints and a measure centered on the districts with a compactness similar to the enacted plan. The Primary ensemble for the legislative ensemble reported in the report was the latter of these two ensembles. The first of these ensembles would be the target distribution of the SMC algorithms from the *rdist* package when it is properly configured with resampling included. We took 4 million steps (proposals the Metropolis-Hastings algorithm) at the spanning tree level and 2 million steps on the other levels. We output maps every 25 steps for a total of 160,000 maps in the 4 million step case and 80,000 map in the 2 million step cases. We interpolated between the different ensembles using between 60 and 100 parallel tempering levels. We proposed switching between the parallel tempering levels every 100 steps. In some cases, we ran a number of clusters together in one sampling run and sometimes we ran them separately or is smaller subgroups in a single run. Generally we ran the larger, more compacted clusters such as Wake or Mecklenburg, in this way.[3] As described in the original report, *independent sample reservoirs* were used to split the 60 to 100 levels into computationally feasible chunks. This also improved the mixing and decorrelation properties of our algorithm. The congressional ensemble was drawn from a measure with a compactness weight against the same tree measure that the resampled rdist algorithm would sample. We used 12 parallel temping levels to move between the distribution without a compactness measure and the finial target distribution with the sampling weight. The number of steps was as specified above. The weights and other parameters used in the different run are specified in the header files of the datasets.

[3]For one run in the Senate, we only ran Granville-Wake for 1 million steps as we had strong evidence that this was sufficient for the parameter values being considered.

I declare under penalty of perjury under the laws of the state of North Carolina that the foregoing is true and correct to the best of my knowledge.

Jonathan Mattingly, 12/28/2021